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QUASIHYPERBOLIC GEOMETRY OF PLANAR DOMAINS

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Dedicated to the memory of Juha Heinonen.

Abstract. Let $k(a,b)$ denote the quasihyperbolic distance between points a,b in a domain $G \subset \mathbf{R}^2$. We show that there is a universal constant $c_0 > 0$ with the following properties: (1) If $k(a,b) < c_0$, then there is only one quasihyperbolic geodesic from a to b. (2) If $k(a,b) < c_0$ and if γ is a quasihyperbolic geodesic from a to b, then there is a prolongation of γ to a quasihyperbolic geodesic γ_1 from a to b_1 with $k(a,b_1) = c_0$. (3) Each quasihyperbolic disk of radius $r < c_0$ is strictly convex in the euclidean metric.

1. Introduction

1.1. Let $G \subset \mathbf{R}^n$ be a domain, $n \ge 2$. We always assume without further notice that $G \ne \mathbf{R}^n$. We recall that the *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ or a path γ in G is the number

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)},$$

where $\delta(x) = \delta_G(x) = d(x, \mathbf{R}^n \setminus G) = d(x, \partial G)$. For $a,b \in G$, the *quasihyperbolic distance* $k(a,b) = k_G(a,b)$ is defined by

$$k(a,b) = \inf l_k(\gamma)$$

where the infimum is taken over all rectifiable arcs γ joining a and b in G.

We write $\gamma\colon a \curvearrowright b$ if γ is an arc from a to b. An arc $\gamma\colon a \curvearrowright b$ is a *quasihyperbolic geodesic* or briefly a *geodesic* if $l_k(\gamma) = k(a,b)$. For quasihyperbolic balls and spheres in G we use the notation

$$B_k(a,r) = \{x \in G\colon k(x,a) < r\},\ \bar{B}_k(a,r) = \{x \in G\colon k(x,a) \le r\},$$
$$S_k(a,r) = \{x \in G\colon k(x,a) = r\}.$$

A domain $D \subset \mathbf{R}^n$ is *strictly convex* if (1) it is convex and (2) D contains the open line segment (x,y) for each pair of boundary points $a,b \in \partial D$. A bounded domain is strictly convex as soon as it satisfies (2). A convex domain is strictly convex iff its boundary does not contain a line segment.

The quasihyperbolic metric of a domain in $\mathbf{R}^n$ was introduced by Gehring and Palka [GP] in 1976, and it has turned out to be a useful tool, for example, in the theory of quasiconformal maps. It is known [GO, Lemma 1] that a quasihyperbolic geodesic between given points always exists. Martin [Ma] proved in 1985 that quasihyperbolic geodesics are C^1 smooth with Lipschitz continuous derivatives. However,

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several questions on the basic quasihyperbolic geometry remain open, for example the following three conjectures:

1.2. Uniqueness conjecture. *There is a universal constant* $c_{\mathrm{U}} > 0$ *such that if* $a, b \in G$ *and* $k(a, b) < c_{\mathrm{U}}$, *then there is only one quasihyperbolic geodesic* $\gamma\colon a \curvearrowright b$.

1.3. Prolongation conjecture. *There is a universal constant* $c_{\mathrm{P}} > 0$ *such that if* $\gamma\colon a \curvearrowright b$ *is a quasihyperbolic geodesic with* $l_k(\gamma) = k(a, b) < c_{\mathrm{P}}$, *then there is a quasihyperbolic geodesic* $\gamma_1\colon a \curvearrowright b_1$ *such that* $\gamma \subset \gamma_1$ *and* $l(\gamma_1) = c_{\mathrm{P}}$.

1.4. Convexity conjecture. *There is a universal constant* $c_{\mathrm{C}} > 0$ *such that the quasihyperbolic ball* $B_k(a, r)$ *is strictly convex for all* $r < c_{\mathrm{C}}$.

If the condition of one of the conjectures, say 1.2, holds for a domain $G \subset \mathbf{R}^n$ with c_{U} replaced by a constant $c > 0$, we say that G *satisfies the conjecture* 1.2 with $c_{\mathrm{U}} = c$.

1.5. In [MV] we proved that convex domains satisfy all three conjectures without any restriction to the quasihyperbolic distance. The domain $G_1 = \mathbf{R}^2 \setminus \{0\}$ and the points $a = -1,\ b = 1$ show that one must have $c_{\mathrm{U}} \le \pi$ and $c_{\mathrm{P}} \le \pi$. Moreover, the same domain shows that $c_{\mathrm{C}} \le 1$; see [Kl, 3.5] or Corollary 3.7 of the present paper.

The main purpose of this paper is to prove that the three conjectures are true in the case $n = 2$.

We show in 2.3 and 2.6 that 1.4 $\Rightarrow$ 1.2 $\Rightarrow$ 1.3 in all dimensions. The rest of the paper is devoted to the proof of the Convexity conjecture for planar domains with the sharp constant $c_{\mathrm{C}} = 1$. This will imply the other two conjectures with $c_{\mathrm{U}} = 2,\ c_{\mathrm{P}} = \pi/2$, which will be improved to $c_{\mathrm{P}} = 2$ in 8.11.

The strategy of the proof is as follows: We start with the case of the punctured plane $G_1 = \mathbf{R}^2 \setminus \{0\}$. This domain was considered in 1986 by Martin and Osgood [MO], who made the important observation that the exponential function e^z transforms euclidean length in $\mathbf{R}^2$ to quasihyperbolic length in G_1. The quasihyperbolic geodesics and disks of G_1 are therefore well understood.

Next we consider the case where $Q = \mathbf{R}^2 \setminus G$ is a finite set. This set defines a *Voronoi diagram* $\operatorname{Vor} Q$, which is the decomposition of the plane into *Voronoi cells*

$$D_q = \{x \in \mathbf{R}^2 \colon |x - q| < |x - p| \text{ for all } p \in Q \setminus \{q\}\},$$

$q \in Q$. See, for example, [OBS]. Each cell is an open polygon, possibly a half plane or a parallel strip, and $\mathbf{R}^2 = \bigcup\{\bar{D}_q \colon q \in Q\}$. If γ is a quasihyperbolic geodesic of G and if α is a component of $\gamma \cap D_q$, then $\bar{\alpha}$ is a geodesic in $\mathbf{R}^2 \setminus \{q\}$ and therefore well known. To study the whole geodesic, we must investigate its behavior at the edges of the polygons D_q. This leads to a combinatorial analysis that will be carried out in Section 5, and we obtain the nonstrict Convexity conjecture for these domains in 5.7 and the strict one in 6.10.

Finally, to prove the Convexity conjecture for an arbitrary domain $G \subset \mathbf{R}^2$, we approximate G by a sequence of domains $G_j = \mathbf{R}^2 \setminus Q_j$ where each Q_j is a finite subset of ∂G. Given $a \in G$ and $r > 0$, the sets Q_j can be chosen so that the quasihyperbolic disk $B_k(a, r)$ of G is the intersection of the quasihyperbolic disks $B_{k_j}(a, r)$ of G_j and therefore convex if $r \le 1$. To obtain strict convexity we need estimates for the strictness of the convexity of the disks $B_{k_j}(a, r)$. The proof will be completed in 7.7. Some further results and conjectures are given in Section 8.

We use the same notation as in [V3]. In particular, arcs are assumed to be oriented, that is, equipped with one of the two possible orderings. We write $\gamma\colon a \curvearrowright b$ if γ is an arc with first point a and last point b. An arc γ is C^1 smooth or briefly *smooth* if it has a unit tangent vector $v(x)$ at every $x \in \gamma$ (one-sided at the endpoints) and if the map $v\colon \gamma \to S(1)$ is continuous. See [V3, 2.7]. We write $\mathring{\gamma} = \gamma \setminus \{a, b\}$.

The affine subspace spanned by a set $A \subset \mathbf{R}^n$ is aff A. For open and closed balls and for spheres in $\mathbf{R}^n$ we use the notation $B(a,r)$, $\bar{B}(a,r)$, $S(a,r)$, where the center a may be omitted if $a = 0$. In particular, $S(1)$ is the unit sphere of $\mathbf{R}^n$.

It is often convenient to parametrize an arc or a path by quasihyperbolic length. We say that $g\colon [0,r] \to G$ is a *quasihyperbolic parametrization* if $l_k(g|[0,t]) = t$ for all $t \in [0,r]$. Then $r = l_k(g)$ and

$$|g'(t)| = \delta(g(t)) \tag{1.6}$$

almost everywhere. Every rectifiable arc $\gamma \subset G$ has a quasihyperbolic parametrization $g\colon [0,r] \to \gamma$, and g satisfies the Lipschitz condition

$$|g(s) - g(t)| \le M|s - t| \tag{1.7}$$

where $M = \max\{\delta(x)\colon x \in \gamma\}$.

If γ is a geodesic, then g is an isometry from $[0,r]$ into the metric space (G,k), and we say that g is a *geodesic path* from $g(0)$ to $g(r)$. Then g is C^1 by [Ma, 4.8] and (1.6) holds for all $t \in [0,r]$. Instead of $[0,r]$, the parametric interval of a geodesic path may be $[t_0, t_0 + r]$ for some $t_0 \in \mathbf{R}$.

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2. General results

In this section we give some results on quasihyperbolic geometry, valid in all dimensions. A domain $D \subset \mathbf{R}^n$ is *strictly starlike* with respect to a point $a \in D$ if D is bounded and if each ray from a meets ∂D at exactly one point. From [V3, 3.11] we get

2.1. Lemma. *If $0 < r < \pi/2$, then every quasihyperbolic ball $B_k(a,r)$ in a domain $G \subset \mathbf{R}^n$ is strictly starlike with respect to a. Hence $S_k(a,r)$ is homeomorphic to the unit sphere $S(1)$. Moreover, if $x \in \bar{B}_k(a,r)$, then the closed ball $\bar{A}(a,x) = \bar{B}((a+x)/2, |a-x|/2)$ lies in G.* □

We next consider sequences of geodesic paths.

2.2. Theorem. *Let $G \subset \mathbf{R}^n$ be a domain and let $g_j\colon [0,r] \to G$ be a sequence of geodesic paths such that $g_j(0) \to a \in G$, $g_j(r) \to b \in G$. Then:*

(1) *There is a convergent subsequence of (g_j).*

(2) *Each convergent subsequence (h_j) of (g_j) converges uniformly to a geodesic path h from a to b.*

(3) *If there is only one geodesic path g from a to b, then (g_j) converges uniformly to g.*

Proof. There is a compact set in G containing all geodesics $\operatorname{im} g_j$. By (1.7) and by Ascoli's theorem, there is a subsequence (h_j) of (g_j) converging uniformly to a path $h\colon [0,r] \to G$. Then $l_k(h) \le \liminf_{j\to\infty} l_k(h_j) = r$ (see [MV, 3.1]), whence $\gamma = \operatorname{im} h$ is a quasihyperbolic geodesic. Repeating the argument on subintervals of $[0,r]$ we see that h is a geodesic path. Hence (1) and (2) are true, and (3) is a corollary of these. □

2.3. Theorem. *If a domain $G \subset \mathbf{R}^n$ satisfies the Convexity conjecture 1.4, it satisfies the Uniqueness conjecture 1.2 with $c_U = 2c_C$.*

Proof. Assume that $a, b \in G$ with $k(a,b) = 2r < 2c_C$ and that $\gamma_1, \gamma_2 \colon a \curvearrowright b$ are quasihyperbolic geodesics. Let $z_j \in \gamma_j$ be the point bisecting the quasihyperbolic length of γ_j. If $z_1 \neq z_2$, then for $y = (z_1 + z_2)/2$ we have $k(a,y) < r$ and $k(b,y) < r$ by 1.4. This implies the contradiction $k(a,b) < 2r = k(a,b)$, whence $z_1 = z_2$. Iteration and continuity prove the theorem. □

2.4. Theorem. *Suppose that $c > 0$ and that $G \subset \mathbf{R}^n$ satisfies the Prolongation conjecture 1.3 with all $c_P < c$. Then it satisfies 1.3 with $c_P = c$.*

Proof. Let $\gamma_0 \colon a \curvearrowright b$ be a quasihyperbolic geodesic in G with $l_k(\gamma_0) = c_0 < c$. Choose a sequence $c_0 < c_1 < \dots$ converging to c. Then there are quasihyperbolic geodesics $\gamma_j \colon a \curvearrowright b_j$ such that $\gamma_j \subset \gamma_{j+1}$ and such that $l_k(\gamma_j) = c_j$. As $k(b_i, b_j) = c_j - c_i$ for $i < j$, the sequence (b_j) is Cauchy and converges to a point $b \in G$ with $k(a,b) = c$. Now the union of all γ_j is a geodesic $\gamma \colon a \curvearrowright b$ with $l_k(\gamma) = c$. □

2.5. Ball convexity and shuttles. We recall from [Ma, 2.2] that quasihyperbolic geodesics are *ball convex*. This means that if B is a euclidean ball in a domain $G \subset \mathbf{R}^n$ and if $\gamma \colon a \curvearrowright b$ is a quasihyperbolic geodesic with $a, b \in \bar{B}$, then $\gamma \subset \bar{B}$. This implies that sufficiently short geodesics are contained in *shuttles*, defined by

$$Y(a,b;R) = \bigcap \{B(z,R) \colon |z-a| = |z-b| = R\},$$
$$\bar{Y}(a,b;R) = \bigcap \{\bar{B}(z,R) \colon |z-a| = |z-b| = R\},$$

where $|a-b| \le 2R$. If $n = 2$, then $Y(a,b;R)$ is a Jordan domain bounded by two circular arcs. The *angle* α of Y is defined by

$$\sin\alpha = |a-b|/2R, \ \ 0 < \alpha \le \pi/2.$$

If $\gamma \colon a \curvearrowright b$ is a geodesic in $G \subset \mathbf{R}^n$ and if $2R \le \delta(x) \vee \delta(y)$, then $\gamma \subset \bar{Y}(a,b;R)$ ([V3, 2.6]).

2.6. Theorem. *If a domain $G \subset \mathbf{R}^n$ satisfies the Uniqueness conjecture 1.2, it satisfies the Prolongation conjecture 1.3 with $c_P = c_U \wedge \frac{\pi}{2}$.*

Proof. Let $a \in G$ and let $0 < r < s < c_P$. For each $x \in S_k(a,s)$ there is a unique quasihyperbolic geodesic $\gamma_x \colon a \curvearrowright x$. Let $f(x)$ be the unique point in $\gamma_x \cap S_k(a,r)$. By 2.4 it suffices to show that $f \colon S_k(a,s) \to S_k(a,r)$ is surjective.

From 2.2 it follows that f is continuous. Since $s < \pi/2$, the quasihyperbolic balls $B_k(a,r)$ and $B_k(a,s)$ are strictly starlike by 2.1. Hence the central projection from a defines a homeomorphism $g \colon S_k(a,s) \to S_k(a,r)$. It suffices to show that f and g are homotopic, because then they have the same degree. For this, it suffices to show that $a \notin [fx, gx]$ for each $x \in S_k(a,s)$.

The ball $\bar{B} = \bar{A}(a,x)$ lies in G by 2.1. By the ball convexity of γ_x (see 2.5), $fx \in B$, and the theorem follows. □

For nonzero vectors $a, b \in \mathbf{R}^n$ we let $\operatorname{ang}(a,b)$ denote the angle between a and b, defined by

$$a \cdot b = |a||b| \cos \operatorname{ang}(a,b), \ 0 \le \operatorname{ang}(a,b) \le \pi.$$

2.7. Lemma. *Let $g\colon [0,r] \to G$ be a geodesic path and let $0 \le t < s \le r$, $s \le t + 1/2$. Then*

$$\operatorname{ang}(g'(t)), g(s) - g(t)) \le 4(s-t).$$

Proof. By a standard estimate we have

$$k(x,y)/2 \le |x-y|/\delta(x) \le 2k(x,y)$$

whenever $x, y \in G$ with either $|x-y| \le \delta(x)/2$ or $k(x,y) \le 1$; see [V1, 2.5] or [V2, 3.9]. For $x = g(t)$, $y = g(s)$ we have $k(x,y) = s - t \le 1/2$, whence $|x-y| \le \delta(x)$. By 2.5 this implies that $g[t,s] \subset \bar{Y}(x,y;\delta(x)/2)$. For $\alpha = \operatorname{ang}(g'(t), y-x)$ we thus have

$$\alpha \le 2 \sin \alpha \le 2|x-y|/\delta(x) \le 4k(x,y) = 4(s-t). \qquad \square$$

We apply 2.7 to show that in a convergent sequence of geodesics also the derivatives converge:

2.8. Theorem. *Suppose that (g_j) is a sequence of geodesic paths $g_j\colon [0,r] \to G \subset \mathbf{R}^n$ converging to a path $g\colon [0,r] \to G$. Then*

(1) *g is a geodesic path and the convergence is uniform,*

(2) *$g_j'(t) \to g'(t)$ uniformly on $[0,r]$.*

Proof. Part (1) follows from 2.2. By (1.6) we have $|g'(t)| = \delta(g(t))$, $|g_j'(t)| = \delta(g_j(t))$. Hence it suffices to show that $\operatorname{ang}(g_j'(t), g'(t)) \to 0$ uniformly on $[0,r]$.

Assume that this is not true. Passing to a subsequence we may assume that there is a number $\theta > 0$ and a sequence (t_j) in $[0,r]$ such that $t_j \to t_0 \in [0,r]$ and such that $\operatorname{ang}(g_j'(t_j), g'(t_j)) \ge \theta$ for all j. Replacing $g(t)$ by $g(r-t)$ if necessary we may assume that $t_0 \ne r \ne t_j$ for all j. Fix a number u such that $0 < u < \theta/10$, $s_0 = t_0 + u \le r$ and $s_j = t_j + u \le r$ for all j.

From 2.7 it follows that the angles $\operatorname{ang}(g'(t_0), g(s_0) - g(t_0))$ and $\operatorname{ang}(g'(t_j), g_j(s_j) - g_j(t_j))$ are less than $2\theta/5$. Furthermore,

$$g'(t_j) \to g'(t_0), \ g_j(s_j) \to g(s_0), \ g_j(t_j) \to g(t_0)$$

as $j \to \infty$, and we obtain the contradiction $\theta \le 4\theta/5$. $\square$

2.9. Normal vectors. We recall the theory of normal vectors from [V3, Sec. 5]. Let $a \in G \subset \mathbf{R}^n$, $r > 0$, and set $S = S_k(a,r)$. A unit vector e is an *inner normal vector* of S at $b \in S$ if

$$\liminf_{x \to b, k(x,a) \ge r} \operatorname{ang}(x-b, e) \ge \pi/2,$$

and a unit vector u is an *outer normal vector* of S at b if

$$\liminf_{x \to b, k(x,a) \le r} \operatorname{ang}(x-b, u) \ge \pi/2.$$

By Theorem 2.10 below, an inner normal vector always exists, but it is possible that S has several inner normal vectors at some point $b \in S$. However, if an outer normal vector u exists, then both are unique and $u = -e$ by [V3, 5.3]. Then we say that u is the *normal vector* of S at b, and $T = b + u^\perp$ is the *tangent hyperplane* of S at b.

2.10. Theorem. *Let $a \in G \subset \mathbf{R}^n$, $r > 0$, let $\gamma\colon a \curvearrowright b \in S = S_k(a,r)$ be a quasihyperbolic geodesic, and let v be the unit tangent vector of γ at b. Then $-v$ is an inner normal vector of S at b. If γ has a prolongation to a geodesic $\gamma_1\colon a \curvearrowright b_1 \neq b$, then v is a normal vector of S at b.*

If G satisfies the Uniqueness and Prolongation conjectures 1.2 and 1.3 with a constant c and if $r < c$, then S is smooth, that is, it has a continuous normal vector.

If $B_k(a,r)$ is convex, then S is smooth.

Proof. The first part of the theorem follows from [V3, 5.4, 5.10].

In the second part, let $u(y)$ be the normal vector of S at $y \in S$. Let (b_j) be a sequence in $S_k(a,r)$ converging to $b \in S$, and let $g_j\colon [0,r] \to G$ be the unique geodesic path from a to b_j. By 2.2, the sequence (g_j) converges to a geodesic path g with $\operatorname{im} g = \gamma$. Then $g_j'(r) \to g'(r)$ by 2.8. It follows that

$$u(b_j) = g_j'(r)/\delta(b_j) \to g'(r)/\delta(b) = u(b),$$

whence u is continuous.

Finally, assume that $D = B_k(a,r)$ is convex. Let T be a supporting hyperplane of D at $y \in S$. Let H be the component of $\mathbf{R}^n \setminus T$ containing D and let $u(y)$ be the unit normal vector of T with $y + u(y) \notin H$. Then $u(y)$ is a normal vector of S at y. To prove that u is continuous in a neighborhood of a point $b \in S$ we may assume that $b = 0$ and that $u(b) = -e_n$. Now there is a neighborhood V of 0 in S such that V is the graph of a convex function $h\colon U \to \mathbf{R}$, defined in an $(n-1)$-dimensional ball U. This function is differentiable at every point and therefore C^1 by a classical result on convex functions; see [Ro, 25.5.1]. Hence u is continuous. $\square$

2.11. Distortion of geodesics. Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in a domain $G \subset \mathbf{R}^n$ and let $v(x)$ be the unit tangent vector of γ at $x \in \gamma$. Then $v\colon \gamma \to S(1)$ is continuous but it need not be differentiable. However, it is Lipschitz, and a sharp estimate for a generalized curvature of γ was given by Martin in [Ma, (4.10),(4.11)]. From this theory it is easy to obtain the sharp estimate $\operatorname{ang}(v(a), v(b)) \le k(a,b)$. We give a slightly modified treatment of Martin's theory.

The following was proved in [Ma, 2.5]; the stronger condition $|x| < (d-r)/2$ is not needed.

2.12. Lemma. *Suppose that $x, z, p \in \mathbf{R}^n$ with $|x| \le d - r$, $|z| = r$, $|p| \ge d$, $d/2 < r < d$. If $x' \in (x,z)$, then*

$$|x-p|/|x'-p| < |x-z|/|x'-z|. \qquad \square$$

2.13. Cap convexity theorem. (cf. [Ma, 2.4]) *Let $G \subset \mathbf{R}^n$ be a domain, let $x_0, z \in G$ with $|z - x_0| = r$, $\delta(x_0) = d$, $d/2 < r < d$. Let $0 < t \le d - r$ and let γ be a quasihyperbolic geodesic with endpoints on the cap $C = S(z,r) \cap \bar{B}(x_0,t)$. Then $\gamma \subset \bar{B}(z,r) \cap \bar{B}(x_0,t)$.*

Proof. We may assume that $x_0 = 0$. By ball convexity 2.5, we have $\gamma \subset \bar{B}(x_0,t)$. Assume that the theorem is false. Then there is a geodesic $\gamma\colon a \curvearrowright b$ with $a, b \in C$, $\gamma \cap \bar{B}(z,r) = \{a,b\}$. Let u be the inversion in the sphere $S(z,r)$. Then $|u'(x)| = r^2/|x-z|^2$. By the convexity of $B(x_0,d)$ we have $[x,z] \subset B(x_0,d) \subset G$ for each $x \in \gamma$. Hence $u\gamma \subset G$. It suffices to show that $l_k(u\gamma) < l_k(\gamma)$.

We have

$$l_k(u\gamma) = \int_\gamma \frac{|u'(x)||dx|}{\delta(ux)} = r^2 \int_\gamma \frac{|dx|}{\delta(ux)|x-z|^2}. \tag{2.14}$$

Choosing $p \in \partial G$ with $|ux - p| = \delta(ux)$ and applying 2.12 with $x' = ux$ we get

$$\frac{\delta(x)}{\delta(ux)} \leq \frac{|x-p|}{|ux-p|} < \frac{|x-z|}{|ux-z|} = \frac{|x-z|^2}{r^2}$$

for all $x \in \mathring{\gamma}$. This and (2.14) yield $l_k(u\gamma) < l(\gamma)$. □

With the aid of cap convexity one can replace the shuttle in 2.5 by a narrower one if the geodesic γ is short:

2.15. Theorem. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in $G \subset \mathbf{R}^n$ such that $s := |a-b| < d/2$ where $d = \delta(a) \vee \delta(b)$. Then $\gamma \subset \bar{Y}(a, b; d-s)$. For the angle $\alpha(s)$ of the shuttle we have*

$$\lim_{s\to 0} \frac{\alpha(s)}{s} = \frac{1}{2d}.$$

Proof. We may assume that $d = \delta(a) \geq \delta(b)$. Set $r = d - s$ and let z be a point with $|z-a| = |z-b| = r$. Now 2.13 gives $\gamma \subset \bar{B}(z, r)$, whence $\gamma \subset \bar{Y}(a, b; r)$. As $\sin\alpha(s) = s/2(d-s)$, the theorem follows. □

2.16. Theorem. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in $G \subset \mathbf{R}^n$ and let $v(x)$ be the unit tangent vector of γ at $x \in \gamma$. Then* $\operatorname{ang}(v(a), v(b)) \leq k(a, b)$.

Proof. Set $\Delta = [0, k(a,b)]$ and let $g\colon \Delta \to \gamma$ be the quasihyperbolic parametrization of γ. Define $\varphi\colon \Delta \to \mathbf{R}$ by $\varphi(t) = \operatorname{ang}(v(g(t)), v(a))$. It suffices to show that

$$\limsup_{t\to t_0} \frac{|\varphi(t) - \varphi(t_0)|}{|t - t_0|} \leq 1 \tag{2.17}$$

for all $t_0 \in \Delta$, since this implies that φ is 1-Lipschitz.

Set

$$x_0 = g(t_0),\ x = g(t),\ d = \delta(x_0),\ s = |x - x_0|,$$

and assume that $s < d/2$. By 2.15, the arc $\gamma[x_0, x]$ lies in a shuttle with chord $[x_0, x]$ and angle $\alpha(s)$. Here $|\varphi(t) - \varphi(t_0)| \leq 2\alpha(s)$. Since $\alpha(s)/s \to 1/2d$ and since $|g'(t_0)| = d$ by (1.6), this implies (2.17). □

From 2.16 we obtain as an easy corollary:

2.18. Theorem. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in $G \subset \mathbf{R}^n$ with $k(a,b) \leq \pi/2$. Let $x, y \in \gamma$, $x \neq y$, and let L be either the tangent of γ at x or the line* $\operatorname{aff}\{x, y\}$*. Then the tangent of γ is nowhere perpendicular to L, and the orthogonal projection of γ into L is injective.* □

3. The domains G_1 and G_2

3.1. In this section we consider domains $G \subset \mathbf{R}^2$ such that $Q = \mathbf{R}^2 \setminus G$ consists of one or two points and start with the punctured plane $G_1 = \mathbf{R}^2 \setminus \{0\}$, which has been studied by Martin and Osgood [MO] in 1986 and recently by Klén [Kl]. We shall make extensive use of the exponential map

$$F\colon \mathbf{R}^2 \to G_1, \quad F(z) = e^z, \quad F(x,y) = (e^x \cos y, e^x \sin y).$$

Recall that F is an angle-preserving covering map with $F(z) = F(z')$ iff $z = z' + 2m\pi i$ for some $m \in \mathbf{Z}$. Thus F is injective in each strip $\{(x,y)\colon a < y < a + 2\pi\}$, $a \in \mathbf{R}$.

Every path $g\colon [t_1,t_2]\to G_1$ has an *F-lift* $g^*\colon [t_1,t_2]\to \mathbf{R}^2$ with $F\circ g^*=g$, and g^* is unique as soon as we fix the point $g^*(t_1)\in F^{-1}\{g(t_1)\}$.

Martin and Osgood made the important observation that F transforms euclidean lengths to quasihyperbolic lengths. We recall its proof.

3.2. Lemma. [MO, p. 38] *Let $g\colon [0,\lambda]\to \mathbf{R}^2$ be a rectifiable path. Then the quasihyperbolic length of $F\circ g$ in G_1 is equal to the euclidean length of g.*

Proof. We may assume that g is a length parametrization with $\lambda=l(g)$. We have $\delta(w)=|w|$ for $w\in G_1$. As $(F\circ g)'(t)=F(g(t))g'(t)$ and $|g'(t)|=1$ a.e., we get

$$l_k(F\circ g)=\int_0^\lambda \frac{|(F\circ g)'(t)|}{|F(g(t))|}\,dt=\int_0^\lambda dt=\lambda. \qquad \square$$

A domain $D\subset \mathbf{R}^2$ is a *Jordan domain* if it is bounded and if ∂D is a Jordan curve (topological circle). A Jordan domain D is *smooth* if ∂D is a C^1 smooth curve.

3.3. Lemma. *Let $D\subset \mathbf{R}^2$ be a Jordan domain with rectifiable boundary. If $0\in D$, then $l_k(\partial D)\ge 2\pi$ in G_1.*

Proof. Choose a path $g\colon [0,\lambda]\to \partial D$ such that $g(0)=g(\lambda)$ and such that $g|[0,\lambda)$ is injective. Let $g^*\colon [0,\lambda]\to \mathbf{R}^2$ be an F-lift of g. As g is not null-homotopic, we have $g^*(0)\ne g^*(\lambda)$. By 3.2 this gives

$$l_k(\partial D)=l(\gamma^*)\ge |g^*(0)-g^*(\lambda)|\ge 2\pi. \qquad \square$$

3.4. Geodesics in G_1. From Lemma 3.2 it follows that a quasihyperbolic geodesic $\gamma\colon a\curvearrowright b$ in G_1 can be found as follows. Choose an arbitrary point $a^*\in F^{-1}\{a\}$ and then $b^*\in F^{-1}\{b\}$ such that $|a^*-b^*|$ is minimal. Then $\gamma=F[a^*,b^*]$.

If $k(a,b)<\pi$, then γ is unique. It is a subarc of a logarithmic spiral, possibly of a circle centered at 0 or a ray emanating from 0; these limiting cases are also regarded as logarithmic spirals. We say that the origin is the *center* of the spiral.

3.5. Quasihyperbolic disks in G_1. From 3.2 it follows that a quasihyperbolic disk in G_1 is obtained from

$$B_k(a,r)=FB(a^*,r),$$

where a^* is an arbitrary point in $F^{-1}\{a\}$.

3.6. Theorem. *If $p\in \mathbf{R}^2$ and $0<r<\pi$, then $D=FB(p,r)$ is a smooth Jordan domain. If $r\le 1$, then D is strictly convex. If $r<1$, then the curvature radius of ∂D at each point has the upper bound*

$$R_0(p,r)=\frac{r|e^p|e^r}{1-r}.$$

If $r>1$, then D is not convex.

Proof. As F is a C^∞ embedding in each strip $s<\operatorname{Im} z<s+2\pi i$, the first part of the theorem is clear. A parametrization for ∂D is given by $g(t)=F(p+re^{it})=ae^{re^{it}}$, $0\le t\le 2\pi$, where $a=e^p$. Suppose that $r\le 1$. Setting $\varphi(t)=\arg g'(t)$ we have

$$\varphi(t)=\arg a+r\sin t+t+\pi/2.$$

Hence $\varphi'(t) = r\cos t + 1 > 0$ except for $r = 1,\ t = \pi$. Consequently, φ is strictly increasing on $[0, 2\pi]$, whence D is strictly convex. Furthermore,

$$|g'(t)| = r|e^p||e^{re^{it}}| \le r|e^p|e^r.$$

If $r < 1$, the curvature of ∂D at a point $g(t)$ is therefore

$$\varphi'(t)/|g'(t)| \ge (1-r)/r|e^p|e^r = 1/R_0(p, r).$$

If $r > 1$, then $\varphi'(t) < 0$ in a neighborhood of π, whence D is not convex. □

By 3.5 we get the result of [Kl, 3.10] for $n = 2$:

3.7. Corollary. *The domain G_1 satisfies the Convexity conjecture 1.4 with the sharp constant $c = 1$.* □

3.8. The domain G_2. We next consider the complement of two points in $\mathbf{R}^2$, which can be normalized as

$$G_2 = \mathbf{R}^2 \setminus \{0, 2\}.$$

The vertical line

$$L_0 = \{z\colon \operatorname{Re} z = 1\}$$

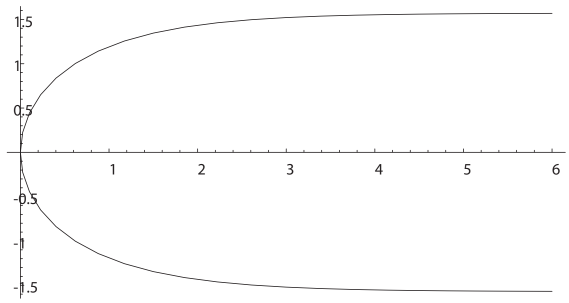
divides the plane into two open half planes $H = \{z\colon \operatorname{Re} z < 1\}$ and $\tilde{H} = \{z\colon \operatorname{Re} z > 1\}$. For $z \in G_2$ we have $\delta(z) = |z| \wedge |z-2|$, whence quasihyperbolic geodesics lying in H or in $\tilde{H}$ are parts of logarithmic spirals as explained in 1.4. To study the behavior of a geodesic meeting L_0 we set

$$C^* = F^{-1}L_0, \quad U = F^{-1}H.$$

Then

$$C^* = C_0 + 2\pi i\mathbf{Z}, \quad C_0 = \{(x, y) \in \mathbf{R}^2\colon -\pi/2 < y < \pi/2,\ x = -\log\cos y\}.$$

The curve C_0 lies in the half strip $x \ge 0,\ -\pi/2 < y < \pi/2$ with horizontal asymptotes $y = \pm\pi/2$. The function $f(y) = -\log\cos y$ is strictly convex.



lc-curve C_0

We say that a set in $\mathbf{R}^2$ is an *lc-curve* (lc for log cos) if it is of the form $C_0 + z_0$ for some $z_0 \in \mathbf{R}^2$. Thus C^* is the union of a countable number of lc-curves $C_m = C_0 + 2\pi im,\ m \in \mathbf{Z}$. The set U is the domain with $\partial U = C^*$. It contains the left half plane $x < 0$ and the closed horizontal strips $\pi/2 + 2m\pi \le y \le 3\pi/2 + 2m\pi,\ m \in \mathbf{Z}$. The map F defines a covering map $\bar{U} \to \bar{H}$, and it maps each C_m homeomorphically onto L_0. If $L \subset \mathbf{R}^2$ is an arbitrary line not containing the origin, we can write $L = e^{z_0}L_0$ for some $z_0 \in \mathbf{R}^2$, and thus $F^{-1}L = C^* + z_0$ consists of lc-curves $C_0 + z_0 + 2\pi im, m \in \mathbf{Z}$.

We next study quasihyperbolic geodesics meeting L_0 but consider only the case $k(a, b) = l_k(\gamma) \le \pi$.

3.8. Lemma. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G_2 such that $l_k(\gamma) \le \pi$ and $a, b \in L_0$. Then $\gamma = [a, b]$.*

Proof. Assume that $\gamma \ne [a, b]$. Passing to a subarc we may assume that $\gamma \cap L_0 = \{a, b\}$. By symmetry we may assume that $\gamma \subset \bar{H}$. Let $\gamma^*\colon a^* \curvearrowright b^*$ be an F-lift of γ with $a^* \in C_0$. As $l(\gamma^*) = l_k(\gamma) \le \pi$ by 3.2, we have $b^* \in C_0$. Let α^* be the subarc of C_0 between a^* and b^*. Then $l(\alpha^*) < l(\gamma^*) = l_k(\gamma)$. Since $l(\alpha^*) = l_k(F\alpha^*) = l_k([a, b])$, this gives a contradiction. □

3.10. Geodesics in $\bar{H}$. Suppose that $a, b \in \bar{H}$ and that $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic in G_2 with $l_k(\gamma) \le \pi$. From 3.9 it follows that $\gamma \in \bar{H}$. Let $\gamma^*\colon a^* \curvearrowright b^*$ be an F-lift of γ. By 3.2, the arc γ^* is a geodesic in the inner metric $d_{\bar{U}}$ of $\bar{U}$, defined by

$$d_{\bar{U}}(a^*, b^*) = \inf\{l(\alpha) \mid \alpha\colon a^* \curvearrowright b^*,\ \alpha \subset \bar{U}\}.$$

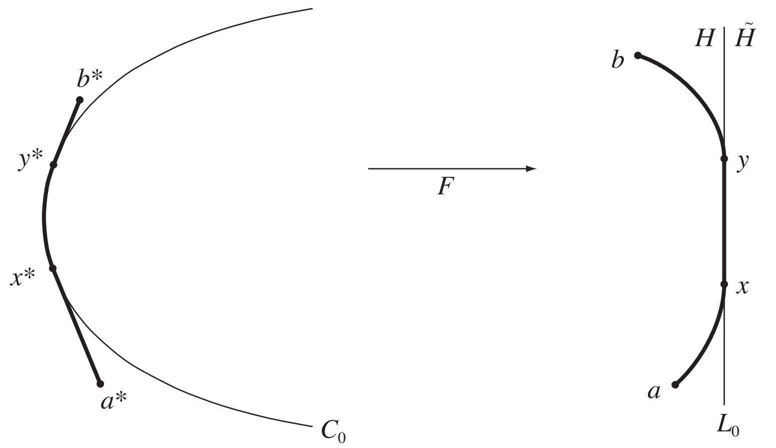
Since $l(\gamma^*) \le \pi$, the arc γ^* meets at most one component of C^*, and we may assume that this is C_0. There are three cases:

Case 1. $[a^*, b^*] \cap C_0 = \varnothing$. Now $\gamma^* = [a^*, b^*]$ and $\gamma \subset H$ is a quasihyperbolic geodesic in G_1, hence a part of a logarithmic spiral.

Case 2. $[a^*, b^*] \cap C_0 = \{z_0\}$ is a singleton. Again γ is a geodesic in G_1, but now L_0 is a tangent of γ at z_0.

Case 3. $[a^*, b^*]$ meets C_0 at two points. If $a, b \in H$, then the $d_{\bar{U}}$-geodesic γ^* is a union $[a^*, x^*] \cup C_0[x^*, y^*] \cup [y^*, b^*]$ where the line segments meet C_0 tangentially at the endpoints x^* and y^*. If $a \in L_0$ or $b \in L_0$, then the corresponding line segment degenerates to a point.

The geodesic $\gamma = F\gamma^*$ consists of the line segment $[F(x^*), F(y^*)] \subset L_0$ and (possibly) of two arcs of logarithmic spirals in $\bar{H}$.



Case 3. A geodesic in $\bar{H}$ containing a segment of L_0

Let $\varrho_0\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in L_0, $\varrho_0(x, y) = (2 - x, y)$. If $\tilde{a}, \tilde{b} \in \tilde{H}$ with $k(\tilde{a}, \tilde{b}) \le \pi$, then a geodesic $\tilde{\gamma}\colon \tilde{a} \curvearrowright \tilde{b}$ is $\varrho_0\gamma$ where γ is a geodesic from $a = \varrho_0\tilde{a}$ to $\varrho_0\tilde{b}$ given above.

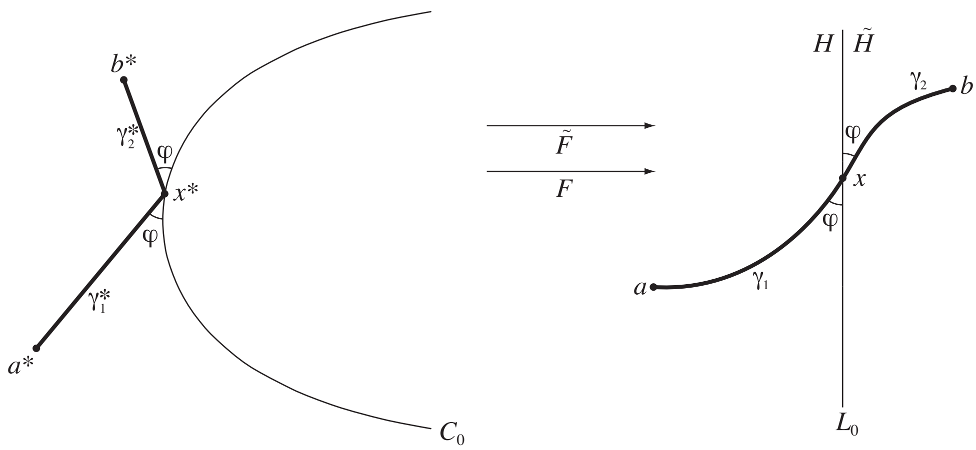
3.11. Geodesics from H to $\tilde{H}$. We shall make use of the sense-reversing angle-preserving covering map

$$\tilde{F} = \varrho_0 \circ F\colon \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{2\}.$$

Observe that $\tilde{F}^{-1}\tilde{H} = U$ and that $\tilde{F} = F$ on C^*.

Suppose that $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic in G_2 with $a \in H$, $b \in \tilde{H}$, $l_k(\gamma) \le \pi$. Let $x, y \in \gamma$ be the first and the last point of γ in L_0, respectively. From 3.9 it follows that $\gamma \cap L_0 = [x, y]$, where the case $x = y$ may occur. Write $\alpha = [x, y]$, $\gamma_1 = \gamma[a, x]$, $\gamma_2 = \gamma[y, b]$. If $x \ne y$, the arc α has a unique F-lift $\alpha^*\colon x^* \curvearrowright y^*$ on C_0, and α^* is also an $\tilde{F}$-lift of α. Furthermore, the arc γ_1 has a unique F-lift to a line segment $\gamma_1^* = [a^*, x^*] \subset \bar{U}$, and γ_2 has a unique $\tilde{F}$-lift to a line segment $\gamma_2^* = [y^*, b^*] \subset \bar{U}$. Let $\varphi \in [0, \pi/2]$ be the angle between $[a^*, x^*]$ and the tangent of C_0 at x^*. We consider three cases.

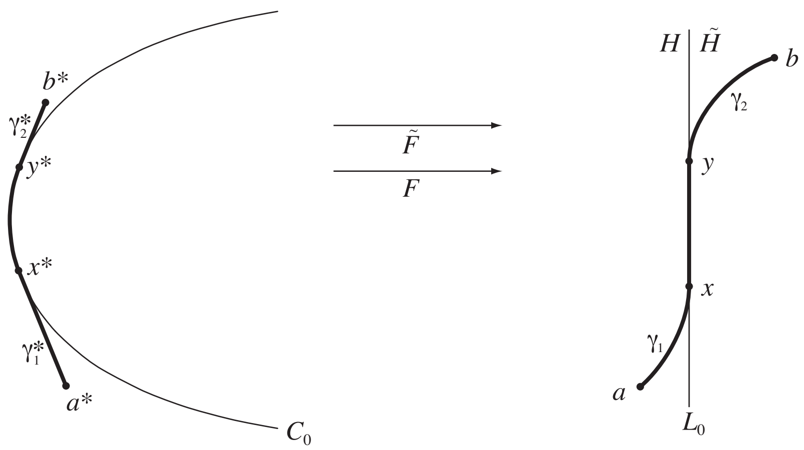
Case 1. $x = y$ and $\varphi > 0$. Now γ crosses L_0 at the point $x = y$, and the angle between L_0 and the tangent of γ at x is φ. Hence also the angle between γ_2^* and the tangent of C_0 at $x^* = y^*$ is φ. We may think that the pair (γ_1^*, γ_2^*) represents a light beam from a^*, which *reflects* from the convex mirror C_0 to b^*. We say that the pair (γ_1^*, γ_2^*) is the $(F, \tilde{F})$-*lift* of γ.



Case 1. Crossing overpass from H to $\tilde{H}$

Case 2. $x = y$ and $\varphi = 0$. This case is almost similar to Case 1, but now $\gamma_1^* \cup \gamma_2^* = [a^*, b^*]$ and γ touches L_0 at the overpass point x.

Case 3. $x \ne y$. Since γ is smooth, the line segments γ_1^* and γ_2^* meet C_0 tangentially, and the arc $\gamma_1^* \cup \alpha \cup \gamma_2^*$ is similar to γ^* in Case 3 of 3.10. The arc γ consists of the line segment $[x, y]$ and two arcs of logarithmic spirals.



Case 3. Sliding overpass from H to $\tilde{H}$

Terminology. In Cases 1,2,3 we say that the overpass from H to $\tilde{H}$ of the geodesic γ is *crossing, touching* or *sliding*, respectively.

3.12. Geodesic germs. We say that two quasihyperbolic geodesics $\gamma\colon a \curvearrowright b$ and $\gamma'\colon a' \curvearrowright b'$ in a domain G are *equivalent* if $a = a'$ and if there is a neighborhood U of a such that $\gamma \cap U = \gamma' \cap U$. An equivalence class is a *geodesic germ* or briefly a *germ*.

We let $[\gamma]$ denote the germ containing γ. Each germ $[\gamma]$ with $\gamma\colon a \curvearrowright b$ has a well-defined *starting point* a and a *direction* $v \in S(1)$, which is the unit tangent vector of γ at a. We want to find all germs in G_2 with given $a \in G_2$ and $v \in S(1)$.

Suppose first that $a \in H$ and that $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic in G_2 with $\gamma \subset H$. Choose a point $a^* \in F^{-1}\{a\}$ and an F-lift $\gamma^*\colon a^* \curvearrowright b^*$ of γ. Then γ^* is a line segment $[a^*, b^*]$, and $b^* - a^*$ has the same direction as $v^* = v/a$. Hence there is precisely one germ starting at a to the direction v.

By symmetry, the same is true if $a \in \tilde{H}$. Furthermore, if $a \in L_0$ and $v \neq \pm e_2$, the same argument proves that the germ from a to the direction v is uniquely determined.

Next assume that $a \in L_0$, $v = e_2$. The discussion in 3.10 shows that there are three germs $[\gamma]$ from a to the direction e_2: one with $\mathring{\gamma} \subset H$, one with $\mathring{\gamma} \subset \tilde{H}$, and one such that γ contains a line segment $[a, a + te_2]$, $t > 0$. The case $v = -e_2$ is similar.

4. Voronoi diagrams and quasihyperbolic geodesics

4.1. Voronoi diagrams. In this section we assume that $Q \subset \mathbf{R}^2$ is a finite set containing at least two points. We shall study quasihyperbolic geodesics in the domain $G_Q = \mathbf{R}^2 \setminus Q$. The section is preparation for Sections 5 and 6 where we prove the basic conjectures for G_Q.

Recall from the introduction that the *Voronoi diagram* $\operatorname{Vor} Q$ of Q is the finite family of *Voronoi cells*

$$D_q = \{x \in \mathbf{R}^2\colon |x - q| < |x - p| \text{ for all } p \in Q \setminus \{q\}\},\ q \in Q.$$

The point q is the *nucleus* of the cell D_q. Each cell is a finite intersection of open half planes, hence a convex domain. If D_q is bounded, it is a convex polygon with m *edges* and m *vertices* for some $m \geq 3$. If D_q is unbounded, there are three possibilities:

(1) D_q is a half plane,

(2) D_q is a parallel strip,

(3) D_q has $m \geq 1$ vertices $v_1, \ldots, v_m \in \mathbf{R}^2$, and ∂D_q consists of $m+1$ edges: the line segments $[v_j, v_{j+1}]$, $1 \leq j \leq m-1$, and two rays, emanating from v_1 and v_m.

The cases (1) and (2) occur only if Q is contained in a line, and then the edges are lines. The domains D_q are disjoint, and their closures $\bar{D}_q$ cover the plane. If $q \neq p$ and $\bar{D}_q$ meets $\bar{D}_p$, then $\bar{D}_q \cap \bar{D}_p$ is a common edge or a common vertex. The latter case can occur only if there is a circle containing four points of Q. We shall use the notation

$X = X(Q) =$ the union of all edges of $\operatorname{Vor} Q$.

We give a generalization of Lemma 3.9:

4.2. Edge theorem. *Let γ be a quasihyperbolic geodesic in G_Q with $l_k(\gamma) \leq \pi$ and let J be an edge of* $\operatorname{Vor} Q$. *Then $\gamma \cap J$ is a singleton or a line segment.*

Proof. If the theorem is false, then there is an edge $J = \bar{D}_p \cap \bar{D}_q$ and a geodesic $\gamma\colon a \curvearrowright b$ such that $a, b \in J$, $l_k(\gamma) \leq \pi, \gamma \neq [a, b]$. Let k_2 be the quasihyperbolic metric of the domain $\mathbf{R}^2 \setminus \{p, q\}$. Since $k \ge k_2$, we obtain by 3.9 the contradiction

$$l_k(\gamma) \ge l_{k_2}(\gamma) > l_{k_2}([a, b]) = l_k([a, b]). \qquad \square$$

4.3. Terminology. Let $U \subset \mathbf{R}^2$ be an open set and let $\gamma \subset \bar{U}$ be an arc in $\bar{U}$ with endpoints a and b. We recall that γ is a *crosscut* of U if $\gamma \cap \partial U = \{a, b\}$ and an *endcut* of U if $\gamma \cap \partial U$ is $\{a\}$ or $\{b\}$.

4.4. The standard decomposition. Let $\gamma\colon a \curvearrowright b$ be a geodesic in G_Q with $l_k(\gamma) \le \pi$. If α is a component of $\gamma \setminus X$, then $\bar{\alpha}$ is a crosscut of a cell $D \in \operatorname{Vor} Q$ or perhaps an endcut if α contains one of the endpoints of γ. By 4.2 we obtain a unique decomposition of γ into subarcs

$$\gamma = \gamma_1 \cup \cdots \cup \gamma_m$$

where $\gamma_\nu = \gamma[x_{\nu-1}, x_\nu],\ x_0 = a,\ x_m = b$, and each γ_ν is either a line segment on some edge of $\operatorname{Vor} Q$ or a crosscut or (if $\nu = 1, m$) an endcut of a cell $D \in \operatorname{Vor} Q$. This is called the *standard decomposition* of γ.

4.5. Orientation. Let $W \subset \mathbf{R}^2$ be a domain and let $\gamma\colon a \curvearrowright b$ be a smooth arc on ∂W such that

(i) $\mathring{\gamma}$ is open in ∂W,

(ii) $\gamma \subset \partial(\mathbf{R}^2 \setminus \bar{W})$.

For $x \in \gamma$ let $v(x)$ be the unit tangent vector of γ at x. The vector $n(x) = iv(x)$ is the *left normal vector* of γ at x. For each $x \in \mathring{\gamma}$ there is $s_x > 0$ such that either (1) $x + tn(x) \in W$ for $0 < t < s_x$ or (2) $x - tn(x) \in W$ for $0 < t < s_x$. Moreover, if (1) holds at some point x, it holds at every $x \in \mathring{\gamma}$. Then we say that W lies on the *left-hand side* of γ and that γ is *positively oriented* in W. In case (2), W lies on the *right-hand side* of γ and γ is *negatively oriented*.

If W is a smooth Jordan domain, an orientation of ∂W can be defined by choosing a continuous unit tangent vector $v(x)$ of $\partial D,\ x \in \partial D$. The orientation is positive if $x + tiv(x) \in W$ for small $t > 0$.

4.6. The direction angle. Let $\gamma\colon a \curvearrowright b$ be a smooth arc in $\mathbf{R}^2$ and let $v(x)$ be the tangent vector as above. The *direction angle* $\varphi(x) = \arg v(x)$ is defined up to a multiple of 2π, and it is uniquely determined as a continuous function as soon as we fix $\varphi(x_0)$ for some $x_0 \in \gamma$.

4.7. The maps F_q. For each $q \in Q$ we define the covering map $F_q\colon \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{q\}$ by

$$F_q(z) = F(z) + q = e^z + q.$$

Setting $U_q = F_q^{-1} D_q$ we obtain covering maps $U_q \to D_q,\ \bar{U}_q \to \bar{D}_q,\ \partial U_q \to \partial D_q$.

Assume that the cell D_q is bounded and let $J_1, \ldots, J_m$ be the successive edges of D_q in the positive orientation. Let L_ν be the line containing J_ν. From 3.8 we see that the preimage $F_q^{-1} L_\nu$ consists of disjoint lc-curves $C_n^\nu = C_0^\nu + 2n\pi i,\ n \in \mathbf{Z}$, where $C_0^\nu = C(z_\nu) = C_0 + z_\nu$ and $F_q(z_\nu)$ is the point of L_ν closest to q. The boundary ∂U_q consists of successive arcs $\ldots, K_{-1}^m, K_0^1, \ldots, K_0^m, K_1^1, \ldots$ such that $F_q K_n^\nu = J_\nu$ and $K_n^\nu \subset C_n^\nu$. Each horizontal line $l_y = \{z\colon \operatorname{Im} z = y\}$ meets ∂U_q at exactly one point $w(y)$, and F_q maps the ray $l_y \cap \bar{U}_q$ onto the line segment $(q, F_q(w(y))]$.

If D_q is unbounded, then ∂U_q again consists of arcs $K_n^\nu \subset C_n^\nu$ but some of these are unbounded and ∂U_q is not connected. In fact, a horizontal line l_y meets ∂U_q iff D_q does not contain the ray $\{q + te^{iy}\colon t \ge 0\}$.

Let d_q be the inner euclidean metric of $\bar{U}_q$, defined by

$$d_q(a,b) = \inf\{l(\gamma) |\ \gamma\colon a \curvearrowright b,\ \gamma \subset \bar{U}_q\}.$$

For given $a,b \in \bar{U}_q$, a d_q-geodesic $\gamma\colon a \curvearrowright b$ always exists and is unique. If $a,b \in \partial U_q$, then γ consists of a finite number of subarcs, each of which is either a subarc of some K_n^ν or a line segment joining two points of ∂U_q in $\bar{U}_q$. Moreover, the projection $\operatorname{Im} z$ is strictly monotone for $z \in \gamma$.

4.8. Lemma. *Suppose that $q \in Q$ and that $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic in G_Q such that $a,b \in \partial D_q$, $\gamma \subset \bar{D}_q$. Then γ consists of a finite number of successive subarcs, each of which is either a line segment on ∂D_q or a subarc of a logarithmic spiral with center q.*

The direction angle φ of γ is monotone on γ and strictly monotone on each component of $\gamma \cap D_q$. The q-component A_q of $D_q \setminus \gamma$ is a convex Jordan domain with $\gamma \subset \partial A_q$, and φ is increasing iff A_q lies on the left-hand side of γ.

Proof. Let $\gamma^*\colon a^* \curvearrowright b^*$ be an F_q-lift of γ. By Lemma 3.2, the arc γ^* is a d_q-geodesic in $\bar{U}_q$ and has the structure explained in 4.7. Hence γ has the required structure. If $\operatorname{Im} a^* < \operatorname{Im} b^*$, then $z \mapsto \operatorname{Im} z$ is strictly increasing on γ^*, whence $x \mapsto \arg(x-q)$ is strictly increasing on γ. The lemma follows. □

4.9. Lemma. *Suppose that $D_q \in \operatorname{Vor} Q$ and that $\gamma\colon a \curvearrowright b$ is a smooth crosscut of $\mathbf{R}^2 \setminus \bar{D}_q$. Let $W \subset \mathbf{R}^2 \setminus \bar{D}_q$ be the Jordan domain bounded by γ and an arc $\alpha\colon a \curvearrowright b$ on ∂D_q. If the direction angle of γ is decreasing, then W lies on the right-hand side of γ.*

Proof. Let J be an edge of D_q containing a subarc of α and let L be the line containing J. Then there is an open half plane H bounded by L such that $D_q \subset \mathbf{R}^2 \setminus \bar{H}$. Now W meets H, and we can choose a point $x \in \gamma \cap H$ such that $d(x,L)$ is maximal. The tangent line L_1 of γ at x is parallel to L and there is a component H_1 of $\mathbf{R}^2 \setminus L_1$ containing W. As φ is decreasing, the points $x + tn(x)$ lie in $\mathbf{R}^2 \setminus H_1$ for $t \geq 0$, and the lemma follows. □

4.10. Terminology. Let $D \in \operatorname{Vor} Q$ and let $\gamma\colon a \curvearrowright b$ be a crosscut of $U = \mathbf{R}^2 \setminus \bar{D}$. Then there is a unique bounded component W of $U \setminus \gamma$. We say that γ *encircles* a point x in U if $x \in W$. We also say that the open arc $\mathring{\gamma}$ encircles x.

4.11. Lemma. *Let $\gamma\colon a \curvearrowright b$ be a crosscut of the half plane $H = \{z\colon \operatorname{Re} z < 1\}$ with $0 \notin \gamma$, and let $\gamma^*\colon a^* \curvearrowright b^*$ be an F-lift of γ (see 3.1). Then the points a^* and b^* belong to different components of $F^{-1}\partial H$ iff γ encircles the origin in H. Moreover, in this case $l_k(\gamma) > \pi$ in the domain $\mathbf{R}^2 \setminus \{0\}$.*

Proof. Let W be the Jordan domain as in 4.10. Then γ does not encircle 0 iff the Jordan curve ∂W is null-homotopic in $\mathbf{R}^2 \setminus \{0\}$. As F is a universal covering map, this holds iff the F-lifts of ∂W are Jordan curves, and the first part of the lemma follows.

If a^* and b^* are in different components of $F^{-1}\partial H$, then 3.2 yields

$$l_k(\gamma) = l(\gamma^*) \geq |a^* - b^*| > \pi. \qquad \square$$

4.12. Theorem. *Let $p \in Q$ and let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G_Q with $\gamma \cap \bar{D}_p = \{a,b\}$. Then γ encircles some point of Q. Moreover, $l_k(\gamma) > \pi$.*

Proof. Assume that the first part of the theorem is false. Then there are p and γ as in the theorem such that γ does not encircle any point of Q. Set $Q_\gamma = \{q \in Q \colon D_q \cap \gamma \neq \varnothing\}$ and $m = \#Q_\gamma$. We may assume that m has the least possible value over all such p and γ. Since γ is smooth, we have $m \geq 1$.

Let W be as in 4.10. We may assume that γ is positively oriented in W. Let $q \in Q_\gamma$. By the minimality of m, the set $\gamma_q = \gamma \cap \bar{D}_q$ is a subarc of γ, and we may apply Lemma 4.8 to γ_q. If the direction angle φ of γ is increasing on γ_q, then γ_q is positively oriented in A_q. Hence $A_q \subset W$, which gives the contradiction $q \in W$.

It follows that φ is decreasing on γ_q and hence on the whole arc γ. By 4.9 this implies that γ is negatively oriented in W, contrary to the assumption. Hence γ encircles a point $q \in Q$.

As D_p is convex, there is a half plane H containing q such that $\bar{H} \cap D_p = \varnothing$. Furthermore, a subarc $\beta = \gamma[a', b']$ is a crosscut of H, and β encircles q in H. Let k' be the quasihyperbolic metric of $\mathbf{R}^2 \setminus \{q\}$. By 4.11 we get $l_k(\gamma) \geq l_{k'}(\beta) > \pi$. □

4.13. Cell theorem. *Let $a, b \in G_Q$ be points with $k(a, b) \leq \pi$ such that $a, b \in \bar{D}$ for some cell $D \in \operatorname{Vor} Q$. If γ is a quasihyperbolic geodesic in G_Q from a to b, then $\gamma \subset \bar{D}$.*

Proof. This follows directly from 4.12. □

4.14. Geodesic germs in G_Q. Let $a \in G_Q$, $v \in S(1)$. We study geodesic germs from a to the direction v; see 3.12. If $a \in D_q$ for some $q \in Q$, we see with the aid of the covering map F_q that there is precisely one germ from a to the direction v. The same is true if $a \in \partial D_q$ and $a + tv \in D_q$ for small $t > 0$. Finally assume that $a \in \partial D_q$ and that $a + tv \in J$ for small $t \geq 0$ where J is an edge of D_q. Then $J = \bar{D}_q \cap \bar{D}_p$ for some $p \in Q \setminus \{q\}$, and the situation is essentially the same as in the domain G_2 with $a \in L_0$, $v = e_2$; see 3.12. Thus there is precisely one germ $[\gamma]$ with $\mathring{\gamma} \subset D_q$, one with $\mathring{\gamma} \subset D_p$, and one such that γ contains a subsegment of J.

5. Basic conjectures in G_Q

In this section we continue the study of the domain $G_Q = \mathbf{R}^2 \setminus Q$ where $Q \subset \mathbf{R}^2$ is a finite set with $\#Q \geq 2$ and show that it satisfies the conjectures 1.2, 1.3 and a nonstrict version of 1.4. Although the Convexity conjecture 1.4 implies the other two by 2.3 and 2.6, we start with the Uniqueness conjecture 1.2, since it and the Prolongation conjecture 1.3 are needed in the proof of 1.4.

5.1. Theorem. *The domain G_Q satisfies the Uniqueness conjecture 1.2 with the constant $c_{\mathrm{U}} = \pi/2$.*

Proof. Assume that the theorem is false. Then there are points $a, b \in G_Q$ with $k(a, b) < \pi/2$ and geodesics $\gamma_1, \gamma_2 \colon a \curvearrowright b$ such that $\gamma_1 \cap \gamma_2 = \{a, b\}$. We may assume that $a = 0$, $b = (b_1, 0)$ with $b_1 > 0$. By Theorem 2.18, the arcs γ_j are graphs of smooth functions $f_j \colon [0, b_1] \to \mathbf{R}$ with $f_j(0) = f_j(b_1) = 0$. We may assume that $f_1(t) < f_2(t)$ for $0 < t < b_1$. Let W be the Jordan domain bounded by $\gamma = \gamma_1 \cup \gamma_2$. If $q \in Q \cap W$, it follows from 3.3 that the quasihyperbolic length of γ in $\mathbf{R}^2 \setminus \{q\}$ is at least 2π, whence $2\pi \leq l_k(\gamma) < \pi$. This contradiction shows that $Q \cap W = \varnothing$.

For $x \in \gamma_j$ let $v_j(x)$ be the unit tangent vector of γ_j at x and set $\varphi_j(x) = \arg v_j(x)$, $-\pi/2 < \varphi_j(x) < \pi/2$. Then $\varphi_1(a) \leq \varphi_2(a)$ and $\varphi_1(b) \leq \varphi_2(b)$. Setting

$$\Delta\varphi_j = \varphi_j(b) - \varphi_j(a)$$

we therefore have $\Delta\varphi_2 \le \Delta\varphi_1$. We shall show that $\Delta\varphi_1 < \Delta\varphi_2$, which will give the desired contradiction.

For $j = 1, 2$ we let Q_j denote the family of all $q \in Q$ such that D_q meets γ_j but not γ_{3-j} and set $Q_{12} = \{q \in Q\colon \gamma_1 \cap D_q \ne \varnothing \ne \gamma_2 \cap D_q\}$. If γ_j meets D_q, then $\gamma_j \cap \bar D_q$ is a subarc $\gamma_j[x, y]$ by 4.13, and we write

$$\Delta_q\varphi_j = \varphi_j(y) - \varphi_j(x).$$

Then

$$\Delta\varphi_j = \sum_{q\in Q_j} \Delta_q\varphi_j + \sum_{q\in Q_{12}} \Delta_q\varphi_j.$$

Observe that φ_j is constant on each subarc of γ_j contained in an edge of $\operatorname{Vor} Q$.

Let $q \in Q$ be such that D_q meets γ and assume first that D_q does not meet $\{a, b\}$. If $q \in Q_1$, then the q-component A_q of $D_q \setminus \gamma_1$ lies on the right-hand side of $\gamma_1 \cap \bar D_q$, since $q \in W$ in the opposite case. By 4.8 this implies that φ_1 is decreasing on $\gamma_1 \cap \bar D_q$, whence $\Delta_q\varphi_1 < 0$. Similarly $\Delta_q\varphi_2 > 0$ for all $q \in Q_2$.

Let $q \in Q_{12}$ and set $\gamma_1[x_1, y_1] = \gamma_1 \cap \bar D_q$, $\gamma_2[x_2, y_2] = \gamma_2 \cap \bar D_q$. These arcs consist of crosscuts of D_q and of line segments on ∂D_q as explained in 4.8. Let A_q^j be the q-component of $D_q \setminus \gamma_j$. Assume first that A_q^1 lies on the left-hand side of $\gamma_1 \cap \bar D_q$. Since $q \notin W$, also A_q^2 lies on the left-hand side of $\gamma_2 \cap \bar D_q$. Moreover, $\gamma_2 \cap \bar D_q$ contains a crosscut $\gamma_2[x_2', y_2']$ of D_q separating q and $\gamma_1 \cap \bar D_q$ in D_q. Setting $\omega_1 = \operatorname{ang}(x_1 - q, y_1 - q)$, $\omega_2 = \operatorname{ang}(x_2' - q, y_2' - q)$ we have $\omega_1 < \omega_2$. By a fundamental property of logarithmic spirals, we have $\omega_1 \ge \Delta_q\varphi_1$, $\omega_2 = \varphi_2(y_2') - \varphi_2(x_2') \le \Delta_q\varphi_2$. Hence $0 < \Delta_q\varphi_1 < \Delta_q\varphi_2$.

If the domains A_q^1 and A_q^2 lie on the right-hand side of $\gamma_1 \cap \bar D_q$ and $\gamma_2 \cap \bar D_q$, respectively, we similarly obtain $0 > \Delta_q\varphi_2 > \Delta_q\varphi_1$.

If D_q contains one of the endpoints a, b, we obtain the inequality $\Delta_q\varphi_1 < \Delta_q\varphi_2$ by an obvious modification of the above arguments. Combining the estimates yields $\Delta\varphi_1 < \Delta\varphi_2$, and the theorem is proved. □

5.2. Theorem. *The domain G_Q satisfies the Prolongation conjecture 1.3 with the constant $c_{\mathrm{P}} = \pi/2$.*

Proof. This follows directly from Theorems 5.1 and 2.6. □

5.3. Theorem. *If $a \in G_Q$ and if $r < \pi/2$, then $B_k(a, r)$ is a smooth Jordan domain.*

Proof. This follows from the results 2.1, 2.10, 5.1, 5.2. □

The proof of the Convexity conjecture will be based on the following local characterization of the strict convexity of a Jordan domain:

5.4. Lemma. *Suppose that $D \subset \mathbf{R}^2$ is a Jordan domain such that*

(1) *D has a tangent T_b at every point $b \in \partial D$,*

(2) *there is a finite set $E \subset \partial D$ such that each point $p \in \partial D \setminus E$ has a neighborhood $V(b)$ with $\bar D \cap V(b) \subset H_b \cup \{b\}$ where H_b is a component of $\mathbf{R}^2 \setminus T_b$.*

Then D is strictly convex.

Proof. First observe that ∂D does not contain any line segment. Furthermore, the neighborhood $V(b)$ can be chosen to be an arbitrarily small disk $B(b, r)$.

Fact 1. Suppose that ∂D is divided into subarcs α and β by points $x, y \in \partial D$ such that

(i) $(x, y) \subset \mathbf{R}^2 \setminus \bar{D}$,

(ii) α is a crosscut of the Jordan domain D' bounded by $[x, y] \cup \beta$.

Let L be the line containing $[x, y]$ and let $z \in \alpha$ be a point where $d(z, L)$ is maximal. Then $z \in E$.

Assume that $z \notin E$. Let $V(z) = B(z, r)$ be a disk given by (2) with $V(z) \subset D'$. Now α is a crosscut of D', and the components of $D' \setminus \alpha$ are D and a Jordan domain U with $\partial U = [x, y] \cup \alpha$. The line T_z is parallel to L. Let H_z be the component of $\mathbf{R}^2 \setminus T_z$ with $H_z \cap L = \varnothing$, and let W be the half disk $V(z) \cap H_z$. Since W does not meet $\alpha \cup \beta = \partial D$, we have either $W \subset D$ or $W \subset \mathbf{R}^2 \setminus \bar{D}$.

If $W \subset D$, then $T_z \cap V(z) \subset \bar{D}$, which is a contradiction by (2). If $W \subset \mathbf{R}^2 \setminus \bar{D}$, then $W \subset D' \setminus \bar{D} = U$, which is impossible, because $d(z, L) \geq d(p, L)$ for all $p \in \bar{U}$. Fact 1 is proved.

Assume that the lemma is false. Then there are $x, y \in \partial D$ such that $(x, y) \subset \mathbf{R}^2 \setminus \bar{D}$. Let α, β, L, z be as in Fact 1. Then $z \in E$ by Fact 1. Since E is finite, there is a disk $B = B(z, r) \subset D'$ such that $B \cap E = \{z\}$. Let H_z and H'_z be the components of $\mathbf{R}^2 \setminus T_z$ where $H_z \cap L = \varnothing$ as in the proof of Fact 1, and set $W = H_z \cap B$. Now $\alpha \subset \bar{H}'_z$. Since α contains no line segment, there is a point $z_1 \in \alpha \cap H'_z$ with $\alpha[z, z_1] \subset B$. As T_z is a tangent of D, there is $z_2 \in (z, z_1] \cap \alpha$ with $(z, z_2) \cap \alpha = \varnothing$.

Again $W \cap \partial D = \varnothing$ and again W cannot lie in $\mathbf{R}^2 \setminus \bar{D}$. Hence $W \subset D$. Let u be the unit vector perpendicular to T_z such that $z + u \in H'_z$. Then u is normal vector of D at z, whence $(z, z_2) \subset \mathbf{R}^2 \setminus \bar{D}$. Setting $\alpha_2 = \alpha[z, z_2]$ and applying Fact 1 with the substitution $(x, y, \alpha) \mapsto (z, z_2, \alpha_2)$ we get the desired contradiction $\mathring{\alpha}_2 \cap E \neq \varnothing$. $\square$

5.5. Harmful arcs. Recall that X is the union of all edges of $\operatorname{Vor} Q$. Moreover, we let X_0 denote the finite set of all vertices of the cells in $\operatorname{Vor} Q$.

In the proof of the convexity of a quasihyperbolic disk $B_k(a, r)$, $r \leq 1$, we study quasihyperbolic geodesics $\gamma\colon a \curvearrowright b \in S_k(a, 1)$ and the behavior of quasihyperbolic circles $S_k(a, u)$, $u \leq 1$, along γ. In order to limit the number of cases and subcases we shall rule out certain arcs with unpleasant properties. We say that an arc $\alpha\colon a_1 \curvearrowright b_1$ in G_Q is *harmful* to a if

(i) α is a subarc of a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ with $l_k(\gamma) = 1$,

(ii) $a_1 \in X_0$,

(iii) $a_1 \neq a$,

(iv) $\alpha \cap X$ does not contain a line segment.

We let $A = A(Q, a)$ denote the union of all arcs harmful to a. If we are working with a fixed a, we briefly say that α is harmful if it is harmful to a.

5.6. Lemma. (1) *The set A is a finite union of harmful arcs,*

(2) *$A \cap S_k(a, u)$ is finite for all $u \in (0, 1]$,*

(3) *$A \cap X$ is finite.*

Proof. Suppose that $\alpha\colon a_1 \curvearrowright b_1$ is harmful to a. Let $\gamma\colon a \curvearrowright b$ be as in (i). Since $l_k(\alpha) \leq l_k(\gamma) = 1$, it follows from (iv) and from the Edge theorem 4.2 that each edge J of $\operatorname{Vor} Q$ contains at most one point of α. Hence $\alpha \cap X$ is finite, and (3) will follow from (1).

By Theorem 5.1, the geodesic $\gamma[a, a_1]$ is uniquely determined by a_1, whence also the tangent vector $v(a_1)$ of α at a_1 is uniquely determined by a_1. Let $\alpha = \alpha_1 \cup \cdots \cup \alpha_m$

be the standard decomposition of α; see 4.4. By (iv), the arcs α_j are crosscuts of Voronoi cells for $j \le m-1$; the arc α_m may be an endcut. From the discussion in 4.14 we see that there are at most two possibilities for α_1 and hence at most two possibilities for $v(a_2)$. Repeating the argument we see that (1) is true, and (2) is an obvious corollary. □

We next give the nonstrict version of the Convexity conjecture for G_Q.

5.7. Theorem. *If $a \in G_Q$ and $0 < r \le 1$, then $B_k(a,r)$ is convex.*

Proof. To simplify notation we write $B_r = B_k(a,r)$, $S_r = S_k(a,r)$. By 5.3, B_r is a smooth Jordan domain. Let $b \in S_k(a,1)$ and let $\gamma\colon a \curvearrowright b$ be the unique quasihyperbolic geodesic. Let $\gamma = \gamma_1 \cup \dots \cup \gamma_m$ be the standard decomposition of γ with $\gamma_\nu = \gamma[x_{\nu-1}, x_\nu]$. Set $\Delta = [0,1]$ and let $g\colon \Delta \to \gamma$ be the quasihyperbolic parametrization. For $t_\nu = k(a, x_\nu)$ and $\Delta_\nu = [t_{\nu-1}, t_\nu]$ we have $\gamma_\nu = g\Delta_\nu$. We express the set $M = \{1, \dots, m\}$ as a disjoint union $M = M_1 \cup M_2$ where $\nu \in M_1$ if γ_ν is a crosscut or and endcut of a Voronoi cell D_ν and $\nu \in M_2$ if γ_ν is a segment of an edge in $\operatorname{Vor} Q$.

Suppose that $\nu \in M_1$. Let $q_\nu \in Q$ be the nucleus of D_ν and write $F_\nu(z) = e^z + q_\nu$, $U_\nu = F_\nu^{-1} D_\nu$. Fix a point $x^*_{\nu-1} \in F_\nu^{-1}\{x_{\nu-1}\}$ and let $g^*_\nu\colon \Delta_\nu \to \bar U_\nu$ be the F_ν-lift of $g_\nu = g|\Delta_\nu$ with $g^*_\nu(x_{\nu-1}) = x^*_{\nu-1}$. Then g^*_ν is an affine euclidean isometry of Δ_ν onto a line segment $\gamma^*_\nu = [x^*_{\nu-1}, x^*_\nu] \subset \bar U_\nu$ with $F_\nu \gamma^*_\nu = \gamma_\nu$. Set

$$v^*_\nu = \frac{x^*_\nu - x^*_{\nu-1}}{|x^*_\nu - x^*_{\nu-1}|}, \quad p_\nu = x^*_{\nu-1} - t_{\nu-1} v^*_\nu.$$

Then

$$|g^*_\nu(u) - p_\nu| = u \tag{5.8}$$

for all $u \in \Delta_\nu$.

If $t_{\nu-1} < u < t_\nu$ and if $Y \subset S_u \cap D_\nu$ is an arc neighborhood of the point $y = g(u)$, then Y has an F_ν-lift to an arc Y^* containing $y^* = g^*_\nu(u)$. We say that the index $\nu \in M$ is *good* (for γ) if

(1) $\nu \in M_1$,

(2) γ_ν is not contained in a harmful arc,

(3) for each $u \in (t_{\nu-1}, t_\nu)$ there is an arc neighborhood Y of $y = g(u)$ in $S_u \cap D_\nu$ such that $Y^* \subset \bar B(p_\nu, u)$.

We shall use induction to show that all indices satisfying (1) and (2) are good.

Fact 1. If $1 \in M_1$, then 1 is a good index.

Conditions (1) and (2) are clearly true. Now $p_1 = x^*_0 \in \bar U_1$. If $x^*_0 \in U_1$, there is an arc neighborhood Y of y such that Y^* is a circular arc of $S(x^*_0, u)$. The same is true if $x^*_0 \in \partial U_1$ and if γ^*_1 is not tangent to an arc in ∂U_1. In the tangential case Y^* can be chosen to be an arc of the d_1-circle $S_{d_1}(x^*_0, u)$ where d_1 is the inner metric of $\bar U_1$, and again $Y^* \subset \bar B(x^*_0, u)$.

Fact 2. If $\nu \ge 2$, $\nu \in M_1$, $\nu - 1 \in M_2$, $x_{\nu-1} \notin X_0$, then ν is good.

Conditions (1) and (2) are again clear. Now $\gamma_{\nu-1} = [x_{\nu-2}, x_{\nu-1}]$ lies on a common edge J of Voronoi cells D and $\tilde D$. Since $x_{\nu-1} \notin X_0$, the arc γ_ν is a crosscut of D or $\tilde D$ (or possibly an endcut if $\nu = m$). We may assume that $\gamma_\nu \subset \bar D$ and thus $D = D_\nu$. Now $y \in D_\nu$ and we may again use the function $F_\nu(z) = e^z + q_\nu$. The F_ν-lift of $\gamma_{\nu-1}$ is a subarc $\gamma^*_{\nu-1} = C[x^*_{\nu-2}, x^*_{\nu-1}]$ of an lc-curve C with $F_\nu C = \operatorname{aff} J$, and the line segment $\gamma^*_\nu = [x^*_{\nu-1}, x^*_\nu]$ is tangent to C at $x^*_{\nu-1}$.

There is an arc neighborhood $Y_1 \subset S_u$ of y such that for each $z \in Y_1$ the geodesic $\gamma_z\colon a \curvearrowright z$ contains a line segment $[x_{\nu-2}, x] \subset J$. The F_ν-lift γ_z^* of $\gamma_z[x_{\nu-2}, z]$ is a geodesic in the inner metric d_ν of $\bar{U}_\nu$, and the F_ν-lift Y_1^* of Y_1 lies on the d_ν-circle $S_{d_\nu}(x_{\nu-2}^*, u - t_{\nu-2})$. The arc Y_1^* can be obtained by taking a thread of length $u - t_{\nu-2}$ with one endpoint at $x_{\nu-2}^*$, keeping it taut and moving the other endpoint so that the thread stays in $\bar{U}_\nu$.

It follows from classical curve theory that Y_1^* is an arc of the involute (= evolvent) of C. The curvature center of Y_1^* at y^* is $x_{\nu-1}^*$ and the curvature radius is $|x_{\nu-1}^* - y^*| = k(x_{\nu-1}, y) < u$. Hence there is an arc neighborhood Y^* of y^* in Y_1^* with $Y^* \subset \bar{B}(p_\nu, u)$, and Fact 2 is proved.

Fact 3. If $\nu \ge 2$, $\nu \in M_1$, $x_{\nu-1} \notin X_0$ and $\nu - 1$ is good, then ν is good.

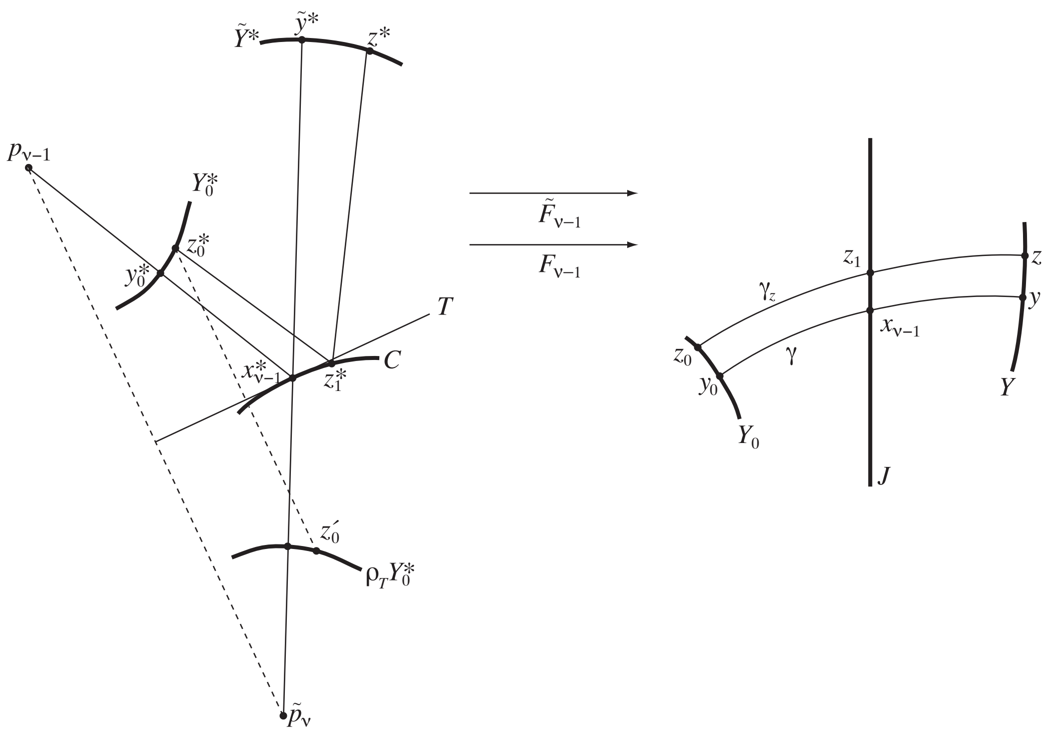
Conditions (1) and (2) are again clear. Now there are unique cells $D_{\nu-1}, D_\nu$ containing $\mathring{\gamma}_{\nu-1}$ and $\mathring{\gamma}_\nu$, respectively. Since $x_{\nu-1} \notin X_0$, there are three possibilities (see 3.11 and 3.10):

(1) $D_{\nu-1} \ne D_\nu$ and γ crosses the common edge $J = \bar{D}_{\nu-1} \cap \bar{D}_\nu$ at $x_{\nu-1}$.

(2) As (1) but γ touches J at $x_{\nu-1}$.

(3) $D_{\nu-1} = D_\nu$, γ touches an edge J at $x_{\nu-1}$ and returns to $D_{\nu-1}$.

We prove case (1) in detail. Let $q_{\nu-1}, q_\nu$ be the nuclei of $D_{\nu-1}, D_\nu$, and let $\varrho\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in the line L containing J. We shall apply the covering map $F_{\nu-1}(z) = e^z + q_{\nu-1}$ in $D_{\nu-1}$ and the map $\tilde{F}_{\nu-1} = \varrho \circ F_{\nu-1}$ in D_ν, modifying in an obvious way the treatment in 3.11, replacing H by $D_{\nu-1}$ and $\tilde{H}$ by D_ν. Fix a point $x_{\nu-1}^* \in F_{\nu-1}^{-1}\{x_{\nu-1}\} = \tilde{F}_{\nu-1}^{-1}\{x_{\nu-1}\}$ and let $g_{\nu-1}^*\colon \Delta_\nu \to \mathbf{R}^2$ and $\tilde{g}_\nu^*$ be the $F_{\nu-1}$-lift of $g_{\nu-1}$ and the $\tilde{F}_{\nu-1}$-lift of g_ν with $g_{\nu-1}^*(t_{\nu-1}) = \tilde{g}_\nu^*(t_{\nu-1}) = x_{\nu-1}^*$. Then the line segments $\gamma_{\nu-1}^* = [x_{\nu-2}^*, x_{\nu-1}^*] = g_{\nu-1}^*\Delta_{\nu-1}$ and $\tilde{\gamma}_\nu^* = [x_{\nu-1}^*, \tilde{x}_\nu^*] = \tilde{g}_\nu^*\Delta_\nu$ represent a light beam that reflects from the convex mirror C, which is an lc-curve in $F_{\nu-1}^{-1}L = \tilde{F}_{\nu-1}^{-1}L$; see 3.11, Case 1, and the figure below.



Fact 3, case (1). The dotted lines are perpendicular to T.

Fix $u_0 \in (t_{\nu-2}, t_{\nu-1})$ and set $y_0 = g(u_0)$, $y_0^* = g_{\nu-1}^*(u_0)$, $\tilde{y}^* = \tilde{g}_{\nu-1}^*(u)$. Let T be the tangent of C at $x_{\nu-1}^*$ and let $\varrho_T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in T. Since $\nu - 1$ is a good index, there is an arc neighborhood $Y_0 \subset S_{u_0}$ of y_0 with $Y_0^* \subset \bar{B}(p_{\nu-1}, u_0)$. From 2.2 it follows that there is an arc neighborhood $Y \subset S_u$ of y such that every geodesic $\gamma_z \colon a \curvearrowright z \in Y$ meets Y_0 at some point z_0. Let $\tilde{Y}^*$ be the $\tilde{F}_{\nu-1}$-lift of Y containing $\tilde{y}^*$ and set

$$\tilde{v}_\nu^* = \frac{\tilde{x}_\nu^* - x_{\nu-1}^*}{|\tilde{x}_\nu^* - x_{\nu-1}^*|}, \quad \tilde{p}_\nu = x_{\nu-1}^* - t_{\nu-1}\tilde{v}_\nu^*.$$

Then $|\tilde{y}^* - p_\nu| = u$ and $\tilde{p}_\nu = \varrho_T p_{\nu-1}$.

Condition (3) in the definition of a good index is easily seen to be equivalent to

$$\tilde{Y}^* \subset \bar{B}(\tilde{p}_\nu, u). \tag{5.9}$$

Indeed, the maps F_ν and $\tilde{F}_{\nu-1}$ are related by $F_\nu = \tilde{F}_{\nu-1} \circ \mu$ where μ is the reflection of $\mathbf{R}^2$ in the horizontal line $\operatorname{Im} z = \arg(q_\nu - q_{\nu-1}) + \pi/2$.

Let $z \in Y$ and let γ_z and $z_0 \in \gamma_z \cap Y_0$ be as above. Let z_1 be the unique point in $\gamma_z \cap J$ and let $[z_0^*, z_1^*]$ and $[z_1^*, z^*]$ give the $(F_{\nu-1}, \tilde{F}_{\nu-1})$-lift of $\gamma_z[z_0, z^*]$. We must show that

$$|\tilde{p}_\nu - z^*| \le u.$$

By the convexity of the lc-curve C, we have $|z_0' - z_1^*| \le |z_0^* - z_1^*|$ where $z_0' = \varrho_T(z_0^*)$. Furthermore,

$$u - u_0 = k(z_0, z) = |z_0^* - z_1^*| + |z_1^* - z^*|.$$

Since $|z_0' - \tilde{p}_\nu| = |z_0^* - p_{\nu-1}^*| \le u_0$, we obtain

$$|\tilde{p}_\nu - z^*| \le |\tilde{p}_\nu - z_0'| + |z_0' - z_1^*| + |z_1^* - z^*| \le u_0 + (u - u_0) = u,$$

and case (1) of Fact 3 is proved. We omit the proofs of cases (2) and (3), which are obtained by combining the proofs of case (1) and Fact 2.

Recall that A denotes the union of all arcs harmful to a; see 5.5.

Fact 4. If $\nu \in M_1$ and $\gamma_\nu \not\subset A$, then ν is a good index.

Assume that ν is not good. Then $\nu \ge 2$ by Fact 1. Since γ_ν is not harmful, we have $x_{\nu-1} \notin X_0$. Hence $\nu - 1 \in M_1$ by Fact 2 and $\nu - 1$ is not good by Fact 3. As $\gamma_\nu \not\subset A$ implies $\gamma_{\nu-1} \not\subset A$, we may proceed inductively and see that $1 \in M_1$ and 1 is not good, which is a contradiction by Fact 1.

From Lemma 5.6 it follows that the set $Z = (A \cap X) \cup X_0$ is finite.

Fact 5. If $0 < t < 1$, and if $S_t \cap Z = \varnothing$, then $S_t \cap X$ is finite.

Assume that $z \in S_t \cap X$. It suffices to show that there is an arc neighborhood $Y \subset S_t$ of z such that $Y \cap X = \{z\}$. Since $X_0 \cap S_t = \varnothing$, z is an interior point of an edge J of $\operatorname{Vor} Q$. Let γ be a quasihyperbolic geodesic from a through z with $l_k(\gamma) = 1$. By Theorem 2.10, the quasihyperbolic circle S_t has a tangent T_z at z, and T_z is perpendicular to the tangent vector $v(z)$ of γ at z. Hence the arc Y exists if $v(z)$ is not perpendicular to J.

Assume that $v(z) \perp J$. Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ be the standard decomposition of γ. Then $z \in \gamma_\nu = \gamma[x_{\nu-1}, x_\nu]$ for some ν, and z is an endpoint of γ_ν, since $\gamma_\nu \not\subset J$. We may choose ν so that $z = x_\nu$. Then $t = t_\nu = k(a, x_\nu)$ and ν is a good index for γ by Fact 4.

The arc γ_ν is a crosscut or (if $\nu = 1$) an endcut of a cell $D_\nu \in \operatorname{Vor} Q$. We use the notation $F_\nu(w) = e^w + q_\nu$ and $g_\nu \colon [t_{\nu-1}, t_\nu] \to \gamma_\nu$ as before. Let g_ν^* be an F_ν-lift of g_ν. Then $\gamma_\nu^* = \operatorname{im} g_\nu$ is a line segment $[x_{\nu-1}^*, x_\nu^*]$, which meets orthogonally a lift J^* of J

lying on an lc-curve C. Fix $u \in (t_{\nu-1}, t_\nu)$ and set $y = g_\nu(u) \in \mathring{\gamma}_\nu$, $y^* = g_\nu^*(u) \in \mathring{\gamma}_\nu^*$. As ν is a good index, there is an arc neighborhood Y_0 of y in S_u with F_ν-lift $Y_0^* \subset \bar{B}(p_\nu, u)$. Then the euclidean distance $d(x^*, C) > t - u$ for all $x^* \in Y_0^* \setminus \{y^*\}$, whence $k(x, J) > t - u$ for all $x \in Y_0 \setminus \{y\}$. Hence there is an arc neighborhood Y of z in S_t with $Y \cap J = \{z\}$, and Fact 5 is proved.

We turn to the proof of Theorem 5.7. Let Z be as above and suppose that $0 < r < 1$ and that $S_r \cap Z = \varnothing$. As Z is finite, it suffices to show that B_r is convex. We show that the conditions of Lemma 5.4 are satisfied with the substitution $D \mapsto B_r$, $E \mapsto (X \cup A) \cap S_t$. This will imply that B_r is in fact strictly convex.

By Fact 5 and Lemma 5.6, the set E is finite. Let $y \in S_r \setminus E$, and let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic containing y with $l_k(\gamma) = 1$. Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ be the standard decomposition of γ. As $y \notin X$, there is ν such that $y \in \mathring{\gamma}_\nu$, and γ is a crosscut or an endcut of a cell D_ν. Furthermore, we have $y \notin A$, whence $\gamma_\nu \not\subset A$. By Fact 4, the index ν is good for γ. With notation as before we obtain an arc neighborhood $Y \subset S_r$ of y such that the F_ν-lift Y^* of Y satisfies $Y^* \subset \bar{B}(p_\nu, r)$.

By Theorem 3.6, the domain $W = F_\nu B(p_\nu, r)$ is a strictly convex smooth Jordan domain. The tangent T of W at y is also a tangent of the arc Y. There is a component H_y of $\mathbf{R}^2 \setminus T$ such that $\bar{W} \subset H_y \cup \{y\}$, and therefore $\bar{B}_r \cap V(y) \subset H_y \cup \{y\}$ for some neighborhood $V(y)$ of y. Hence B_r is convex by 5.4. □

6. Strict convexity in G_Q

In this section we show that a quasihyperbolic disc $B_k(a, r)$ in $G_Q = \mathbf{R}^2 \setminus Q$ is strictly convex for $r < 1$. Moreover, we give an estimate for the strictness, which is needed in the next section to obtain the result for arbitrary domains in $\mathbf{R}^2$. As a quasihyperbolic circle $S_k(a, r)$ need not be C^2 smooth, it does not always have a curvature in the ordinary sense. We must therefore introduce a more general notion, called outer curvature radius. First, an elementary lemma:

6.1. Lemma. *Suppose that $0 < t_0 < R$ and that $f\colon [-t_0, t_0] \to \mathbf{R}$ is a convex C^1 function such that $f(0) = 0$ and such that*

$$f(x) \ge g(x) := R - \sqrt{R^2 - x^2}$$

for all $x \in [-t_0, t_0]$. Then $|f'(x)| \ge |x|/2R$ for all $x \in [-t_0, t_0]$.

Proof. We may assume that $x > 0$. Since f is convex and since $g(x) > x^2/2R$, we get

$$f'(x) > f(x)/x \ge g(x)/x > x/2R. \qquad \square$$

6.2. Outer curvature radius. Suppose that $W \subset \mathbf{R}^2$ is a convex Jordan domain and that $\gamma = \partial W$ is C^1 smooth. For this it suffices to know that W has a tangent at every point of γ, since a convex differentiable function is C^1; see [Ro, 25.5.1]. Assume that γ is positively oriented in W and let $v(x)$ be the unit tangent vector of γ at $x \in \gamma$. Then the left normal vector $n(x) = iv(x)$ is directed into W; see 4.5. The *outer curvature radius* $R(x)$ of γ at x is the infimum of all numbers $r > 0$ such that there is an arc neighborhood $Y \subset \gamma$ of x contained in $\bar{B}(x + rn(x), r)$. If there is no such r, we set $R(x) = \infty$. If γ is C^2 smooth, then $R(x)$ is equal to the ordinary curvature radius.

6.3. Lemma. *Suppose that W is a convex Jordan domain with smooth boundary $\gamma = \partial W$, that $E \subset \gamma$ is a finite set and that $R(x) \le R_0 < \infty$ for all $x \in \gamma \setminus E$. Let $\alpha\colon a \curvearrowright b$ be a positively oriented arc on γ. Then*

$$\varphi(b) - \varphi(a) \ge l(\alpha)/2R_0$$

where $\varphi(x) = \arg v(x)$ is the direction angle of γ.

Proof. Assume first that $E = \varnothing$. Set $\lambda = l(\alpha)$ and let $g\colon [0,\lambda] \to \alpha$ be the length parametrization of α. Write $\varphi(s) = \varphi(g(s))$ and let $0 \le t \le \lambda$. It suffices to show that

$$\liminf_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} \ge \frac{1}{2R_0}, \tag{6.4}$$

since a bisection argument then gives $\varphi(s_2) - \varphi(s_1) \ge (s_2 - s_1)/2R_0$ for $0 \le s_1 < s_2 \le \lambda$.

We normalize the situation so that $g(t) = 0$, $\varphi(t) = 0$. Let $R_1 > R_0$. By the definition of outer curvature, there is an arc neighborhood Y of 0 in ∂W that is a graph of a convex C^1 function $f\colon [-\delta,\delta] \to \mathbf{R}$ such that $f(x) \ge R_1 - \sqrt{R_1^2 - x^2}$ for all $x \in [-\delta,\delta]$. By Lemma 6.1 we have $f'(x) \ge |x|/2R_1$ for all $|x| \le \delta$.

Let $0 < h \le \delta$. There is $x_h \in (0,h)$ such that $g(h) = (x_h, f(x_h))$. Then $f'(x_h) = \tan\varphi(h)$, and we obtain

$$\frac{\varphi(h)}{h} \ge \frac{\varphi(h)}{\tan\varphi(h)} \frac{1}{2R_1} \frac{x_h}{h}.$$

As $h \to 0$, we have $x_h/h \to 1$ and $\varphi(h)/\tan\varphi(h) \to 1$, whence

$$\liminf_{h\to 0+} \frac{\varphi(h)}{h} \ge \frac{1}{2R_1}.$$

The case $h \to 0-$ is treated similarly. As $R_1 \to R_0$, we get (6.4).

The case $E \subset \{a,b\}$ follows by a limiting argument. The general case is proved applying the special case to each component of $\alpha \setminus E$. □

6.5. Lemma. *Let W and $\alpha \subset \gamma = \partial W$ be as in 6.3. Then there is $z \in \alpha$ such that*

$$d(z, \operatorname{aff}\{a,b\}) \ge \frac{|a-b|^2}{4(1+2R_0)^2}.$$

Proof. Let $z \in \alpha$ be the point where $h = d(z, \operatorname{aff}\{a,b\})$ is maximal. We normalize the situation so that $z = 0$ and $\varphi(z) = 0$. Then $a_2 = b_2 = h$ for the second coordinates. We may assume that $|a_1| \le |b_1|$ and that $b_1 \ge 0$. It suffices to show that

$$b_1 \le (1+2R_0)\sqrt{h}. \tag{6.6}$$

If $b_1 \le \sqrt{h}$, this is clearly true. Assume that $b_1 > \sqrt{h}$ and let $x \in \alpha(0,b)$ be the unique point with $x_1 = \sqrt{h}$. Now

$$b_1 \le \sqrt{h} + h/\tan\varphi(x) \le \sqrt{h} + h/\varphi(x).$$

By 6.3 we have $\varphi(x) \ge \sqrt{h}/2R_0$, and (6.6) follows. □

We return to the domain $G_Q = \mathbf{R}^2 \setminus Q$. Let $a \in G_Q$. As in 5.5 we let A denote the union of all arcs harmful to a. Moreover, we set $Z = (A \cap X) \cup X_0$ as in Fact 5 of 5.7. The set Z is finite by Lemma 5.6.

6.7. Lemma. *Let $a \in G_Q$, $0 < r < 1$, and suppose that $S_k(a,r) \cap Z = \varnothing$. Then there is a finite set $E \subset S_k(a,r)$ such that the outer curvature radius of $S_k(a,r)$ is at most $K(r)\delta(a)$ for all $y \in S_k(a,r) \setminus E$ where*

$$K(r) = \frac{re^{3r}}{1-r}.$$

Proof. We show that the lemma holds with $E = (X \cup A) \cap S_k(a,r)$, which is finite by Fact 5 of 5.7 and by Lemma 5.6. Let $y \in S_k(a,r) \setminus E$ and choose a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ such that $y \in \gamma$ and such that $l_k(\gamma) = 1$. Using the notation of the proof of 5.7 we consider the covering map $F_\nu(z) = e^z + q_\nu$ where q_ν is the nucleus of the Voronoi cell D_ν containing y. We find an arc neighborhood Y of y in $S_k(a,r)$ such that the F_ν-lift Y^* of Y is contained in a disk $\bar{B}(p_\nu, r)$ where p_ν is a point with $|p_\nu - y^*| = r = k(a,y)$.

By Theorem 3.6, the domain $F_\nu B(p_\nu, r)$ is a smooth strictly convex Jordan domain, and the curvature radius of its boundary at each point is at most

$$R_0(p_\nu, r) = \frac{r|e^{p_\nu}|e^r}{1-r}.$$

Consequently, it suffices to show that

$$|e^{p_\nu}| \le e^{2r}\delta(a). \tag{6.8}$$

Let k_ν be the quasihyperbolic metric of $\mathbf{R}^2 \setminus \{q_\nu\}$. By Lemma 3.2 we get

$$r = |p_\nu - y^*| = k_\nu(F_\nu(p_\nu), F_\nu(y^*)) = k_\nu(e^{p_\nu} + q_\nu, y) \ge \log \frac{|e^{p_\nu}|}{|y - q_\nu|} = \log \frac{|e^{p_\nu}|}{\delta(y)}.$$

Since $r = k(a,y) \ge \log \frac{\delta(y)}{\delta(a)}$, this implies (6.8). □

6.9. Lemma. *Let $a \in G_Q$, $0 < r < 1$ and let $\alpha\colon x \curvearrowright y$ be an arc in $S_k(a,r)$. Then there is $z \in \alpha$ such that*

$$d(z, \operatorname{aff}\{x,y\}) \ge \frac{|x-y|^2}{4(1 + 2K(r)\delta(a))^2}.$$

Proof. If $S_k(a,r) \cap Z = \varnothing$, the estimate holds by 6.5 and 6.7. As Z is finite, the case $S_k(a,r) \cap Z \ne \varnothing$ follows by an easy limiting process. □

6.10. Theorem. *The domain G_Q satisfies the Convexity conjecture 1.4 with the sharp constant $c_{\mathrm{C}} = 1$.*

Proof. This follows from 5.7 and 6.9. □

7. Arbitrary planar domains

In this section we prove the main results of the paper.

7.1. Approximation. Let $G \subset \mathbf{R}^2$ be a bounded domain. For each positive integer j we choose a finite set $Q_j \subset \partial G$ such that

(1) $Q_j \subset Q_{j+1}$,

(2) $d(x, Q_j) < 1/j$ for all $x \in \partial G$.
Writing $G_j = \mathbf{R}^2 \setminus Q_j$ and

$$\delta_j = \delta_{G_j}, \quad k_j = k_{G_j}, \quad \delta = \delta_G, \quad k = k_G$$

we have

$$\delta(x) \le \delta_{j+1}(x) \le \delta_j(x) \le \delta(x) + 1/j \tag{7.2}$$

for all $j \in \mathbf{N}$, $x \in G$. Hence

$$\delta_j(x) \searrow \delta(x)$$

uniformly in G. For a rectifiable arc $\gamma \subset G \subset G_j$ we have

$$l_{k_j}(\gamma) \nearrow l_k(\gamma). \tag{7.3}$$

We next show that

$$k_j(a, b) \nearrow k(a, b) \tag{7.4}$$

for all $a, b \in G$.

Since $G \subset G_{j+1} \subset G_j$, the sequence $(k_j(a, b))$ is increasing and $\lim_{j\to\infty} k_j(a, b) \le k(a, b)$. To prove the converse inequality we choose for each $j \in \mathbf{N}$ a quasihyperbolic geodesic $\gamma_j : a \curvearrowright b$ in G_j. For each $x \in \gamma_j$ we have

$$k(a, b) \ge k_j(a, b) \ge k_j(a, x) \ge \log \frac{\delta_j(a)}{\delta_j(x)} \ge \log \frac{\delta(a)}{\delta_j(x)}.$$

Thus

$$\delta_j(x) \ge \delta(a)e^{-k(a,b)} =: s > 0.$$

If $1/j < s/2$, then (7.2) implies that $\gamma_j \subset G$ and that $1/\delta(x) - 1/\delta_j(x) \le 1/js\delta(x)$ for all $x \in \gamma_j$. Consequently,

$$k_j(a, b) = \int_{\gamma_j} \frac{|dx|}{\delta_j(x)} \ge (1 - 1/js)l_k(\gamma_j) \ge (1 - 1/js)k(a, b),$$

and (7.4) follows.

7.5. Theorem. *Let $G \subset \mathbf{R}^2$ be a domain and let $a \in G$, $0 < r \le 1$. Then $B_k(a, r)$ is a convex smooth Jordan domain.*

Proof. Replacing G by a component of $G \cap B(a, R)$ with a large R we may assume that G is bounded. Let $G_j = \mathbf{R}^2 \setminus Q_j$ be as in 7.1. By (7.4) we have

$$\bar{B}_k(a, r) = \bigcap \{\bar{B}_{k_j}(a, r) : j \in \mathbf{N}\}.$$

As the domains $B_{k_j}(a, r)$ are convex by 5.7, the set $\bar{B}_k(a, r)$ is convex. Since $B_k(a, r)$ is a Jordan domain by 2, it is convex, and the smoothness follows from the last part of 2.10. $\square$

7.6. Lemma. *Let $G \subset \mathbf{R}^2$ be a domain and let $a \in G$, $0 < r < 1$. Let $\alpha : x \curvearrowright y$ be an arc in $S_k(a, r)$. Then there is $z \in \alpha$ such that*

$$d(z, L) \ge \frac{|x - y|^2}{4(1 + 2K(r)\delta(a))^2},$$

where $L = \operatorname{aff} \{x, y\}$ and $K(r)$ is as in 6.7.

Proof. We may again assume that G is bounded. For each j we choose points $x_j, y_j \in S_{k_j}(a,r) \cap L$. Let $\alpha_j\colon x_j \curvearrowright y_j$ be the subarc of $S_{k_j}(a,r)$ for which $\mathring{\alpha}$ is contained in the Jordan domain bounded by $\alpha_j \cup [x_j, y_j]$. By 6.9 we find a point $z_j \in \alpha_j$ for which

$$d(z_j, L) \geq \frac{|x_j - y_j|^2}{4(1 + 2K(r)\delta_j(a))^2}.$$

Passing to a subsequence we may assume that (z_j) converges to a point $z \in \alpha$. Now z satisfies the lemma. □

7.7. Main theorem. *Let $G \subset \mathbf{R}^2$ be a domain.*
(1) *The Convexity conjecture 1.4 holds for G with the sharp constant $c_{\mathrm{C}} = 1$.*
(2) *The Uniqueness conjecture 1.2 holds for G with $c_{\mathrm{U}} = 2$.*
(3) *The Prolongation conjecture 1.3 holds for G with $c_{\mathrm{P}} = 2$.*

Proof. Part (1) follows from 7.5 and 7.6. By Theorems 2.3 and 2.6, this implies (2) and (3) with $c_{\mathrm{U}} = 2$, $c_{\mathrm{P}} = \pi/2$. The improvement $c_{\mathrm{P}} = 2$ will be proved in 8.11. □

7.8. Sharpness. I do not know whether $B_k(a,1)$ is always strictly convex. The constants in (2) and (3) are presumably not sharp. The punctured plane G_1 gives the upper bounds $c_{\mathrm{U}} \leq \pi$, $c_{\mathrm{P}} \leq \pi$, and it is possible that the constants $c_{\mathrm{U}} = c_{\mathrm{P}} = \pi$ are valid for all planar domains.

The dimensions $n \geq 3$ remain open, but the following example shows that the uniqueness constant c_{U} must be less than π in $\mathbf{R}^3$. Let $G = \mathbf{R}^3 \setminus \{-e_3, e_3\}$ and let $a = -2e_1$, $b = 2e_1$. Explicit calculation shows that $l_k([a,b]) = 2\log(2+\sqrt{5}) > 2.88$ and $l_k(\gamma) = 2\pi/\sqrt{5} < 2.81$ for the semicircle $\gamma\colon a \curvearrowright b$, $\gamma \subset \mathbf{R}^2$. Hence $[a,b]$ is not a quasihyperbolic geodesic in G. If $\alpha\colon a \curvearrowright b$ is a geodesic, then $\varrho\alpha$ is another geodesic where $\varrho\colon \mathbf{R}^3 \to \mathbf{R}^3$ is the reflection in the line $\operatorname{span} e_1$. Thus $c_{\mathrm{U}} < 2.81$ for this domain.

7.9. Theorem. *If $a \in G \subset \mathbf{R}^2$ and $r < \pi/2$, then the quasihyperbolic disk $B_k(a,r)$ is a smooth Jordan domain.*

Proof. By 2.1, $B_k(a,r)$ is a Jordan domain, and the smoothness follows from 2.10 and 7.7. □

8. Other topics

In this section we give some further results and make some conjectures on the quasihyperbolic geometry of domains in $\mathbf{R}^n$.

8.1. Quasihyperbolic convexity. Let $G \subset \mathbf{R}^n$ be a domain. A set $A \subset G$ is *quasihyperbolically convex* in G if $\gamma \subset A$ whenever $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic in G with $a, b \in A$.

8.2. Quasihyperbolic convexity conjecture. *There is a universal constant $c_{\mathrm{QH}} > 0$ such that the quasihyperbolic ball $B_k(a,r)$ is quasihyperbolically convex for all $r < c_{\mathrm{QH}}$.*

8.3. Remark. Using the function $F(z) = e^z$ as in Section 3 it is easy to show that the punctured plane satisfies 8.2 with $c_{\mathrm{QH}} = \pi/2$.

8.4. Local geodesics. Let $G \subset \mathbf{R}^n$ be a domain and let $\Delta \subset \mathbf{R}$ be a closed interval, possibly unbounded. A map $g\colon \Delta \to G$ is a *locally geodesic path* if each $t_0 \in \Delta$ has an interval neighborhood $\Delta_0 \subset \Delta$ such that $g|\Delta_0$ is a geodesic path. An arc $\gamma \subset G$ is a *local geodesic* if $\gamma = \operatorname{im} g$ for some injective locally geodesic path.

For example, the map $g\colon \mathbf{R} \to G_1 = \mathbf{R}^2 \setminus \{0\}$, $g(t) = Me^{it}$, $M > 0$, is a locally geodesic path.

Suppose that $g\colon \Delta \to G \subset \mathbf{R}^n$ is a locally geodesic path. Then g is C^1 with $|g'(t)| = \delta(g(t))$ for all $t \in \Delta$. The vector $v(t) = g'(t)/\delta(g(t))$ is the unit tangent vector of g at $g(t)$. From 2.16 we see that

$$\operatorname{ang}(v(s), v(t)) = \operatorname{ang}(g'(s), g'(t)) \le |s - t| \tag{8.5}$$

for all $s, t \in \Delta$. More generally, the total variation of $t \mapsto \operatorname{ang}(v(t_0), v(t))$ is at most $s - t_0$ on any subinterval $[t_0, s] \subset \Delta$.

The Prolongation theorem 7.7(3) implies:

8.6. Theorem. *Every geodesic path $g\colon [t_1, t_2] \to G \subset \mathbf{R}^2$ can be extended to a locally geodesic path $g_1\colon \mathbf{R} \to G$.* □

8.7. Theorem. *If $g\colon [0, r] \to G \subset \mathbf{R}^n$ is a locally geodesic path and if $r \le \pi$, then g is injective.*

Proof. Assume that g is not injective. We may assume that $g(0) = g(r) = 0$. Let $t_0 \in (0, r)$ be the point where $|g(t)|$ is maximal, set $x_0 = g(t_0)$ and $u = g'(t_0)$. Then $u \cdot x_0 = 0$. Let $t_1 \in (0, t_0)$ be the point where $u \cdot g(t)$ is minimal and let $t_2 \in (t_0, r)$ be the point where $u \cdot g(t)$ is maximal. Then $u \cdot g'(t_j) = 0$ for $j = 1, 2$, and (8.5) yields the contradiction

$$\pi \ge l_k(g) > l_k(g|[t_1, t_0]) + l_k(g|[t_0, t_2]) \ge \pi/2 + \pi/2 = \pi. \qquad \square$$

8.8. Local geodesic conjecture. *There is a universal constant $c_{\mathrm{LG}} > 0$ such that if $g\colon \Delta \to G \subset \mathbf{R}^n$ is a locally geodesic path with $l_k(g) \le c_{\mathrm{LG}}$, then g is a geodesic path.*

We show in 8.10 that the conjecture holds for $n = 2$. First we show that two short geodesics can be joined together.

8.9. Theorem. *Let $s_1, s_2 \in (0, 1]$, let $g\colon [0, s_1 + s_2] \to G \subset \mathbf{R}^2$ be a path such that the restrictions $g_1 = g|[0, s_1]$ and $g_2 = g|[s_1, s_1 + s_2]$ are geodesic paths and such that g is differentiable at s_1. Then g is a geodesic path.*

Proof. Suppose first that $s_1 < 1$, $s_2 < 1$. Set $a = g(0)$, $z = g(s_1)$, $b = g(s_1 + s_2)$, and let $L = z + g'(s_1)^{\perp}$ be the normal of g at z. By 2.10, the line L is a common tangent of the quasihyperbolic disks $B_k(a, s_1)$ and $B_k(b, s_2)$ at z. Since these disks are strictly convex by 7.7(1), their closures meet only at z. Consequently, the unique quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ contains z, whence $\gamma[a, z] = \operatorname{im} g_1$, $\gamma[z, b] = \operatorname{im} g_2$ and therefore $\gamma = \operatorname{im} g$.

The case where $s_1 = 1$ or $s_2 = 1$ follows by an easy limiting process. □

8.10. Theorem. *The Local geodesic conjecture 8.8 holds for all planar domains with $c_{\mathrm{LG}} = 2$.*

Proof. Let $0 < r \le 2$ and let $g\colon [0, r] \to G \subset \mathbf{R}^2$ be a locally geodesic path. There is a subdivision of $[0, r]$ by points $0 = t_0 < t_1 < \cdots < t_m = r$ such that

$g|[t_{\nu-1}, t_\nu]$ is a geodesic path for each ν and such that $r/2$ is one of the points t_ν. By successive applications of 8.9 we see that $g|[0, r/2]$ and $g|[r/2, r]$ are geodesic paths, and the theorem follows by one further application. □

8.11. Theorem. *The Prolongation conjecture 1.3 holds for all planar domains with* $c_{\mathrm{P}} = 2$.

Proof. This follows from 8.6 and 8.10. □

We finally show that the Prolongation conjecture implies a limiting version.

8.12. Theorem. *Let* $G \subset \mathbf{R}^n$ *be a domain satisfying the Prolongation conjecture 1.3 with a constant* c. *Let* $a \in G$ *and let* $v \in S(1)$. *Then there is a quasihyperbolic geodesic* $\gamma\colon a \curvearrowright b$ *such that* $l_k(\gamma) = c$ *and such that* v *is the unit tangent vector of* γ *at* a.

Proof. Let $c \wedge (1/2) > r_1 > r_2 > \ldots$ be a sequence converging to 0. For each j there is a geodesic path $g_j\colon [0, c] \to G$ with $g_j(0) = a, g_j(r_j) = a + |g_j(r_j)|v$. The geodesics $\operatorname{im} g_j$ lie in the compact set $\bar{B}(a, c)$. Passing to a subsequence we may assume by Ascoli's theorem and by 2.2 that (g_j) converges to a geodesic path $g\colon [0, c] \to G$. By 2.7 we have $\operatorname{ang}(g_j'(0), v) \le 4r_j$. By 2.8 this yields $\operatorname{ang}(g'(0), v) = 0$. □

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