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BELTRAMI OPERATORS, NON-SYMMETRIC ELLIPTIC EQUATIONS AND QUANTITATIVE JACOBIAN BOUNDS

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Abstract. In recent studies on the G-convergence of Beltrami operators, a number of issues arouse concerning injectivity properties of families of quasiconformal mappings. Bojarski, D'Onofrio, Iwaniec and Sbordone formulated a conjecture based on the existence of a so-called primary pair. Very recently, Bojarski proved the existence of one such pair. We provide a general, constructive, procedure for obtaining a new rich class of such primary pairs.

This proof is obtained as a slight adaptation of previous work by the authors concerning the nonvanishing of the Jacobian of pairs of solutions of elliptic equations in divergence form in the plane. It is proven here that the results previously obtained when the coefficient matrix is symmetric also extend to the non-symmetric case. We also prove a much stronger result giving a quantitative bound for the Jacobian determinant of the so-called *periodic* σ-harmonic sense preserving homeomorphisms of $\mathbf{C}$ onto itself.

1. Introduction

In order to explain the results of this paper and their motivations, it is necessary to introduce a number of topics, and to illustrate their mutual relationships. These topics are Beltrami operators and their associated concept of G-convergence, non-symmetric elliptic operators in divergence form and H-convergence, σ-harmonic mappings.

1.1. The G-convergence of Beltrami operators and the $K > 3$ conjecture. Recently Iwaniec et al. [27] and Bojarski et al. [16], introduced a notion of G-convergence for Beltrami operators, aimed at generalizing to this context the well-known theory of G-convergence initiated by Spagnolo [40] and De Giorgi [21]. Let us recall their definitions and the main conjecture in [16]. Let Ω be a bounded, simply connected open subset of $\mathbf{R}^2$, and, as usual, let us identify points $x = (x_1, x_2) \in \mathbf{R}^2$ with points $z \in \mathbf{C}$ through the relation $z = x_1 + ix_2$. Let ν and μ be two complex valued measurable functions defined on Ω and satisfying, for some $K \geq 1$, the following ellipticity condition

$$|\mu| + |\nu| \leq \frac{K-1}{K+1}. \tag{1.1}$$

Consider the following first order non homogeneous Beltrami equation

$$f_{\overline{z}} - \mu f_z - \nu \overline{f_z} = g. \tag{1.2}$$

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Given a sequence of pairs of Beltrami coefficients (μ_j, ν_j) and an extra pair (μ, ν) all satisfying (1.1), for a fixed $K \geq 1$, one denotes by $\mathscr{B}_j$, $\mathscr{B}$ the differential operators defined as follows

$$\mathscr{B}_j := \frac{\partial}{\partial \bar{z}} - \mu_j \frac{\partial}{\partial z} - \nu_j \overline{\frac{\partial}{\partial z}}, \tag{1.3}$$

$$\mathscr{B} := \frac{\partial}{\partial \bar{z}} - \mu \frac{\partial}{\partial z} - \nu \overline{\frac{\partial}{\partial z}}, \tag{1.4}$$

so that (1.2) can be rewritten as

$$\mathscr{B} f = g. \tag{1.5}$$

The authors in [27] introduce the following definition, and prove Theorem 1.2 below.

Definition 1.1. The sequence of differential operators $\mathscr{B}_j$ is said to G-converge to $\mathscr{B}$ if, for any sequence $f_j \in W^{1,2}(\Omega; \mathbf{C})$ which converges weakly to $f \in W^{1,2}(\Omega; \mathbf{C})$, and such that $\mathscr{B}_j f_j$ converges strongly in $L^2(\Omega; \mathbf{C})$, one has

$$\lim_{j \to +\infty} \mathscr{B}_j f_j = \mathscr{B} f \tag{1.6}$$

strongly in $L^2(\Omega; \mathbf{C})$.

Theorem 1.2. ([27]) *For any $K \in [1, 3]$, the family of Beltrami operators defined by (1.4) and satisfying (1.1) is G-compact.*

In order to explain our new main results and to put the previous one into context, let us begin by explaining the main point in the proof of Theorem 1.2.

As previously outlined one of the main results in [27] is a compactness result obtained under an assumption of *small ellipticity*, that is, $K \leq 3$ in (1.1).

The key to this result relies on the following issue. Let Ω be a bounded, open and convex set. Let (μ, ν) be a Beltrami pair satisfying (1.1) and let Φ and Ψ be the solutions to

$$\begin{cases} \Phi_{\bar{z}} = \mu \Phi_z + \nu \overline{\Phi_z} & \text{in } \Omega, \\ \mathfrak{Re}\Phi = x_1 & \text{on } \partial\Omega, \\ \Psi_{\bar{z}} = \mu \Psi_z + \nu \overline{\Psi_z} & \text{in } \Omega, \\ \mathfrak{Re}\Psi = x_2 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where the boundary conditions are understood in the sense of $W^{1,2}(\Omega)$ traces. The pair (Φ, Ψ) is called a primary pair. In [16] the authors formulate the following conjecture.

Conjecture 1.3. Let (μ, ν) be complex valued measurable coefficients satisfying (1.1). Then the pair of quasiconformal mappings Φ and Ψ defined by (1.7) satisfies the following pointwise inequality:

$$\mathfrak{Im}(\Phi_z \overline{\Psi_z}) > 0 \quad \text{almost everywhere in } \Omega. \tag{1.8}$$

In Section 12 of [27], it is proven that, if Conjecture 1.3 holds, then Theorem 1.4 follows. See also [27, 16].

As a consequence of our results we prove that (1.8) holds and therefore we obtain the following result.

Theorem 1.4. *For any $K \in [1, +\infty)$, the family of Beltrami operators defined by (1.4) and satisfying (1.1) is G-compact.*

Very recently, Bojarski [15] has proved a result which also implies Theorem 1.4 but does not solve Conjecture 1.3. More precisely he has proven that given Ω and a Beltrami pair (μ, ν) satisfying (1.1) there exists a *primary pair* (Φ, Ψ) so that Φ and Ψ are quasiconformal mappings of the complex plane onto itself satisfying the Beltrami equations with coefficients μ and ν and satisfy (1.8). Bojarski's primary pair is obtained by requiring the so-called *hydrodynamical normalization*, that is, by looking for a globally homeomorphic solution of $\mathbf{C}$ onto itself obtained as follows. First extend (μ, ν) to be zero in the complement of Ω. Then look for a solution of the new Beltrami equation defined on $\mathbf{C}$. Such a solution will be holomorphic near infinity. Then normalize the behaviour at infinity of such function. By the seminal work of Bojarski (see the references of [15]), it is known that one obtains a quasiconformal mapping of $\mathbf{C}$ onto itself. This beautiful construction however does not set the question of whether the Dirichlet data in (1.7) will provide us with a primary pair. We prove that this is the case in Theorem 1.10. In fact we provide a large class of Dirichlet boundary data achieving the desired task. We use the combination of Theorem 2.4 and Theorem 3.1. See Corollary 3.2.

1.2. Second order equations in divergence form, ellipticity and H-convergence. It is well known that Beltrami equations with complex dilatations ν and μ give rise in a very natural way to second order elliptic operators whose coefficient matrices σ depend in an explicit way upon ν and μ and conversely. A brief review will be offered in the following subsection. The authors in [27, 16] use the notion of G-convergence for Beltrami operators also to induce a concept of G-convergence for second order non-symmetric operators in divergence form (see Definition 2 in [16]) and to treat the G-convergence of second order non-divergence equations (see [27]). We shall not enter such issues in this note, however we observe that it is also instructive to recall the notion H-convergence introduced by Murat and Tartar for possibly non-symmetric, elliptic operators in divergence form. An easily accessible reference is [36]. The original work dates back to 1977 (see the quoted reference for more details).

Definition 1.5. Consider a bounded, open, simply connected set $\Omega \subset \mathbf{R}^2$. Given positive constants α and β, we say that a measurable function σ, defined on Ω with values into the space of 2×2 matrices, belongs to the class $\mathscr{M}(\alpha, \beta, \Omega)$ if one has

$$\begin{aligned} \sigma(z)\xi \cdot \xi &\geq \alpha|\xi|^2 \quad \text{for every } \xi \in \mathbf{R}^2 \text{ and for a.e. } z \in \Omega, \\ \sigma^{-1}(z)\xi \cdot \xi &\geq \beta^{-1}|\xi|^2 \quad \text{for every } \xi \in \mathbf{R}^2 \text{ and for a.e. } z \in \Omega. \end{aligned} \tag{1.9}$$

It is obvious that, for $\lambda = \alpha$ and for some $M > 0$, such bounds are equivalent to the usual ellipticity bounds for second order elliptic operators, see for instance [28, Chapter 8]

$$\begin{aligned} \sigma(z)\xi \cdot \xi &\geq \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbf{R}^2 \text{ and for a.e. } z \in \Omega, \\ \sum_{i,j=1}^{2} |\sigma_{ij}(z)|^2 &\leq M \quad \text{for a.e. } z \in \Omega. \end{aligned} \tag{1.10}$$

Yet another notion, originally used for the H-convergence is the following.

Definition 1.6. A matrix σ with measurable entries belongs to $M(\lambda,\Lambda,\Omega)$ if

$$\begin{aligned} \sigma(z)\xi\cdot\xi &\geq \lambda|\xi|^2 \quad \text{for every } \xi\in\mathbf{R}^2 \text{ and for a.e. } z\in\Omega,\\ |\sigma(z)\xi| &\leq \Lambda|\xi| \quad \text{for every } \xi\in\mathbf{R}^2 \text{ and for a.e. } z\in\Omega. \end{aligned} \tag{1.11}$$

However, different ways of bounding sets of matrices σ may or may not give rise to compact classes with respect to convergences of weak type. To explain this let us recall the notion of H-convergence [36].

Definition 1.7. We say that a sequence of elliptic matrices $\sigma_j \in \mathscr{M}(\alpha,\beta,\Omega)$ H-converges to $\sigma_0 \in \mathscr{M}(\alpha,\beta,\Omega)$ if for any $f\in H^{-1}(\Omega)$ the weak solution u_j to

$$-\operatorname{div}(\sigma_j\nabla u_j) = f \quad \text{in } \Omega, \quad u_j \in W_0^{1,2}(\Omega), \tag{1.12}$$

satisfies the following properties

$$\begin{cases} u_j \rightharpoonup u_0 & \text{weakly in } W^{1,2}(\Omega),\\ \sigma_j\nabla u_j \rightharpoonup \sigma_0\nabla u_0 & \text{weakly in } L^2(\Omega), \end{cases} \tag{1.13}$$

where u_0 denotes the weak solution to

$$-\operatorname{div}(\sigma\nabla u_0) = f \quad \text{in } \Omega, \quad u_0 \in W_0^{1,2}(\Omega). \tag{1.14}$$

One of the main results in this theory is compactness. Given any sequence $\{\sigma_j\}\subset \mathscr{M}(\alpha,\beta,\Omega)$ there exists a subsequence which H-converges to some element of $\mathscr{M}(\alpha,\beta,\Omega)$. It is worth noting here that the compactness does indeed depend on the specific character of the ellipticity bounds given by Murat and Tartar. For instance, it is known that the set of matrices in $M(\lambda,\Lambda,\Omega)$, that is the set constrained by (1.11), is *not* compact for H-convergence. Murat and Tartar proved that a sequence of matrices in $M(\lambda,\Lambda,\Omega)$ admits (up to subsequence) an H-limit in the class $M\left(\lambda,\frac{\Lambda^2}{\lambda},\Omega\right)$. An explicit example given by Marcellini in [34] shows that there exist a sequence $\{\sigma_j\}\subset M(\lambda,\Lambda,\Omega)$ such that its H-limit σ_0 is constant (with respect to position) and satisfies

$$\inf_{|\xi|=1}\sigma_0\xi\cdot\xi = \lambda, \quad \sup_{|\xi|=1}|\sigma_0\xi| = (\Lambda^2/\lambda).$$

Let us also recall that the approach of Murat and Tartar has been later extended to larger classes of operators (under the name of G-convergence) by Dal Maso, Chiadò-Piat and Defranceschi [20].

1.3. Beltrami equations, second order equations in divergence form and ellipticity. Let us recall now the basic algebraic relationship between second order elliptic equations in divergence form and linear first order systems. Given $\sigma\in\mathscr{M}(\alpha,\beta,\Omega)$, let $u\in W^{1,2}_{\text{loc}}(\Omega)$ be a weak solution to

$$\operatorname{div}(\sigma\nabla u) = 0 \quad \text{in } \Omega. \tag{1.15}$$

Then there exists $\tilde u \in W^{1,2}_{\text{loc}}(\Omega)$, called the *stream function* of u, such that one has

$$\nabla\tilde u = J\sigma\nabla u \quad \text{in } \Omega, \quad J := \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}. \tag{1.16}$$

Setting

$$F = u + i\tilde u \tag{1.17}$$

one has $F = u + i\tilde{u} \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ and one writes, in complex notations,

$$F_{\bar{z}} = \mu F_z + \nu \bar{F}_z \quad \text{in } \Omega, \tag{1.18}$$

where, the so called complex dilatations μ, ν are given by

$$\mu = \frac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{1 + \operatorname{Tr} \sigma + \det \sigma}, \quad \nu = \frac{1 - \det \sigma + i(\sigma_{12} - \sigma_{21})}{1 + \operatorname{Tr} \sigma + \det \sigma}, \tag{1.19}$$

and satisfy (1.1) for some $K \geq 1$ only depending on α, β, or in other words F is a quasiregular mapping.

In this paper we are interested in the opposite route, as well. Given measurable complex valued functions μ and ν satisfying (1.1), consider the matrix σ defined as follows

$$\sigma := \begin{pmatrix} \frac{|1-\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} & \frac{2\Im\mathfrak{m}(\nu - \mu)}{|1+\nu|^2 - |\mu|^2} \\ \\ \frac{-2\Im\mathfrak{m}(\nu + \mu)}{|1+\nu|^2 - |\mu|^2} & \frac{|1+\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} \end{pmatrix}, \tag{1.20}$$

which is obtained just by inverting the algebraic system (1.19). One can check [10] that if (1.1) holds for some for given $K \geq 1$, then there exists $\alpha, \beta > 0$ such that (1.9) holds for σ as defined in (1.20). In short, ellipticity in the Beltrami sense implies ellipticity in the Murat and Tartar sense.

The exact relationship between K and (α, β) will not play a crucial role here. However, we shall prove the following.

Proposition 1.8. *Let (μ, ν) satisfy the ellipticity condition (1.1), let σ be defined via (1.20). Then σ satisfies (1.9) with*

$$\alpha = \frac{1}{K} \quad \text{and} \quad \beta = K. \tag{1.21}$$

Conversely assume that $\sigma \in \mathscr{M}(\lambda, \frac{1}{\lambda}, \Omega)$ for some $\lambda \in (0, 1]$ and let (μ, ν) be defined by (1.19). Then (μ, ν) satisfy the ellipticity condition (1.1) with K defined as follows

$$K = \frac{1 + \sqrt{1 - \lambda^2}}{\lambda}. \tag{1.22}$$

See Section 5 for a proof, which also shows the optimality of these bounds.

1.4. Quasiconformal solutions to (1.16). A question that is crucial in the mere formulation of Conjecture 1.3 is the following.

Is it possible to prescribe a Dirichlet boundary data g on the real part of F as defined in (1.17) so that the solution to (1.18) with that boundary data is globally one-to-one?

Or, equivalently, for $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$, consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

Under which condition on g the mapping $F = u + i\tilde{u}$ is one-to-one?

We recall that solutions to the Beltrami equation (1.18) are K-quasiregular mapping, therefore the question can be rephrased as requiring a boundary data which give rise to a global quasiconformal solution.

Such issues turned out to be very important in applications of very different character [5, 10, 33, 8, 23] and were addressed already in past years.

The relevant notion in this context is *unimodality.* Assume that $\partial\Omega$ is a simple closed curve. We say that a continuous, real valued function g on $\partial\Omega$ is *unimodal* if $\partial\Omega$ can be split into two simple arcs on which g is separately monotone (increasing on one arc and decreasing on the other, once the orientation on $\partial\Omega$ is fixed). We shall also say that g is *strictly unimodal* if it is strictly monotone on the same arcs. We shall prove the following.

Theorem 1.9. *Let $F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ be a solution to* (1.18) *such that $u = \mathfrak{Re} F \in C(\overline{\Omega})$. If $g = u|_{\partial\Omega}$ is unimodal then F is one-to-one in Ω.*

The above statement summarizes a circle of reasonings which, in the last two decades, has been repeatedly used in various contexts [4, 5, 10, 7]. See in particular [10, Proposition 3.7], where indeed an interior Hölder bound for F^{-1} is obtained. A sketch of a proof is given, for the convenience of the reader in Section 5.

The first result in this direction we are aware of is due to Leonetti and Nesi [33, Theorem 5]. Indeed they proved a stronger statement.

If g is strictly unimodal and $F \in C(\overline{\Omega}; \mathbf{C})$ then F is one-to-one in $\overline{\Omega}$.

In fact, in [33] there are two additional assumptions, that Ω is a disk, and that σ is symmetric, that is, in other words, $\mathfrak{Im}\,\nu = 0$. However, such assumptions are indeed immaterial, in fact we can always reduce to the case that Ω is a disk by a conformal mapping, and if F solves (1.18) then, as is well-known, it also solves a similar equation with $\nu = 0$ and μ replaced by

$$\tilde{\mu} = \mu + \frac{\overline{F_z}}{F_z}\nu. \tag{1.24}$$

Later, a result of the same sort was proven also in [16, Theorem 6.1]. In this case the assumptions are that $F \in W^{1,2}(\Omega, \mathbf{C})$ and that $g = \mathfrak{Re} F_0$ where F_0 is a given quasiconformal mapping whose one-to-one image is a convex domain. It is worth noticing that this last set of hypotheses clearly *implies* both $F \in C(\overline{\Omega}; \mathbf{C})$ and the unimodality of g.

1.5. σ-harmonic mappings. Now we review several known results about the so-called σ-harmonic mappings. We close this subsection by reformulating Conjecture 1.3 in the language of σ-harmonic mappings and stating Theorem 1.10 which proves Conjecture 1.3. Possibly because of a slightly different language, several results which were published before [16, 27] may have escaped the authors' attention. We review here those of more immediate relevance for Conjecture 1.3 and postpone a few of them to the following Sections. In order to rephrase what is already known it is convenient to use the following notation. We fix $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and we denote by $U = (u_1, u_2)$ the $W^{1,2}(\Omega, \mathbf{R}^2)$ solution to

$$\begin{cases} \operatorname{div}(\sigma \nabla u_1) = 0 & \text{in } \Omega, \\ u_1 = x_1 & \text{on } \partial\Omega, \\ \operatorname{div}(\sigma \nabla u_2) = 0 & \text{in } \Omega, \\ u_2 = x_2 & \text{on } \partial\Omega. \end{cases} \tag{1.25}$$

Finally we define the stream functions associated to u_1 and u_2 to be $\tilde{u}_1$ and $\tilde{u}_2$ respectively. Using these notations and recalling (1.7), we have the identities

$$\Phi \equiv u_1 + i\tilde{u}_1, \quad \Psi \equiv u_2 + i\tilde{u}_2. \tag{1.26}$$

Alessandrini and Nesi use the terms σ-harmonic functions and σ-harmonic mapping for u_1, u_2 and U respectively. With this language, one can compute

$$\mathfrak{Im}(\Phi_z \overline{\Psi_z}) = \frac{1 + \operatorname{Tr}\sigma + \det\sigma}{4} \det DU. \tag{1.27}$$

Note also that (1.9) implies

$$\operatorname{Tr}\sigma \geq 2\alpha, \quad \frac{\operatorname{Tr}\sigma}{\det\sigma} \geq 2\beta^{-1},$$

and hence

$$(1 + \operatorname{Tr}\sigma + \det\sigma) > 0. \tag{1.28}$$

The interest of these calculations shall be evident after the following Theorem and Remark.

Theorem 1.10. *Let $\sigma \in \mathscr{M}(K^{-1}, K, \Omega)$. If Ω is convex, then the σ-harmonic mapping U defined by (1.25) satisfies*

$$\det DU > 0 \quad \textit{almost everywhere in } \Omega. \tag{1.29}$$

Remark 1.11. It is a straightforward matter to conclude that, by (1.27) and (1.28), Theorem 1.10 proves Conjecture 1.3 and, consequently, Theorem 1.4.

A proof of Theorem 1.10 will be given in Section 3.

The first result towards Theorem 1.10 was proven by Bauman, Marini and Nesi [13]. They proved the assertion under the assumption that σ is symmetric and of class C^α. A further advance was obtained by Alessandrini and Nesi [7] under the assumption that σ is symmetric with measurable entries. The two papers follow a common scheme, first one proves that under suitable conditions on the boundary data (which are indeed satisfied for the problem (1.25) when Ω is convex) the mapping U is one-to-one. Here the guiding light is a conjecture by Radò [38], which was first proved by Kneser [30] and later, independently, by Choquet [17], in the case when U is harmonic. See Theorem 2.4 below, for further details. Second, one proves that if U is locally injective, and sense preserving, then $\det DU > 0$ almost everywhere. In this case the paradigmatic result, in the harmonic setting, is due to Lewy [32]. Actually, in the harmonic case, and in the case $\sigma \in C^\alpha$, one obtains that $\det DU$ is strictly positive, uniformly on compact subsets. In the case when σ has measurable entries, such uniform bound cannot hold true. Instead, in [7] it is proven that for any subset D compactly contained in Ω one has

$$\log(\det DU) \in \mathrm{BMO}(D) \tag{1.30}$$

which, as is well-known implies that there exist $C, \varepsilon > 0$ such that in any square $Q \subset \Omega$ one has

$$\left(\frac{1}{|Q|}\int_Q (\det DU)^{\varepsilon}\, dx\right)\left(\frac{1}{|Q|}\int_Q (\det DU)^{-\varepsilon}\, dx\right) \leq C \tag{1.31}$$

which clearly implies Theorem 1.10.

Therefore, when σ is symmetric, the tools to prove Conjecture 1.3 were already available. Later Bojarski, D'Onofrio, Iwaniec and Sbordone addressed the more general question in the case when σ is not necessarily symmetric. They proved Conjecture 1.3 in two cases. First when the coefficients are Hölder continuous so extending the results by Bauman et al. to the non-symmetric case. Second they proved the result when $K \leq 3$ so extending the result of Alessandrini and Nesi to the non-symmetric case in that regime.

In the next two Sections we shall show that the procedure outlined above for the symmetric case and developed by the authors in [7] also apply to the non-symmetric case. In fact these proofs already appeared in 2003 as a part of the *Laurea Thesis* of Natascia Fumolo [24], an undergraduate student of the first author. In this paper we present a much shorter version by outlining the very few slight changes needed to adapt the arguments in [7]. On the other hand, some more delicate issues concerning the precise ellipticity constants, like in Proposition 1.8 are treated in a more efficient way here.

In Section 2 below, we summarize some of the results obtained in [7] which extend to the non-symmetric case in a straightforward fashion.

Section 3 contains the core results of this paper, the main result being Theorem 3.1. From the standpoint of primary pairs the main implication is Corollary 3.2.

In Section 4 we discuss consequences and improvements to Theorem 3.1 in the case of periodic conductivities σ, which is relevant in the context of homogenization and also in connection to issues concerning the *rigidity* of gradient fields where quasiconvex hulls are defined either by using affine or periodic boundary conditions. We refer to [23], [3], [2], [1] for more details. The main result here is Theorem 4.1, which provides a novel, stronger, quantitative formulation of the non-vanishing of the Jacobian determinant, in terms of Muckenhoupt weights.

Section 5 contains proofs of some auxiliary results.

The final Section 6 collects further developments, remarks and connections with various relevant areas and applications. In § 6 we extend some area formulas first discussed in [9]. In § 6 we lay a bridge towards the theory of *correctors* in homogenization. Finally § 6 develops an application of the Theorem by Astala [11], generalizing results in [33] and [9].

2. Preliminaries

In this Section, Ω is a simply connected open subset of $\mathbf{R}^2$ and, for applications which will be discussed in Section 4, we also admit here that Ω be unbounded, possibly the whole $\mathbf{R}^2$. We consider matrix valued functions $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ as defined in (1.9).

Notation 2.1. Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $U = (u_1, u_2) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ be σ-harmonic. We denote by $\tilde{U} := (\tilde{u}_1, \tilde{u}_2)$ the *vectorial stream function* associated to U. Moreover, for any given non zero constant vector ξ we set $f = U \cdot \xi + i\,\tilde{U} \cdot \xi$.

Proposition 2.2. *Let $\Omega \subseteq \mathbf{R}^2$ be simply connected and open. Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $U = (u_1, u_2) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ be σ-harmonic. If for every non zero ξ, f is univalent, then U is univalent.*

The proof is identical to the proof of Proposition 1 in [7]. In the latter symmetry of σ was assumed but never used. Details can be found in [24].

Theorem 2.3. *Let $\Omega \subseteq \mathbf{R}^2$ be a simply connected and open set. Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $U = (u_1, u_2) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ be σ-harmonic. Adopt the Notation 2.1. We have that the following properties are equivalent:*

$$\begin{array}{ll} \text{(i)} & f \text{ is locally one-to-one for every non zero vector } \xi, \\ \text{(ii)} & U \text{ is locally one-to-one for every non zero vector } \xi, \\ \text{(iii)} & \tilde{U} \text{ is locally one-to-one for every non zero vector } \xi. \end{array} \tag{2.1}$$

Also in this case, the proof is identical to the proof of Theorem 3 in [7], since symmetry of σ was assumed but never used. In fact, additional equivalent conditions to (i)–(iii) were stated in [7], which involve the notion of *geometrical critical point*, we omit them here for the sake of simplicity. Details can be found in [24].

Theorem 2.4. *Let Ω be a bounded open set whose boundary is a simple closed curve and let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$. Let $\phi = (\phi_1, \phi_2)\colon \partial\Omega \to \mathbf{R}^2$ be a sense preserving homeomorphism of $\partial\Omega$ onto a simple closed curve Γ which is the boundary of a* convex *domain D. Let $U \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbf{R}^2) \cap C^0(\overline{\Omega}; \mathbf{R}^2)$ be the σ-harmonic mapping with components u_1 and u_2 solving*

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u_i(x)) = 0 & \text{in } \Omega, \ i = 1, 2, \\ u_i = \phi_i & \text{on } \partial\Omega, \ i = 1, 2. \end{cases} \tag{2.2}$$

Then

$$U \text{ is a sense preserving homeomorphism of } \overline{\Omega} \text{ onto } \overline{D}. \tag{2.3}$$

Again, the proof is identical to the proof of Theorem 4 in [7], and details can be found in [24]. Theorem 2.4 generalizes to the measurable, non-symmetric, context the celebrated result of Kneser [30] who solved a problem raised by Radò [38].

3. Jacobian of a σ-harmonic mapping: the BMO bound

The main subject of this Section is the proof of Theorem 1.10. We will preliminarily proof a much more general result, namely Theorem 3.1.

We recall that, given an open set $D \subset \mathbf{R}^2$, $\phi \in L^1_{\mathrm{loc}}(D)$ belongs to $\mathrm{BMO}(D)$ if

$$\|\phi\|_* = \sup_{Q \subset D} \left(\frac{1}{|\, Q \,|} \int_Q |\, \phi - \phi_Q \,| \right) < \infty$$

where Q is any square in D and $\phi_Q = \frac{1}{|Q|} \int_Q \phi$. Recall also that the normed space $(\mathrm{BMO}(D), \| \cdot \|_*)$ is in fact a Banach space. The main object of this Section is the following.

Theorem 3.1. *Let Ω be an open subset of $\mathbf{R}^2$, let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^2)$ be a σ-harmonic mapping which is locally one-to-one and sense preserving. For every $D \subset\subset \Omega$ we have*

$$\log(\det DU) \in \mathrm{BMO}(D). \tag{3.1}$$

Corollary 3.2. *Let (μ, ν) be a Beltrami pair satisfying (1.1) and let Φ and Ψ be the solutions to*

$$\begin{cases} \Phi_{\bar{z}} = \mu \Phi_z + \nu \overline{\Phi_z} & \text{in } \Omega, \\ \mathfrak{Re}\Phi = \phi_1 & \text{on } \partial\Omega, \\ \Psi_{\bar{z}} = \mu \Psi_z + \nu \overline{\Psi_z} & \text{in } \Omega, \\ \mathfrak{Re}\Psi = \phi_2 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\phi = (\phi_1, \phi_2)$, as in Theorem 2.4, defines the convex set D. Then Φ and Ψ are quasiconformal mappings defined on Ω which satisfy the inequality

$$\mathfrak{Im}(\Phi_z \overline{\Psi_z}) > 0 \quad \text{almost everywhere in } \Omega. \tag{3.3}$$

The proof of Theorem 3.1 needs some preparation. It will be presented at the end of this Section. This part requires slightly more extended changes with respect to the work in [7]. For this reason more details will be given.

We recall below two fundamental results, Theorems 3.3 and 3.4, which will be needed for a proof of Theorem 3.1.

Theorem 3.3. (Reimann [39]) *Let f be a quasiregular mapping on the open set $D \subset \mathbf{R}^2$, then for every $D' \subset\subset D$*

$$\log(\det Df) \in \mathrm{BMO}(D').$$

Proof. See [39, Theorem 1, Remark 2]. □

Theorem 3.4. (Reimann [39]) *Let $f\colon D \to G$ be a quasiconformal mapping, $D, G \subset \mathbf{R}^2$. For every $D' \subset\subset D$, there exists $C > 0$ such that*

$$\|v \circ f\|_* \leq C\|v\|_* \quad \text{for every } v \in \mathrm{BMO}(f(D')).$$

Proof. See [39, Theorem 4] and also [29, p. 58]. □

The next Theorem requires the notion of adjoint equation for a nondivergence elliptic operator. Let $G \subset \mathbf{R}^2$ be an open set. Let $a \in \mathscr{M}(\alpha, \beta, G)$. Set

$$L = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

We say that $v \in L^1_{\mathrm{loc}}(G)$ is a weak solution of the adjoint equation

$$L^* v = 0 \quad \text{in } G, \tag{3.4}$$

if

$$\int_G vLu = 0 \quad \text{for every } u \in W_0^{2,2}(G).$$

We remark that, usually, the ellipticity bounds for a are expressed in the form (1.10), rather than (1.9), but this plays no role here.

Theorem 3.5. (Bauman [12], Fabes and Strook [22]) *For every $w \in L^2_{\mathrm{loc}}(G)$, $w \geq 0$, which is a weak solution of the adjoint equation (3.4) we have*

$$\left(\frac{1}{|Q|}\int_Q w^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|Q|}\int_Q w\right) \tag{3.5}$$

for every square Q such that $2Q \subset G$. Here $C > 0$ only depends on the ellipticity constants α and β.

Proof. This Theorem is a slight adaptation between [12, Theorem 3.3] and [22, Theorem 2.1]. A proof is readily obtained by following the arguments in [22]. The only additional ingredient which is needed here, is the observation that, with no need of any smoothness assumption on the coefficients of L, for the special case when the dimension is two (which is of interest here), for any ball $B \subset G$ and any $f \in L^2(B)$ there exists and it is unique, the strong solution

$$u \in W^{2,2}(B) \cap W_0^{1,2}(B)$$

to the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

see [41, Theorem 3]. □

Proof of Theorem 3.1. Preparation. Let $U = (u_1, u_2)$ satisfy the hypotheses of Theorem 3.1 and let

$$f = u_1 + i\tilde{u}_1 \tag{3.6}$$

be the quasiregular mapping introduced in Notation 2.1 with $\xi = (1, 0)$. In view of Theorem 2.3, for every $z \in \Omega$, we can find a neighborhood D of z, $D \subset\subset \Omega$ such that $U|_D$ and $f|_D$ (i.e. the restrictions of U and f to D) are univalent. Therefore, for the proof of Theorem 3.1, it suffices to show that (3.1) holds for any sufficiently small $D \subset\subset \Omega$, such that $U|_D$ and $f|_D$ are univalent. We set

$$G = f|_D(D)$$

and $V : G \to \mathbf{R}^2$ given by

$$V = U|_D \circ (f|_D)^{-1} \tag{3.7}$$

where, by definition $(f|_D)^{-1} : G \to D$. From now on, with a slight abuse of notation, we will drop the subscripts denoting restrictions to D. We have $DU = (DV \circ f)Df$, and hence

$$\log(\det DU) = \log(\det DV) \circ f + \log(\det Df)\ . \tag{3.8}$$

In view of Theorems 3.3 and 3.4, the thesis will be proven as soon as we show that $\log(\det DV)$ belongs to BMO on compact subsets of G. The advantage in replacing U by V, lies in the observation that, in contrast with $\det DU$, $\det DV$ satisfies an equation of the type (3.4) for a suitable choice of the operator L^*.

In fact, letting v_1 and $\tilde{v}_1$ be the first component of V and its stream function respectively, we can compute

$$\begin{aligned} v_1(z) &= u_1 \circ f^{-1}(z) = u_1 \circ (u_1 + i\tilde{u}_1)^{-1}(z) = x_1, \\ \tilde{v}_1(z) &= \tilde{u}_1 \circ f^{-1}(z) = \tilde{u}_1 \circ (u_1 + i\tilde{u}_1)^{-1}(z) = x_2. \end{aligned} \tag{3.9}$$

Moreover, by definition,

$$\nabla \tilde{v}_1 = J\tau\nabla v_1, \tag{3.10}$$

where

$$\tau = T_f\sigma = \frac{Df\sigma Df^T}{\det Df} \circ f^{-1}. \tag{3.11}$$

Hence, using (3.9) and (3.10)

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

that is

$$\tau = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \tag{3.12}$$

where, by construction,

$$\begin{aligned} c &= \det\tau = \det(\sigma \circ f^{-1}) \in L^\infty(G), \\ b &= \tau_{12} = (\sigma_{12} - \sigma_{21}) \circ f^{-1} \in L^\infty(G). \end{aligned} \tag{3.13}$$

For a given σ, let us denote

$$\begin{aligned} \alpha_\sigma &= \operatorname*{ess\,inf}_{z\in\Omega} \left\{ \sigma(z)\xi\cdot\xi \text{ such that } \xi \in \mathbf{R}^2,\ |\xi| = 1 \right\}, \\ \frac{1}{\beta_\sigma} &= \operatorname*{ess\,inf}_{z\in\Omega} \left\{ (\sigma(z))^{-1}\xi\cdot\xi \text{ such that } \xi \in \mathbf{R}^2,\ |\xi| = 1 \right\}, \end{aligned} \tag{3.14}$$

that is, $\alpha_\sigma, \beta_\sigma$ are the best ellipticity constants α, β for which $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ holds. We restrict our attention to the case when $\alpha_\sigma = {\beta_\sigma}^{-1} := K^{-1}$. There is no loss of generality in this choice in view of a rescaling argument. See also Remark 4.6 for further details.

A calculation that we omit shows that, if α_τ, β_τ are defined accordingly for τ in G, we have

$$\begin{aligned} \alpha_\tau &= \operatorname*{ess\,inf}_{z\in G} \left\{ \frac{c(z) + 1 - \sqrt{(c(z)-1)^2 + b(z)^2}}{2} \right\}, \\ \frac{1}{\beta_\tau} &= \operatorname*{ess\,inf}_{z\in G} \left\{ \frac{c(z) + 1 - \sqrt{(c(z)-1)^2 + b(z)^2}}{2c(z)} \right\}. \end{aligned} \tag{3.15}$$

That is τ is elliptic in the sense of (1.9) and a calculation shows that, in fact, one can take

$$\alpha_\tau = \frac{1}{\beta_\tau} = 1 - \sqrt{1 - \frac{1}{K^2}}. \tag{3.16}$$

See Section 5 for a proof. Furthermore, by (3.7) and (3.9),

$$\det DV = \frac{\partial v_2}{\partial x_2} \in L^2(G). \tag{3.17}$$

Consequently, v_2 satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_1} + b\frac{\partial v_2}{\partial x_2} \right) + \frac{\partial}{\partial x_2}\left(c\frac{\partial v_2}{\partial x_2} \right) = 0 \quad \text{weakly in } G.$$

Differentiating the equation above with respect to x_2, we see that $w = \det DV$ is a distributional solution of

$$\frac{\partial^2}{\partial x_1^2} w + \frac{\partial^2}{\partial x_1 \partial x_2}(bw) + \frac{\partial^2}{\partial x_2^2}(cw) = 0 \quad \text{in } G,$$

that is, it is a distributional solution to the adjoint equation

$$L^* w = 0 \quad \text{in } G, \tag{3.18}$$

where

$$L = \frac{\partial^2}{\partial x_1^2} + b\frac{\partial^2}{\partial x_1 \partial x_2} + c\frac{\partial^2}{\partial x_2^2}.$$

On use of (3.18) and (3.15) we may now apply Theorem 3.5.

We summarize the resulting statement below.

Proposition 3.6. *For every square Q such that $2Q \subset G$, we have*

$$\left(\frac{1}{\mid Q \mid}\int_Q (\det DV)^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{\mid Q \mid}\int_Q \det DV\right), \tag{3.19}$$

where $C > 0$ only depends on α and β.

Proof of Theorem 3.1. Conclusion. A well known characterization of BMO in terms of the reverse Hölder inequality (see, for instance, [25, Theorem 2.11 and Corollary 2.18]), shows that Proposition 3.6 implies $\log(\det DV) \in \mathrm{BMO}(G')$ for every $G' \subset\subset G$. Thus, possibly after replacing D with $D' = f^{-1}(G')$, we have, by (3.8) and Theorems 3.3 and 3.4 that $\log(\det DU) \in \mathrm{BMO}(D)$. □

Proof of Theorem 1.10. Apply Theorem 2.4 with $\phi_1 = x_1, \phi_2 = x_2$ and $D = \Omega$, which, by assumption, is convex. Then use Theorem 3.1. □

Remark 3.7. We recall now that, in view of Remark 1.11, the proof of Theorem 1.10 concludes also the proof of Conjecture 1.3 and of Theorem 1.4. The proof of Corollary 3.2 is also immediate.

4. The periodic case

In the homogenization theory, operators with periodic coefficients play an important role. We refer to the wide literature on the subject, see for instance [14] and [35]. We want to remark here that our result has two interesting consequences in that particular setting. We set $Q = (0,1) \times (0,1)$ and we shall deal with functions which are 1-periodic with respect to each of its variables x and y, which we will call *Q-periodic*, or for short, *periodic*. For a given 2×2 matrix A, we write $U \in W^{1,2}_{\sharp,A}(Q;\mathbf{R}^2)$ for the space of zero average (on Q) vector fields U such that $U - Ax \in W^{1,2}_{\sharp}(Q;\mathbf{R}^2)$, where $W^{1,2}_{\sharp}(Q;\mathbf{R}^2)$ denotes the completion of Q-periodic function with respect to the $W^{1,2}$ norm (see [19] for more details).

We are especially interested in boundary conditions of periodic type because of their central role in homogenization and in particular in the so-called G-closure problems. In fact, our starting point for this investigation has its origin in such type of applications. Given a 2×2 matrix A, we denote by $U^A = (u_1^A, u_2^A)$ a solution (unique because of our normalization) of

$$\begin{cases} \operatorname{div}(\sigma \nabla u_1^A) = 0 & \text{in } \mathbf{R}^2, \\ \operatorname{div}(\sigma \nabla u_2^A) = 0 & \text{in } \mathbf{R}^2, \\ U^A \in W^{1,2}_{\sharp,A}(\mathbf{R}^2, \mathbf{R}^2). \end{cases} \tag{4.1}$$

The auxiliary problem (4.1) is usually called the cell problem. Solutions to (4.1) will be called, with a slight abuse of language, *periodic* σ-harmonic mappings.

In the sequel, $\alpha, \beta > 0$ and $\sigma \in \mathscr{M}(\alpha, \beta, \mathbf{R}^2)$ and Q-periodic are given.

Theorem 4.1. *Let A be a non singular 2×2 matrix and let U^A be a solution to (4.1). Then we have*

$$U^A \text{ is a homeomorphism of } \mathbf{R}^2 \text{ onto itself.} \tag{4.2}$$

Moreover there exists positive constants C, δ only depending on α and β such that, for every square $P \subset \mathbf{R}^2$ and any measurable set $E \subset P$ we have

$$\int_E \frac{\det DU^A}{\det A} \geq C \left(\frac{|E|}{|P|}\right)^\delta \int_P \frac{\det DU^A}{\det A}. \tag{4.3}$$

Here, and in the sequel, integration is meant with respect to two-dimensional Lebesgue measure.

Remark 4.2. It is worth observing that, when $P = Q$, the unit square, and $E \subset Q$, we obtain

$$\frac{|U^A(E)|}{|\det A|} \geq C|E|^\delta. \tag{4.4}$$

Which also trivially implies

$$\frac{\det DU^A}{\det A} > 0 \quad \text{almost everywhere in } \mathbf{R}^2. \tag{4.5}$$

In fact, for any σ-harmonic homeomorphism U the area formula

$$|U(E)| = \int_E |\det DU| \tag{4.6}$$

holds, see [9, Proposition 4.2], for a proof in the symmetric case, which however applies equally well to the present context. See also the discussion in the Section 6 below.

Remark 4.3. It is anticipated that quantitative Jacobian bounds, like the one obtained in (4.4), are useful to prove new bounds for effective conductivity, i.e., for classes of H-limits. See [37] and [3]. In particular [37, Theorem 3.4] gives an explicit improved bound in terms of the constants C and δ appearing in (4.4). Note the relevance of (4.4) in [37, Definition 3.7] (thanks to the preceding discussion about the role of the boundary conditions in Section 2 of that paper). However, all such developments would require a careful derivation of bounds for C and δ and are beyond the scope of this note.

Before beginning the proof Theorem 4.1, let us recall some basic facts about Muckenhoupt weights.

Definition 4.4. A non negative measurable function $w = w(z)$ with $z \in \mathbf{C}$ is an A_∞-weight if

(i) there exist constants $C, \delta > 0$ such that for every square P and every measurable set $E \subset P$ we have

$$\frac{\int_E w}{\int_P w} \leq C \left(\frac{|E|}{|P|}\right)^\delta. \tag{4.7}$$

Thus, as is well-known, the A_∞ condition is a property of absolute continuity, *uniform at all scales*, of the measure $w dx$ with respect to Lebesgue measure $\mathrm{d}x$. The following characterizations of A_∞ are also well-known, see for instance [18, Lemma 5].

Lemma 4.5. *Condition (i) above is equivalent to (ii) and (iii) below.*

(ii) *There exist constants $N, \theta > 0$ such that for every square P*

$$\left(\frac{1}{|P|}\int_P w^{1+\theta}\right)^{\frac{1}{1+\theta}} \le N\left(\frac{1}{|P|}\int_P w\right). \tag{4.8}$$

(iii) *There exists constants $M, \eta > 0$ such that for every square P and every measurable set $E \subset P$, we have*

$$\frac{\int_E w}{\int_P w} \ge M\left(\frac{|E|}{|P|}\right)^{\eta}. \tag{4.9}$$

We observe that the quantitative relationships among the pairs of constants (C, δ), (N, θ) and (M, η) appearing in the equivalent characterizations of A_∞ can be constructively evaluated, see Vessella [42].

We shall also make use of the following observation.

Remark 4.6. Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let u be σ-harmonic in Ω. Then, up to a multiplicative scaling, we have that u is also $\tilde{\sigma}$-harmonic with

$$\tilde{\sigma} = \sqrt{\frac{\beta}{\alpha}}\sigma \in \mathscr{M}\left(\sqrt{\frac{\alpha}{\beta}}, \sqrt{\frac{\beta}{\alpha}}, \Omega\right). \tag{4.10}$$

Thus in the proof below, we may assume, without loss of generality, $\sigma \in \mathscr{M}(K^{-1}, K, \Omega)$ with $K = \sqrt{\beta/\alpha}$.

Proof of Theorem 4.1. It suffices to treat the case when A is the identity matrix I because $U^A = AU^I$. From now on, for simplicity, we omit the superscript I. The proof of (4.2) follows with no substantial changes the one in [7, Theorem 1]. The proof of (4.3) consist of showing that $\det DU$ is a Muckenhoupt weight. We observe that the arguments of Theorem 3.1 tell us that $(\det DU)^\varepsilon$ is a Muckenhoupt weight for some sufficiently small $\varepsilon > 0$. Here we improve the result and show that this is true also for $\varepsilon = 1$.

By Remark 4.6, we may assume $\sigma \in \mathscr{M}(K^{-1}, K, \Omega)$ with $K = \sqrt{\beta/\alpha}$.

Using the notation of Section 3, we have $U = V \circ f$ where f now is a K-quasiconformal homeomorphism of $\mathbf{C}$ onto itself. Moreover V satisfies (3.19) for all squares in $\mathbf{C}$. Recall also that V is a τ-harmonic homeomorphism of $\mathbf{C}$ onto itself with τ given by (3.12), hence we also have that area formulas of the type (4.6) also apply to V, and obviously to f because of its quasiconformality.

By (3.19) we deduce that $\det DV$ is an A_∞-weight, and for suitable $M, \eta > 0$ only depending on K, we have

$$\int_F \det DV \ge M\left(\frac{|F|}{|P|}\right)^{\eta}\int_P \det DV \tag{4.11}$$

for any square P and any measurable set $F \subset P$.

Since f is K-quasiconformal, we have that f satisfies the following condition, which can be viewed as one of the many manifestations of the bounded distortion property of quasiconformal mappings.

There exist $q \in (0, 1)$ depending on K only such that for every square $P \subset \mathbf{C}$, there exists a square $P' \subset \mathbf{C}$ such that

$$qP' \subset f(P) \subset P'. \tag{4.12}$$

Here, if l is the length of the side of P', we denote by qP' the square concentric to P' with side $q \cdot l$. We refer to [31, Proof of Theorem 9.1] for a proof.

Therefore, we have $f(E) \subset f(P) \subset P'$ and hence

$$|U(E)| = |V(f(E))| \geq M \left(\frac{|f(E)|}{|P'|} \right)^{\eta} |V(P')|. \tag{4.13}$$

Obviously,

$$|V(P')| \geq |V(f(P))| \quad \text{and} \quad |P'| = \frac{1}{q^2} |q\, P'| \leq \frac{1}{q^2} |f(P)|.$$

Therefore

$$|U(E)| \geq Q\, q^{2\eta} \left(\frac{|f(E)|}{|f(P)|} \right)^{\eta} |U(P)|. \tag{4.14}$$

By Gehring's Theorem [26], we have that $\det Df$ satisfies a reverse Hölder inequality of the form (ii) in Lemma 4.5, with constants only depending on K. By (iii) in Lemma 4.5, there exists $L, \rho > 0$ only depending on K such that

$$\frac{|f(E)|}{|f(P)|} \geq L \left(\frac{|E|}{|P|} \right)^{\rho} \tag{4.15}$$

and finally, by (4.14) and (4.15)

$$|U(E)| \geq Q (q^2\, L)^{\eta} \left(\frac{|E|}{|P|} \right)^{\eta \rho} |U(B)|. \tag{4.16}$$

Thus (4.3) follows. □

Remark 4.7. The A_∞-property of the Jacobian determinant, obtained in Theorem 4.1 for the periodic setting, is indeed an improvement of the BMO bound obtained previously and which applies to the wider context of locally injective σ-harmonic mappings. Local versions of a bound like (4.3) could be obtained as well for locally injective σ-harmonic mappings, however it is expected that a quantitative evaluation of the constants might be more involved in this case.

5. Miscellaneous proofs

Proof of Theorem 1.9 (Sketch). By the well-known Stoïlow representation, see for instance [31, Chapter VI], there exists a quasiconformal mapping $\chi \colon \mathbf{C} \to \mathbf{C}$ such that F factorizes as $F = H \circ \chi$ with H holomorphic in $\chi(\Omega)$. Thus, up to the change of variable χ, one can assume w.l.o.g. $\mu = \nu = 0$. Then u is harmonic and $\tilde{u}$ is its harmonic conjugate. Being g unimodal, u has no critical point inside Ω [4, 6], moreover, by the maximum principle, for every $t \in (\min g, \max g)$ the level set $\{u > t\}$ is connected and the level line $\{u = t\}$ in Ω is a simple open arc. On $\{u = t\}$, $\tilde{u}$ has nonzero tangential derivative, hence it is strictly monotone there. Consequently, F is one-to-one on Ω. □

Proof of Proposition 1.8. The proof of this Proposition is a calculus matter regarding matrices σ and complex numbers μ, ν linked by the relations (1.19), or equivalently (1.20). The dependence on the space variables $z = x_1 + ix_2$ plays no role at this point, and thus we can neglect it. The inequalities (1.9) can be viewed as lower bounds on the eigenvalues of the symmetric matrices $\frac{\sigma+\sigma^T}{2}$ and $\frac{\sigma^{-1}+(\sigma^{-1})^T}{2}$. In terms of μ,ν, the lower eigenvalues of such matrices are given by

$$\frac{(1-|\mu|)^2-|\nu|^2}{|1+\nu|^2-|\mu|^2}, \quad \frac{(1-|\mu|)^2-|\nu|^2}{|1-\nu|^2-|\mu|^2}, \tag{5.1}$$

respectively. By computing the minima of such expressions as $\mu,\nu\in\mathbf{C}$ satisfy (1.1) we obtain (1.21). It is worth noticing that such minima are achieved when $\nu=|\nu|$ in the first case, and when $\nu=-|\nu|$ in the second case. In either case, the corresponding σ turns out to be symmetric.

Viceversa, if we constrain μ,ν to satisfy both limitations

$$\frac{(1-|\mu|)^2-|\nu|^2}{|1+\nu|^2-|\mu|^2}\geq\lambda, \quad \frac{(1-|\mu|)^2-|\nu|^2}{|1-\nu|^2-|\mu|^2}\geq\lambda, \tag{5.2}$$

then the maximum of $|\mu|+|\nu|$ turns out to be $\sqrt{\frac{1-\lambda}{1+\lambda}}$ and (1.22) follows. Note that in this case the maximum is achieved with μ,ν satisfying $\mu=0$ and $\mathfrak{Re}\nu=0$ which means

$$\sigma=\begin{pmatrix} a & b\\ -b & a\end{pmatrix} \quad \text{with} \quad a=\lambda,\ b=\pm\sqrt{1-\lambda^2}. \tag{5.3}$$

Let us also recall the well-known fact that, if we a-priori assume σ symmetric, then, under the constraints (5.2), the maximum of $|\mu|+|\nu|$ becomes $\frac{1-\lambda}{1+\lambda}$, that is $K=\frac{1}{\lambda}$. □

Proof of (3.16). As in the Proof of Proposition 1.8, we can neglect the dependence on the space variables $z=x_1+ix_2$. The task here is to evaluate the minimum eigenvalue of the symmetric part of the matrices τ and of τ^{-1}. It suffices to consider the case $\det\sigma\leq 1$. Indeed, up to replacing σ with σ^{-1} we can always reduce to this case. Set $D=\det\sigma$, $T=\mathrm{Tr}\sigma$ and $H=(\sigma_{12}-\sigma_{21})^2$. Elementary computations lead us to minimize the functions

$$F(D,H)=\frac{D+1-\sqrt{(D-1)^2+H}}{2}, \tag{5.4}$$

$$G(F,H)=\frac{F(D,H)}{D}, \tag{5.5}$$

subject to the constraints

$$\frac{T-\sqrt{T^2+H-4D}}{2}\geq\frac{1}{K}, \tag{5.6}$$

$$\frac{T-\sqrt{T^2+H-4D}}{2D}\geq\frac{1}{K}. \tag{5.7}$$

Note that, being $D\leq 1$, we have that (5.6) is always satisfied if (5.7) holds and also that $G(D,H)\geq F(D,H)$ with equality when $D=1$. Thus we are reduced to compute

$$\min\{F(D,H)\mid 0\leq D\leq 1,\ H,T\geq 0,(5.6)\text{ holds}\}=1+\sqrt{1-\frac{1}{K^2}}.$$

The minimum is achieved when

$$T=\frac{2}{K}, \quad D=1, \quad \text{and} \quad H=1-\frac{1}{K^2}, \tag{5.8}$$

which implies that σ has the form (5.3) with $\lambda = 1/K$. This proves that α_τ as defined in (3.15) satisfies (3.16). Consequently, by (5.5) and (5.8) we also obtain $\beta_\tau = \frac{1}{\alpha_\tau}$, proving (3.16). □

6. Further results and connections

6.1. Area formulas for σ-harmonic mappings. One of the original motivations to the study of Theorem 1.9 came from homogenization and in particular the study of bounds for *effective conductivity*, that is, H-limits. So let $\sigma \in \mathscr{M}(\alpha, \beta, \mathbf{R}^2)$ be Q-periodic ($Q = (0,1) \times (0,1)$). By its associated H-limit we mean the constant matrix σ_{eff} also called the effective conductivity defined as the H-limit of $\sigma^\varepsilon(z) := \sigma(\frac{z}{\varepsilon})$ which, as is well-known, it is defined via cell problems as follows. For any vector $\xi \in \mathbf{R}^2$, one has

$$\sigma_{\text{eff}}\xi \cdot \xi = \min\left\{ \int_Q \sigma \nabla u \cdot \nabla u \mid u - \xi \cdot x \in W^{1,2}_\sharp(Q; \mathbf{R}) \right\}. \tag{6.1}$$

Let u^ξ be the minimizer of (6.1) and let $\tilde{u}^\xi$ be its stream function. Using the notation of Section 4, we have $u^\xi = U^I \cdot \xi$. Set $f^\xi = u^\xi + i\tilde{u}^\xi$. Notice that this quasiconformal mapping coincides with the one introduced in Notation 2.1 when $U = U^I$. Here we use the superscript ξ just in order to emphasize this dependence.

Theorem 6.1. *For any nonzero vector $\xi \in \mathbf{R}^2$ one has*

$$\sigma_{\text{eff}}\xi \cdot \xi = |f^\xi(Q)|. \tag{6.2}$$

Proof. We refer to [9, Proposition 4.1]. Again in that context σ was assumed to be symmetric but the hypotheses was not used. □

The previous result transforms the problem of the calculation of the effective conductivity into a geometrical one, finding the area of the set $f^\xi(Q)$.

Next result has already been invoked in Section 3.

Theorem 6.2. *Let Ω be a bounded, open, simply connected set. Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $U \in W^{1,2}(\Omega; \mathbf{R}^2)$ be a univalent σ-harmonic mapping onto an open set D. For any measurable set $E \subset \Omega$ and any function $\phi \in L^1(D; \mathbf{R})$ one has*

$$\int_E \phi(U(x)) |\det DU(x)|\, dx = \int_{U(E)} \phi(y)\, dy. \tag{6.3}$$

Proof. We refer to [9, Proposition 4.2]. Again in that context σ was assumed to be symmetric but the hypotheses was not used. □

6.2. Correctors and H-convergence. In order to explain the meaning of our results in the context of H-convergence we need to recall the notion of correctors. It is convenient to use the operator Div which acts as the usual div operator on the rows of 2×2 matrices.

Definition 6.3. Let σ_ε be a sequence in $\mathscr{M}(\alpha, \beta, \Omega)$ which is H-converging to σ_0. Set $P^\varepsilon = DU^\varepsilon$ where, for ω open with $\omega \subset\subset \Omega$, one has that U^ε satisfies the following properties

$$\begin{cases} U^\varepsilon \in W^{1,2}(\omega; \mathbf{R}^2), & \\ U^\varepsilon \rightharpoonup \mathrm{Id} & \text{weakly in } W^{1,2}(\omega; \mathbf{R}^{2\times 2}), \\ -\mathrm{Div}(DU^\varepsilon (\sigma_\varepsilon)^T) \to -\mathrm{Div}(\sigma_0^T) & \text{strongly in } W^{-1,2}(\omega; \mathbf{R}^2). \end{cases} \tag{6.4}$$

Then P^ε is called a corrector associated with $(\sigma_\varepsilon, \sigma_0)$.

For the main properties of the correctors we refer to [36]. Let us just recall here that they exist and that, for a given sequence, σ_ε which is H-converging to σ_0, the difference between two such correctors converges strongly to zero in $L^2_{\mathrm{loc}}(\Omega; \mathbf{R}^{2\times 2})$. Our interest in this context is given by the following result.

Proposition 6.4. (Murat and Tartar [36]) *Let σ_ε be a sequence in $\mathscr{M}(\alpha, \beta, \Omega)$ which is H-converging to σ_0. Set $U^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in H^1(\Omega; \mathbf{R}^2)$ to be the unique solution to*

$$\begin{cases} \mathrm{Div}(DU^\varepsilon \sigma_\varepsilon^T) = \mathrm{Div}(\sigma_0^T) & \text{in } \Omega, \\ (u_1^\varepsilon, u_2^\varepsilon) = (x_1, x_2) & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Then $P^\varepsilon = DU^\varepsilon$ is a corrector associated with $(\sigma_\varepsilon, \sigma_0)$.

Proposition 6.4 has a particularly simple interpretation in our language when σ_0 does not depend on position. In this case (which is of fundamental importance in the so called G-closure problems), (6.5) is nothing else than a reformulation of the boundary value problem (1.25), or equivalently of (1.7), with $\sigma = \sigma_\varepsilon$ and Proposition 6.4 says that the corrector can be identified, up to an L^2 strong remainder as the Jacobian matrix of an appropriate σ-harmonic mapping.

6.3. Exponent of higher integrability. As a concluding remark, we observe a straightforward corollary to Proposition 1.8 which we state as a Theorem for the reader's convenience.

Theorem 6.5. (Astala) *Let $\sigma \in \mathscr{M}(\alpha, \beta, \Omega)$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a σ-harmonic function. Set*

$$K = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{\beta - \alpha}{\alpha}}. \tag{6.6}$$

Then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for any

$$p \in \left[2, \frac{2K}{K-1}\right).$$

Proof. As we noted already in Remark 4.6, u is also $\tilde{\sigma}$-harmonic with $\tilde{\sigma}$ given by (4.10), which belongs to $\mathscr{M}(\lambda, \lambda^{-1}, \Omega)$ and $\lambda = \sqrt{\alpha/\beta}$. By Proposition 1.8, $f = u + i\tilde{u}$ is K-quasiregular with K given by (6.6). Then one applies the celebrated Astala's Theorem [11]. $\square$

Let us emphasize here that the only, possibly new, observation is of algebraic nature. In the case when σ is symmetric the *algebraically* optimal bound is known as was pointed out in [33] and [9] and achieved for some σ's. Astala states explicitly in his paper fundamental paper [11] that the exact exponent for the σ-harmonic function seems to depend in a non obvious and complicated way on the entries of σ. Our calculation seems to set the algebraically optimal bound in the most general case of non-symmetric σ. Optimality, in the sense of the existence of a σ showing that the exponent of higher integrability cannot be improved, in the context of non symmetric σ's seems to be an open problem. Indeed, by the optimality conditions (5.3), the extremal σ cannot be symmetric almost everywhere. Therefore it appears that the putative example must be of a new type.

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