Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 437–446

# NECESSARY CONDITIONS FOR WEIGHTED POINTWISE HARDY INEQUALITIES

**Juha Lehrbäck**

University of Jyväskylä, Department of Mathematics and Statistics
P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland; juhaleh@maths.jyu.fi

**Abstract.** We establish necessary conditions for domains $\Omega \subset \mathbf{R}^n$ which admit the pointwise $(p, \beta)$-Hardy inequality

$$|u(x)| \leq C d_\Omega(x)^{1-\beta/p} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x), \quad u \in C_0^\infty(\Omega),$$

where $1 < q < p$, $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and $M_{R,q}$ is a maximal operator. In particular, the complement of such a domain must have, even locally, Hausdorff dimension strictly greater than $n - p + \beta$.

## 1. Introduction

In this paper, we consider pointwise $(p, \beta)$-Hardy inequalities for functions $u \in C_0^\infty(\Omega)$. That is, for given $1 < p < \infty$ and $\beta \in \mathbf{R}$ we ask for some exponent $1 < q < p$ and a constant $C > 0$ such that the inequality

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \left( \sup_{r < 2d_\Omega(x)} \fint_{B(x,r)} |\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}} \, dy \right)^{1/q} \tag{1}$$

holds at $x \in \Omega$. Here we denote $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. Such weighted inequalities were introduced in [5] following the considerations in the unweighted case $\beta = 0$, conducted by Hajłasz [2] and Kinnunen and Martio [4]. It is easy to see, using the boundedness of maximal operators, that if the pointwise $(p, \beta)$-Hardy inequality (1) holds for a function $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with constants $1 < q < p$ and $C_1 > 0$, then $u$ satisfies the usual (weighted) $(p, \beta)$-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx$$

with a constant $C = C(C_1, n, p, q) > 0$. See [5] and references therein for more results and the origins of these Hardy inequalities.

If $\Omega \subsetneq \mathbf{R}^n$ is a domain and (1) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$, with same constants $1 < q < p$ and $C_\Omega > 0$, we say that $\Omega$ *admits* the pointwise $(p, \beta)$-Hardy inequality. In [5], sufficient conditions for a domain to admit the pointwise $(p, \beta)$-Hardy inequality were given. These were closely related to the (local) Hausdorff dimension of the boundary (or the complement) of $\Omega$. We mention, for example, that each simply connected planar John domain admits the pointwise $(p, \beta)$-Hardy

2000 Mathematics Subject Classification: Primary 46E35, 26D15, 28A78.
Key words: Pointwise Hardy inequality, maximal function, Hausdorff content, Hausdorff dimension, Minkowski dimension.
The author was supported in part by the Academy of Finland.

inequality whenever $1 < p < \infty$ and $\beta < p - 1$, and a von Koch -type snowflake domain $\Omega \subset \mathbf{R}^2$ admits the pointwise $(p, \beta)$-Hardy inequality if (and only if) $1 < p < \infty$ and $\beta < p - 2 + \dim(\partial\Omega)$.

The main purpose of this paper is to show that size estimates of the above type are indeed necessary for weighted pointwise Hardy inequalities; the unweighted case has been considered in [6]. Our main result can be stated as follows:

**Theorem 1.1.** *Suppose that a domain $\Omega \subset \mathbf{R}^n$ admits the pointwise $(p, \beta)$-Hardy inequality. Then there exist a constant $C > 0$ and an exponent $\lambda > n - p + \beta$ such that*

$$\mathscr{H}^\lambda_\infty\big(B(w, r) \cap \Omega^c\big) \geq C r^\lambda \tag{2}$$

*holds for every $w \in \Omega^c$ and all $r > 0$.*

Here $\mathscr{H}^\lambda_\infty(A)$ is the $\lambda$-dimensional Hausdorff content of the set $A \subset \mathbf{R}^n$, see Section 2. It is immediate that if (2) holds for some $w$ and $r$, then the Hausdorff dimension of $B(w, r) \cap \Omega^c$ is at least $\lambda$. Moreover, if $\beta < p - 1$ (so that $p - \beta > 1$) and (2) holds for all $w \in \Omega^c$ and all $r > 0$, then it is well-known that $\Omega^c$ satisfies a uniform capacity density condition: $\Omega^c$ is uniformly $(p - \beta)$-fat (see e.g. [7] or [4] for the definition). Notice also that the fact that we must have $\beta < p$ in pointwise Hardy inequalities is implicit in Theorem 1.1, since (2) can not hold in $\mathbf{R}^n$ for any $\lambda > n$. Hence, for a fixed $1 < p < \infty$, the relevant values of $\beta$ in pointwise Hardy inequalities lie between $p - n$ and $p$, as every proper subdomain $\Omega \subsetneq \mathbf{R}^n$ admits the pointwise $(p, \beta)$-Hardy inequality when $\beta < p - n$ (cf. [5]).

Interestingly, if the pointwise $(p, \beta)$-Hardy inequality holds in a domain $\Omega \subset \mathbf{R}^n$, we not only obtain the conclusion of Theorem 1.1—a uniform density condition for the complement of $\Omega$—but also a stronger density condition where the complement of $\Omega$ is considered only as "seen" from within the points inside the domain. To this end, we let $D(x)$ denote the $x$-component of $B(x, 2d_\Omega(x)) \cap \Omega$ for points $x \in \Omega$. Then, if $\Omega$ admits the pointwise $(p, \beta)$-Hardy inequality, there exists some $\lambda > n - p + \beta$ and a constant $C > 0$ such that

$$\mathscr{H}^\lambda_\infty\big(\partial D(x) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \tag{3}$$

for every $x \in \Omega$; see Theorem 3.1. This complements the results in [5] on sufficient conditions for pointwise Hardy inequalities. We refer to estimates of the type (3) as inner boundary density conditions.

In order to obtain the above results in the case $\beta < 0$ we need a measure theoretic result which is given in Lemma 4.1, and could also be of independent interest. The claim is, roughly, that a uniform Minkowski-type density for some $\lambda_0 > 0$ implies uniform $\lambda$-Hausdorff content densities for every $\lambda < \lambda_0$. See Section 2 for definitions and Lemma 4.1 for the precise statement.

This paper is organized as follows. In Section 2, we introduce the notation and terminology used in the rest of the paper and also record some preliminary results. Then, in Section 3, we prove our main results, and in fact give some more quantitative formulations of the necessary conditions for pointwise Hardy inequalities. Finally, Section 4 is devoted to the statement and the proof of Lemma 4.1 mentioned above.

## 2. Preliminaries

Our notation is pretty standard. The open ball with center $x \in \mathbf{R}^n$ and radius $r > 0$ is denoted $B(x, r)$, and the corresponding closed ball is $\overline{B}(x, r)$. If $B = B(x, r)$ is a ball and $L > 0$, we denote $LB = B(x, Lr)$. When $A \subset \mathbf{R}^n$, $|A|$ is the $n$-dimensional Lebesgue measure of $A$, $\partial A$ is the boundary of $A$, and the complement of $A$ is $A^c = \mathbf{R}^n \setminus A$. If $0 < |A| < \infty$ and $f \in L^1(A)$, we denote $\fint_A f\,dx = \frac{1}{|A|}\int_A f\,dx$. Also, $\chi_A\colon \mathbf{R}^n \to \{0, 1\}$ is the characteristic function of $A$. The Euclidean distance between two points, or a point and a set, is denoted $d(\cdot, \cdot)$. When $\Omega \subsetneq \mathbf{R}^n$ is a domain, i.e. an open and connected set, and $x \in \Omega$, we also use notation $d_\Omega(x) = d(x, \partial\Omega)$. In the rest of the paper we always assume that $\Omega \subsetneq \mathbf{R}^n$, so that $\partial\Omega \neq \emptyset$. The support of a function $u\colon \Omega \to \mathbf{R}$, $\operatorname{spt}(u)$, is the closure of the set where $u$ is non-zero. We let $C > 0$ denote various positive constants which may vary from expression to expression.

The $\lambda$-Hausdorff content of a set $A \subset \mathbf{R}^n$ is defined by

$$\mathscr{H}^\lambda_\infty(A) = \inf\left\{ \sum_{i=1}^{\infty} r_i^\lambda : A \subset \bigcup_{i=1}^{\infty} B(z_i, r_i),\ z_i \in A \right\},$$

and the Hausdorff dimension of $A \subset \mathbf{R}^n$ is then

$$\dim_{\mathscr{H}}(A) = \inf\left\{ \lambda > 0 : \mathscr{H}^\lambda_\infty(A) = 0 \right\}.$$

As it turns out, we need a similar notion for the case where all the covering balls are required to be of the same radius. Notice that in the following our terminology differs a bit from the standard one. When $A \subset \mathbf{R}^n$ and $r > 0$, we denote

$$\mathscr{M}^\lambda_r(A) = \inf\left\{ N r^\lambda : E \subset \bigcup_{i=1}^{N} B(z_i, r),\ z_i \in A \right\}.$$

Using this notation, we define, in analog with the $\lambda$-Hausdorff content, the $\lambda$-Minkowski content of $A \subset \mathbf{R}^n$ by

$$\mathscr{M}^\lambda_\infty(A) = \inf_{r>0} \mathscr{M}^\lambda_r(A).$$

The corresponding dimension, the usual lower Minkowski dimension, is given by

$$\underline{\dim}_{\mathscr{M}}(A) = \inf\left\{ \lambda > 0 : \mathscr{M}^\lambda_\infty(A) = 0 \right\}.$$

For the record, we recall that the upper Minkowski dimension of $A \subset \mathbf{R}^n$ is

$$\overline{\dim}_{\mathscr{M}}(A) = \inf\left\{ \lambda > 0 : \limsup_{r\to 0} \mathscr{M}^\lambda_r(A) = 0 \right\}.$$

Note that always $\dim_{\mathscr{H}}(A) \le \underline{\dim}_{\mathscr{M}}(A) \le \overline{\dim}_{\mathscr{M}}(A)$, and that both of these inequalities can be strict; cf. [8, Ch. 5].

Let us extend the notation $D(x)$, used in the Introduction, in the following way: When $x \in \Omega$ and $L \ge 1$, we let $D_L(x)$ denote the $x$-component of the open set $B(x, L d_\Omega(x)) \cap \Omega$; thus always $B(x, d_\Omega(x)) \subset D_L(x) \subset B(x, L d_\Omega(x))$. This notation is used e.g. in the following lemma, which is similar to a part of the main result from [6]. Here the result is reformulated for Minkowski contents instead of Hausdorff contents. The proof, which we omit here, is however almost identical to the proof in [6] up to the obvious modifications.

**Lemma 2.1.** *Let $\Omega \subset \mathbf{R}^n$ be a domain and assume that there exists a constant $C_0 > 0$ such that, for some $L > 1$ and some $0 \le \lambda \le n$,*

$$\mathscr{M}^\lambda_\infty\big(\partial D_L(x) \cap \partial\Omega\big) \ge C_0 d_\Omega(x)^\lambda$$

*for every $x \in \Omega$. Then there exists a constant $C = C(C_0, L, n, \lambda) > 0$ such that*

$$\mathscr{M}^\lambda_\infty\big(B(w,r) \cap \Omega^c\big) \ge C r^\lambda$$

*for every $w \in \Omega^c$ and all $r > 0$.*

Actually, by the assumption $\lambda \le n$, we may choose the constant $C$ in Lemma 2.1 to be independent of $\lambda$.

To simplify the notation of pointwise Hardy inequalities we recall the definitions of maximal functions. The classical restricted Hardy–Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is defined by

$$M_R f(x) = \sup_{0<r<R} \fint_{B(x,r)} |f(y)|\, dy\,,$$

where $0 < R \le \infty$ may depend on $x$. The well-known maximal theorem of Hardy, Littlewood and Wiener (see e.g. [9]) states that if $1 < p < \infty$, we have $||M_R f||_p \le C(n,p)||f||_p$ for all $0 < R \le \infty$.

When $1 < q < \infty$, we define $M_{R,q} f = \big(M_R |f|^q\big)^{1/q}$. With the help of maximal functions the pointwise $(p,\beta)$-Hardy inequality (1), for a function $u \in C_0^\infty(\Omega)$, now reads

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x), \tag{4}$$

where $1 < q < p$.

Let us now begin the considerations on necessary conditions for these pointwise Hardy inequalities. The next lemma records the fact that (4) makes sense only if $\beta < p$.

**Lemma 2.2.** *Let $1 < p < \infty$ and let $x_0 \in \Omega$. If $\beta \in \mathbf{R}$ is such that the pointwise $(p,\beta)$-Hardy inequality (4) holds at $x_0$ with constants $1 < q < p$ and $C_0 > 0$ for all $u \in C_0^\infty(\Omega)$, then $\beta < p$.*

*Proof.* To prove the lemma, it is enough to show that the pointwise $(p,p)$-Hardy inequality fails at $x_0 \in \Omega$, since then, by [5], the pointwise $(p,\beta)$-Hardy inequality can not hold for any $\beta \ge p$.

Pointwise Hardy inequalities are local, so we may assume that $\Omega$ is a bounded domain; if this is not the case, we may instead consider $\Omega \cap B(0,R)$ for some $R > 0$ large enough. Denote $A_j = \big\{x \in \Omega : 2^{-j} \le d_\Omega(x) < 2^{-j+1}\big\}$ for $j \in \mathbf{N}$, and define

$$u_j(x) = \min\big\{1, 2^j \max\{0, d_\Omega(x) - 2^{-j}\}\big\},$$

so that $u_j$ is a Lipschitz function with a compact support in $\Omega$, and, moreover, $|\nabla u_j(x)| \le 2^j$ for a.e. $x \in A_j$, and elsewhere $|\nabla u_j(x)| = 0$. Also, for $j$ large enough, $u_j(x_0) = 1$. Since $\Omega$ is bounded, it is clear that $|A_j| \to 0$ as $j \to \infty$. Thus the right-hand side of the pointwise $(p,p)$-Hardy inequality for $u_j$ at $x_0$, with $j \in \mathbf{N}$ so large that $2^{-j} < d_\Omega(x_0)/2$, could be estimated as follows:

$$\begin{aligned} &d_\Omega(x)^{1-\frac{p}{p}}\left(\sup_{r<2d_\Omega(x)} \fint_{B(x,r)} |\nabla u_j(y)|^q d_\Omega(y)^{p\frac{q}{p}}\,dy\right)^{1/q} \\ &\le C d_\Omega(x_0)^{-n/q}\left(\int_{A_j} |\nabla u_j(y)|^q d_\Omega(y)^q\,dy\right)^{1/q} \\ &\le C d_\Omega(x_0)^{-n/q}\big(|A_j|\,2^{jq}\,2^{-jq}\big)^{1/q} \le C d_\Omega(x_0)^{-n/q}|A_j|^{1/q} \xrightarrow{j\to\infty} 0. \end{aligned}$$

But $u_j(x_0) = 1$ for large $j$, so the pointwise $(p,p)$-Hardy inequality fails to hold for the functions $u_j$ with a uniform constant. Using standard approximation, and the fact that functions $u_j$ are constant in a neighborhood of $x_0$, it is now easy to find smooth test functions for which the pointwise $(p,p)$-Hardy inequality does not hold with a uniform constant either. □

On the other hand, if $n \ge 2$, $1 < p < \infty$ and $\beta < p$ are given, there exists, by the results in [5], a domain $\Omega \subset \mathbf{R}^n$ which admits the pointwise $(p,\beta)$-Hardy inequality. Hence the conclusion of Lemma 2.2 is in this sense the best possible. For instance, in the plane one can choose such a domain $\Omega$ to be a snowflake-type domain with $\dim_{\mathscr{H}}(\partial\Omega) > 2 - p + \beta$.

## 3. Main results

In this section we give the precise formulations and proofs of our main results. The key point here is that the pointwise $(p,\beta)$-Hardy inequality in $\Omega \subset \mathbf{R}^n$ implies that $\partial\Omega$ satisfies an inner density condition for some exponent $\lambda > n - p + \beta$ (Theorem 3.1). The density of the complement of $\Omega$ (Theorem 1.1) is then obtained as a consequence of the boundary density, as explained at the end of this section.

**Theorem 3.1.** *Suppose that a domain $\Omega \subset \mathbf{R}^n$ admits the pointwise $(p,\beta)$-Hardy inequality. Then there exist an exponent $\lambda > n-p+\beta$ and a constant $C > 0$, both depending only on $n$ and the data associated with the pointwise $(p,\beta)$-Hardy inequality, such that*

$$\mathscr{H}^\lambda_\infty\big(\partial D_2(x) \cap \partial\Omega\big) \ge C d_\Omega(x)^\lambda \tag{5}$$

*for every $x \in \Omega$. In particular,*

$$\dim_{\mathscr{H}}(\partial D_2(x) \cap \partial\Omega) > n - p + \beta$$

*for every $x \in \Omega$.*

The proof of Theorem 3.1 is somewhat different depending whether $\beta \ge 0$ or $\beta < 0$. In the the former case the theorem follows from the next quantitative lemma. The estimate (6) below is an improvement on the results in [6] even in the unweighted case.

**Lemma 3.2.** *Let $1 < p < \infty$ and $\beta \ge 0$, and let $x_0 \in \Omega$. Suppose that the pointwise $(p,\beta)$-Hardy inequality (4) holds at $x_0$ for all $u \in C_0^\infty(\Omega)$ with constants $1 < q < p$ and $C_0 > 0$. Then there exists a constant $C = C(C_0,n,p,\beta) > 0$ such that*

$$\mathscr{H}^\lambda_\infty\big(\partial D_3(x_0) \cap \partial\Omega\big) \ge C d_\Omega(x_0)^\lambda, \tag{6}$$

*where $\lambda = n - q + \frac{q}{p}\beta > n - p + \beta$.*

*Proof.* By Lemma 2.2 we must have $\beta < p$. Using this fact it is easy to see that $p - \beta > q - \frac{q}{p}\beta$, and thus $\lambda = n - q + \frac{q}{p}\beta > n - p + \beta$.

Then let $x_0 \in \Omega$ be as in the assumptions of the lemma. Denote $E = \partial D_3(x_0) \cap \partial\Omega$, $R_0 = d_\Omega(x_0)$, and let $\{B_i\}_{i=1}^N$, where $B_i = B(w_i, r_i)$ with $w_i \in E$ and $r_i > 0$, be a covering of $E$; we may assume that the covering is finite by the compactness of $E$.

It is now enough to show that there exists a constant $C > 0$, independent of the particular covering, such that

$$\sum_{i=1}^N r_i{}^\lambda \geq C R_0{}^\lambda. \tag{7}$$

But if $r_i \geq R_0/4$ for some $1 \leq i \leq N$, then (7) holds e.g. with the constant $C = 4^{-n}$, and the claim follows.

We may hence assume that $r_i < R_0/4$ for all $1 \leq i \leq N$. Now, let us define a function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ by

$$\varphi(x) = \min_{1 \leq i \leq N} \bigl\{1,\, r_i^{-1} d(x, 2B_i)\bigr\}$$

and let $\psi \in C_0^\infty(B(x_0, 3R_0))$ be such that $0 \leq \psi \leq 1$ and $\psi(x) = 1$ for all $x \in B(x_0, 2R_0)$. Then $u = \psi\,\varphi\chi_{D_3(x_0)}$ is a Lipschitz function with a compact support in $\Omega$. Since $r_i < R_0/4$ for all $1 \leq i \leq N$, we have that

$$d(x_0, 3B_i) > R_0/4 \tag{8}$$

for all $1 \leq i \leq N$, and thus $u(x_0) = 1$ by the definition of $u$.

Using standard approximation we can find smooth test functions $v_j \in C_0^\infty(\Omega)$ such that $v_j(x_0) = u(x_0) = 1$ for all $j \in \mathbf{N}$ and, by the facts that the Lipschitz function $u$ is constant in $B(x_0, R_0/4)$ and has a compact support,

$$\limsup_{j\to\infty} M_{2d_\Omega(x),q}\bigl(|\nabla v_j| d_\Omega{}^{\beta/p}\bigr)(x_0) \leq 2M_{2d_\Omega(x),q}\bigl(|\nabla u| d_\Omega{}^{\beta/p}\bigr)(x_0).$$

It follows that the pointwise $(p,\beta)$-Hardy inequality (4) also holds for $u$ at $x_0$, with a constant depending only on $C_0$.

We shall now show, with the help of (4) for $u$, that the estimate (7) holds. First, denote $A_i = 3\overline{B}_i \setminus 2B_i$. Then

$$\operatorname{spt}(|\nabla u|) \cap B(x_0, 2R_0) \subset \bigcup_{i=1}^N A_i$$

and, in fact,

$$|\nabla u(y)|^q \leq \sum_{i=1}^N r_i^{-q} \chi_{A_i}(y) \tag{9}$$

for a.e. $y \in B(x_0, 2R_0)$. But if $\chi_{A_i}(y) \neq 0$ for some $1 \leq i \leq N$, we must have that $d_\Omega(y) \leq 3r_i$, and hence, by the assumption $\beta \geq 0$, we obtain from (9) that

$$|\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}} \leq C \sum_{i=1}^N r_i^{-q+\beta\frac{q}{p}} \chi_{A_i}(y) \tag{10}$$

for a.e. $y \in B(x_0, 2R_0)$ with $C = 3^\beta$.

Then observe that since $\operatorname{spt}(|\nabla u|) \cap B(x_0, 2R_0) \subset \bigcup_{i=1}^N 3\overline{B}_i$, it follows from (8) that we must have $r > \frac{1}{4}R_0$ in order to obtain something positive when estimating the maximal function of $|\nabla u|^q {d_\Omega}^{\beta\frac{q}{p}}$ at $x_0$. Thus the pointwise $(p,\beta)$-Hardy inequality and (10) imply that (recall $\lambda = n - q + \beta\frac{q}{p}$)

$$
\begin{aligned}
1 = |u(x_0)|^q &\le {C_0}^q {R_0}^{q-\frac{q}{p}\beta} M_{2R_0}\big(|\nabla u|^q {d_\Omega}^{\beta\frac{q}{p}}\big)(x_0) \\
&\le C{R_0}^{q-\frac{q}{p}\beta} \sup_{\frac{1}{4}R_0 \le r \le 2R_0} \left( r^{-n} \int_{B(x_0,r)} |\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}}\, dy \right) \\
&\le C{R_0}^{q-\frac{q}{p}\beta-n} \int_{B(x_0,2R_0)} |\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}}\, dy \qquad (11) \\
&\le C{R_0}^{-\lambda} \sum_{i=1}^N |A_i|\, {r_i}^{-q+\beta\frac{q}{p}} \;\le\; C{R_0}^{-\lambda} \sum_{i=1}^N {r_i}^{\lambda}.
\end{aligned}
$$

Since $q < p$, it is easy to see that we may choose the constant $C$ in (11) so that $C = C(C_0, n, p, \beta) > 0$. This proves that the estimate (6) holds at $x_0$ with the exponent $\lambda$. □

**Remark.** It is obvious from the proof that we may replace $D_3(x_0)$ in the lemma by any $D_L(x_0)$, where $L > 2$.

In the case $\beta < 0$ we obtain first, as in Lemma 3.2, the following *a priori* weaker result for Minkowski contents. To prove Theorem 3.1, we then use Lemma 4.1, which is postponed until the next section, to pass from Minkowski contents to Hausdorff contents.

**Lemma 3.3.** *Let $1 < p < \infty$ and $\beta < 0$, and let $x_0 \in \Omega$. Suppose that the pointwise $(p,\beta)$-Hardy inequality (4) holds at $x_0$ for all $u \in C_0^\infty(\Omega)$ with constants $1 < q < p$ and $C_0 > 0$. Then there exists a constant $C = C(C_0, n, p, \beta) > 0$ such that*

$$\mathscr{M}^\lambda_\infty\big(\partial D_3(x_0) \cap \partial\Omega\big) \ge C d_\Omega(x_0)^\lambda$$

*where $\lambda = n - q + \frac{q}{p}\beta > n - p + \beta$.*

*Proof.* First, it is now obvious that $\lambda > n - p + \beta$ since $\beta < 0$. We proceed as in the proof of Lemma 3.2, but now we cover the set $E = \partial D_3(x_0) \cap \partial\Omega$ by balls $B_i = B(w_i, r)$, all of the same radius $r > 0$ and with center points $w_i \in E$ for $1 \le i \le N$. We may again assume that $r < d_\Omega(x_0)/4$. After defining the function $u$ as in the proof of Lemma 3.2 we obtain that

$$|\nabla u(y)|^q \le \sum_{i=1}^N r^{-q} \chi_{A_i}(y)$$

for a.e. $y \in B(x_0, 2d_\Omega(x))$. But now, if $|\nabla u(y)| \ne 0$, we have by the definition of $u$ that $d_\Omega(y) \ge r$. Since $\beta < 0$, it follows that

$$|\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}} \le C \sum_{i=1}^N r^{-q+\beta\frac{q}{p}} \chi_{A_i}(y) \tag{12}$$

for a.e. $y \in B(x_0, 2d_\Omega(x))$; recall that $A_i = 3\overline{B}_i \setminus 2B_i$. Hence, using the pointwise $(p,\beta)$-Hardy inequality and (12) just as in the proof of Lemma 3.2, we conclude that

$$1 = |u(x_0)|^q \le {C_0}^q d_\Omega(x_0)^{q-\frac{q}{p}\beta} M_{2d_\Omega(x_0)}\big(|\nabla u|^q {d_\Omega}^{\beta\frac{q}{p}}\big)(x_0) \le \cdots \le C d_\Omega(x_0)^{-\lambda} \sum_{i=1}^{N} r^\lambda,$$

where $C = C(C_0, n, p, \beta) > 0$ is independent of $r > 0$ and the particular covering. This yields the desired Minkowski content estimate. □

*Proof of Theorem 3.1.* Let us first remark that if

$$\mathscr{H}^\lambda_\infty\big(\partial D_L(x) \cap \partial\Omega\big) \ge C_0 d_\Omega(x)^\lambda, \tag{13}$$

where $L > 1$, holds for every $x \in \Omega$, and if $L' > 1$, then (13), but with $L$ replaced by $L'$, holds for every $x \in \Omega$ as well, with a constant $C = C(C_0, L, L') > 0$. This is trivial if $L' \ge L$. On the other hand, if $L' < L$ and $x_0 \in \Omega$, take $w \in \partial\Omega$ such that $d(x_0, w) = d_\Omega(x_0)$, and choose $x = x_0 + \frac{L-L'}{L-1}(w - x_0)$. Then $D_L(x) \subset D_{L'}(x_0)$, and the claim follows with simple calculations.

In particular, if $\beta \ge 0$, and $\Omega \subset \mathbf{R}^n$ admits the pointwise $(p, \beta)$-Hardy inequality, it follows from Lemma 3.2 that there exists an exponent $\lambda > n - p + \beta$ such that (13), with $L = 3$, holds for every $x \in \Omega$. Hence also the estimate (5) (i.e. (13) with $L = 2$) holds for every $x \in \Omega$ with this same exponent $\lambda$ and a constant depending only on $n$ and the given data.

In the case $\beta < 0$, the pointwise $(p, \beta)$-Hardy inequality implies, by Lemma 3.3 and Lemma 2.1, that there exists $C_1 = C_1(C_0, n, p, \beta) > 0$ such that

$$\mathscr{M}^{\lambda_0}_\infty\big(B(w, r) \cap \Omega^c\big) \ge C_1 r^{\lambda_0} \tag{14}$$

for every $w \in \Omega^c$ and $r > 0$, where $\lambda_0 = n - q + \frac{q}{p}\beta$. Now choose

$$\varepsilon = \lambda_0 - (n - p + \beta) = p - q - \beta + \beta\tfrac{q}{p} > 0$$

and take $\lambda$ satisfying $\lambda_0 - \varepsilon/2 < \lambda < \lambda_0$. Using Lemma 4.1 we obtain from (14) that

$$\mathscr{H}^\lambda_\infty\big(B(w, r) \cap \Omega^c\big) \ge C_2\, r^\lambda \tag{15}$$

for every $w \in \Omega^c$ and all $r > 0$, where $C_2 = C_2(C_1, n, p, q, \beta) > 0$. In particular, it follows from (15) that $\Omega^c$ is uniformly $(p-\beta-\varepsilon/2)$-fat (cf. for example [6] and notice that $p - \beta - \varepsilon/2 > 1$ by the choice of $\varepsilon$). Hence, by the results in [2], $\Omega$ admits the pointwise $(p - \beta - \varepsilon/2, 0)$-Hardy inequality. But now we are back in the case $\beta \ge 0$, and by the first part of the proof we conclude that the inner boundary density (5) holds for every $x \in \Omega$ with the exponent $\lambda = n - p + \beta + \varepsilon/2$ and a constant $C > 0$, both depending only on $n$ and the associated data. □

Regarding Theorem 1.1, the case $\beta < 0$ was already proved as a part of the proof of Theorem 3.1, see equation (15). For $\beta \ge 0$, Theorem 1.1 follows from Theorem 3.1 and the fact that Lemma 2.1 also holds when $\mathscr{M}^\lambda_\infty$ is replaced by $\mathscr{H}^\lambda_\infty$; this is in fact the original result from [6].

## 4. From Minkowski to Hausdorff

Here we explain how to obtain uniform density conditions for Hausdorff contents if one already has such a condition for some $\lambda_0$-Minkowski content. For our purposes it is sufficient to acquire Hausdorff content estimates for all exponents $\lambda < \lambda_0$, as is the case in the next lemma. Nevertheless, it would be interesting to know if it is possible to extend this result also to include the end-point exponent $\lambda_0$. We remark that a

$\lambda$-Hausdorff content density condition trivially implies a similar density condition for the $\lambda$-Minkowski content.

**Lemma 4.1.** *Let $E \subset \mathbf{R}^n$ be a closed set. Assume that there exist $0 < \lambda_0 \le n$ and $C_0 > 0$ such that*

$$\mathscr{M}_\infty^{\lambda_0}\big(B(w,r) \cap E\big) \ge C_0\, r^{\lambda_0} \tag{16}$$

*for every $w \in E$ and all $r > 0$. Then, for every $0 < \lambda < \lambda_0$, there exists a constant $C = C(C_0, \lambda_0, \lambda, n) > 0$ such that*

$$\mathscr{H}_\infty^{\lambda}\big(B(w,r) \cap E\big) \ge C\, r^{\lambda} \tag{17}$$

*for every $w \in E$ and all $r > 0$.*

*Proof.* The essential idea of the proof is similar to the proof of [3, Thm. 4.1]. Namely, we construct, using (16) repeatedly, a Cantor-type subset which is then shown to satisfy the $\lambda$-Hausdorff density condition (17).

To begin with, we fix $0 < \lambda < \lambda_0$ and then choose $K \in \mathbf{N}$ so large that

$$\lambda < \frac{\lambda_0 \log K}{\log K - \log C_0 + \log 10^{\lambda_0}}$$

(notice that we may assume $C_0 < 1$). We also denote $m = 10(K/C_0)^{1/\lambda_0}$, so that $\lambda < \log K/\log m < \lambda_0$. Now let $w \in E$ and $R > 0$, and take $B_0 = \overline{B}(w,R)$. It suffices to show that (17) holds for this closed ball with a constant independent of $w$ and $R$, since then the claim follows easily for all open balls as well. Using the standard $5r$-covering theorem (cf. [9, pp. 9–10]) and the assumption (16), we find closed balls $B_i = \overline{B}(z_i, r_1)$, $i = 1, 2, \dots, n_0$, with $z_i \in E \cap B_0$ and $r_1 = Rm^{-1}$, such that the balls $2B_i$ are pairwise disjoint, $E \cap B_0 \subset \bigcup_i 10B_i$, and, by (16),

$$n_0(10r_1)^{\lambda_0} \ge C_0 R^{\lambda_0}.$$

By the choices of $r_1 = Rm^{-1}$ and $K$ we see that $n_0 \ge K$. We then proceed with the balls $B_i$ for $i = 1, \dots, K$.

In the next step we find balls $B_{i_1 i_2} = \overline{B}(z_{i_1 i_2}, r_2)$, $i_1 = 1, 2, \dots, K$ and $i_2 = 1, 2, \dots, n_{i_1}$, where $z_{i_1 i_2} \in E \cap B_{i_1}$ and $r_2 = Rm^{-2}$, such that, for each $i_1 = 1, 2, \dots, K$, the balls $2B_{i_1 i_2}$ are pairwise disjoint, $E \cap B_{i_1} \subset \bigcup_{i_2} 10B_{i_1 i_2}$, and, by (16),

$$n_{i_1}(10r_2)^{\lambda_0} \ge C_0 {r_1}^{\lambda_0}.$$

Again, $n_{i_1} \ge K$ for every $i_1 = 1, \dots, K$, and we continue with the balls $B_{i_1 i_2} = \overline{B}(z_{i_1 i_2}, r_2)$, where now $i_1, i_2 = 1, 2, \dots, K$. Notice that since $2B_{i_1} \cap 2B_{j_1} = \emptyset$ whenever $i_1 \ne j_1$, and clearly $2B_{i_1 i_2} \subset 2B_{i_1}$ for every $i_1, i_2 = 1, \dots, K$, we have in fact that all the balls $2B_{i_1 i_2}$, $i_1, i_2 = 1, \dots, K$, are pairwise disjoint.

Continuing in this way recursively, we find in the $k$:th step of the construction a collection of closed balls $B_{i_1 i_2 \dots i_k}$, where $i_j = 1, \dots, K$ for $j \in \{1, \dots, k-1\}$ and $i_k = 1, \dots, n_{i_1 i_2 \dots i_{k-1}}$, with center points $z_{i_1 i_2 \dots i_k} \in E \cap B_{i_1 i_2 \dots i_{k-1}}$ and all of radius $r_k = Rm^{-k}$, satisfying the following properties: The balls $2B_{i_1 i_2 \dots i_k}$ are pairwise disjoint,

$$E \cap B_{i_1 i_2 \dots i_{k-1}} \subset \bigcup_{i_k} 10B_{i_1 i_2 \dots i_k},$$

and, by (16),

$$n_{i_1 i_2 \dots i_{k-1}}(10r_k)^{\lambda_0} \ge C_0 {r_{k-1}}^{\lambda_0}.$$

Since $(r_{k-1}/r_k)^{\lambda_0} = m^{\lambda_0}$, we have, like before, that $n_{i_1 i_2 \dots i_{k-1}} \geq K$ for all $i_1 i_2 \dots i_{k-1}$, where $i_j = 1, \dots, K$ for $j \in \{1, \dots, k-1\}$. We continue with the balls $B_{i_1 i_2 \dots i_k}$, where now $i_1, i_2, \dots, i_k = 1, 2, \dots, K$.

We then define

$$\tilde{E} = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\dots,i_k=1}^{K} B_{i_1 i_2 \dots i_k},$$

so that $\tilde{E} \subset E \cap B_0$ is a compact Cantor-type set. Proceeding as in the proof of Theorem 4.1. in [3] we let $\mu$ denote the equally distributed probability measure on $\tilde{E}$ (see also [1, pp. 13–14]). In particular, $\mu(E \cap B_{i_1 i_2 \dots i_k}) = K^{-k}$ for every $i_1, i_2, \dots, i_k = 1, 2, \dots, K$. Now, if $x \in \mathbf{R}^n$ and $r < R$, we choose $k \in \mathbf{N}$ such that $Rm^{-k} \leq r < Rm^{-k+1}$. Then there exists a constant $C_1 = C_1(n, m) > 0$ such that $B(x, r)$ intersects at most $C_1$ of the balls $B_{i_1 i_2 \dots i_k}$ from the $k$:th step of the construction. Thus, by the definition of $\mu$ and the choice of $k$,

$$\mu(B(x,r)) \leq C_1 K^{-k} \leq C_1 m^{-k \log K/\log m} \leq C_1 (r/R)^{\lambda}, \tag{18}$$

where we have used the fact $\lambda < \log K/\log m$.

Finally, let $\{B(z_i, r_i)\}_i$ be a covering of $E \cap \overline{B}(w, R)$. We may clearly assume that $r_i < R$ for each $i$. Hence, using the properties of the measure $\mu$, especially (18), we conclude that

$$1 = \mu\big(\overline{B}(w,R) \cap E\big) \leq \sum_i \mu(B(z_i, r_i)) \leq \sum_i C_1 \left(\frac{r_i}{R}\right)^{\lambda}.$$

It is then clear that

$$\mathscr{H}^{\lambda}_{\infty}\big(\overline{B}(w,R) \cap E\big) \geq C R^{\lambda},$$

where $C = C_1^{-1} > 0$ now depends only on $C_0$, $\lambda_0$, $\lambda$, and $n$. □

*Acknowledgements.* The author wishes to express his gratitude to Pekka Koskela for helpful discussions and suggestions concerning the contents of this paper, and for reading the manuscript. Thanks are also due to Juha Kinnunen for valuable remarks which led to some improvements in our results.

## References

[1] Falconer, K.: Fractal geometry. Mathematical foundations and applications. - John Wiley & Sons, Ltd., Chichester, 1990.

[2] Hajłasz, P.: Pointwise Hardy inequalities, - Proc. Amer. Math. Soc. 127:2, 1999, 417–423.

[3] Järvi, P., and M. Vuorinen: Uniformly perfect sets and quasiregular mappings. - J. London Math. Soc. (2) 54:3, 1996, 515–529.

[4] Kinnunen, J., and O. Martio: Hardy's inequalities for Sobolev functions. - Math. Res. Lett. 4:4, 1997, 489–500.

[5] Koskela, P., and J. Lehrbäck: Weighted pointwise Hardy inequalities. - J. London Math. Soc. 79:3, 2009, 757–779.

[6] Lehrbäck, J.: Pointwise Hardy inequalities and uniformly fat sets. - Proc. Amer. Math. Soc. 136:6, 2008, 2193–2200.

[7] Lewis, J. L.: Uniformly fat sets. - Trans. Amer. Math. Soc. 308:1, 1988, 177–196.

[8] Mattila, P.: Geometry of sets and measures in euclidean spaces. - Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[9] Stein, E. M.: Singular integrals and differentiability properties of functions. - Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, N.J., 1970.

Received 27 May 2008