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HURWITZ' THEOREM AND A GENERALIZATION FOR HOLOMORPHIC MAPS OF CLOSED RIEMANN SURFACES

Masaharu Tanabe

Tokyo Institute of Technology, Department of Mathematics
Ohokayama, Meguro, Tokyo, 152-8551, Japan; tanabe@math.titech.ac.jp

Abstract. Let X be a closed Riemann surface of genus greater than one. Hurwitz showed that an automorphism of X is completely determined by the induced automorphism on $H_1(X,\mathbf{Z})$. We study this theorem in the context of $H^1(X,\mathbf{Z})$ and we prove the following as a generalization. Let $\widetilde{X}$, X_1, X_2 be closed Riemann surfaces of genera greater than one and let $f_i\colon \widetilde{X} \to X_i\,(i=1,\,2)$ be non-constant holomorphic maps. Assume that there exist $a_i,\,b_i \in H^1(X_i,\mathbf{Z})\,(i=1,\,2)$ so that $\iint_{X_i} a_i \wedge b_i = 1\,(i=1,\,2)$ and that $f_1^*\,a_1 = f_2^*\,a_2$ and $f_1^*\,b_1 = f_2^*\,b_2$ in $H^1(\widetilde{X},\mathbf{Z})$. Then there exists a conformal map $h\colon X_1 \to X_2$ which satisfies $f_2 = h\circ f_1$.

1. Introduction

Hurwitz [4] showed that if an automorphism of a closed Riemann surface of genus greater than one induces the identity on the first homology group then the automorphism is the identity. Martens [6] observed this theorem in the context of Jacobian varieites and generalized it for holomorphic maps of closed Riemann surfaces.

Theorem 1. (Martens) *Let $\widetilde{X}$, X_1, X_2 be closed Riemann surfaces of genera ≥ 1 and let $f_i\colon \widetilde{X} \to X_i$ $(i=1,2)$ be non-constant holomorphic maps. Assume that there exists a homomorphism H of the first homology groups from $H_1(X_1,\mathbf{Z})$ onto $H_1(X_2,\mathbf{Z})$ which commutes with the induced homomorpisms $f_{i*}\colon H_1(\widetilde{X},\mathbf{Z}) \to H_1(X_i,\mathbf{Z})$ $(i=1,2)$, i.e. $f_{2*} = H\circ f_{1*}$. Then there exists a unique (modulo a translation in genus 1) holomorphic map $h\colon X_1 \to X_2$ with $f_2 = h\circ f_1$.*

Other generalizations of Hurwitz' theorem are due to Accola [1] and Gilman [3]. They studied automorphisms of Riemann surfaces and proved several theorems concerning rigidity of automorphisms in terms of homology groups. One of their results interesting is the following which firstly proved by Accola and later Gilman proved a theorem which includes it as a corollary.

Theorem 2. (Accola) *Let X be a closed Riemann surface of genus greater than one. Let T be an automorphism of X. Suppose that there exist four independent cycles χ_1, χ_2, χ_3, χ_4 so that $\chi_1\cdot\chi_3 = 1$, $\chi_2\cdot\chi_4 = 1$, otherwise $\chi_i\cdot\chi_j = 0$ and that $T(\chi_i) = \chi_i$ for $i=1,2,3,4$. Then T is the identity.*

Martens [7] proposed some problems in the theory of closed Riemann surfaces and one topic was about their results. He wrote that it would be interesting to try and interpret their results in the context of Jacobian varieties. In this paper, we will

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generalize Theorem 1 and 2 for holomorphic maps of closed Riemann surfaces. We will interpret these theorems in terms of $H^1(X,\mathbf{Z})$ (the dual space for $H_1(X,\mathbf{Z})$, that is to say we will study in the context of dual Jacobian varieties rather than Jacobian varieties). We will show

Theorem 3. *Let $\widetilde{X}$, X_1, X_2 be closed Riemann surfaces of genera greater than one and let $f_i\colon \widetilde{X} \to X_i\ (i=1,\,2)$ be non-constant holomorphic maps. Assume that there exist $a_i,\, b_i \in H^1(X_i,\mathbf{Z})$ $(i=1,2)$ so that $\iint_{X_i} a_i \wedge b_i = 1$ $(i=1,2)$ and that $f_1^* a_1 = f_2^* a_2$ and $f_1^* b_1 = f_2^* b_2$ in $H^1(\widetilde{X},\mathbf{Z})$. Then there exists a conformal map $h\colon X_1 \to X_2$ which satisfies $f_2 = h \circ f_1$.*

The method of the proof is to construct Riemann surfaces which reflect the properties of given two cohomology classes.

2. Preliminaries

In the following, all of the Riemann surfaces are closed and of gerera greater than one. Let X be a Riemann surface of genus g. Any basis for $H_1(X,\mathbf{Z})$ (say $\{\chi_1,\dots,\chi_{2g}\}$), with intersection matrix (that is a matrix whose (k,j)-entry is given by the intersection number $\chi_k \cdot \chi_j$) $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, will be called a canonical homology basis, where E is the $g \times g$ identity matrix. For a canonical homology basis $\{\chi_1,\dots,\chi_{2g}\}$, there is a unique dual basis $\{a_1,\dots,a_{2g}\}$ for $H^1(X,\mathbf{Z})$, namely

$$\langle a_k, \chi_j \rangle = \int_{\chi_j} a_k = \delta_{jk} \quad (j,k = 1,\dots,2g).$$

Furthermore, the matrix whose (k,j)-entry is given by $\iint_X a_k \wedge a_j$ is of the form J above. Conversely, taking a basis $\{a_1,\dots,a_{2g}\}$ for $H^1(X,\mathbf{Z})$ which satisfies that the matrix whose (k,j)-entry is given by $\iint_X a_k \wedge a_j$ is of the form J, a homology basis dual to $\{a_1,\dots,a_{2g}\}$ must be a canonical homology basis (for the details, see, e.g., [2, Ch. 3]).

Let $\{\chi_1',\dots,\chi_{2\gamma}'\}$ be a canonical homology basis for $H_1(Y,\mathbf{Z})$ and let $\{a_1',\dots,a_{2\gamma}'\}$ be its dual basis for $H^1(Y,\mathbf{Z})$. Let $f\colon X \to Y$ be a holomorphic map. Then f induces a homomorphism $f_*\colon H_1(X,\mathbf{Z}) \to H_1(Y,\mathbf{Z})$. Let $M = (m_{kj}) \in M(2\gamma, 2g; \mathbf{Z})$, where $f_*(\chi_j) = \sum_{k=1}^{2\gamma} m_{kj}\chi_k'$. (We denote by $M(m,n;\mathbf{Z})$ the set of $m \times n$ matrices with integral coefficients.) We will call M the matrix representation of f_* or f with respect to the canonical homology bases. There is another interpretation of M. Denote by f^*a_k' the pull back of a_k' by f. Considering an equality

$$\langle f^*a_k', \chi_j \rangle = \langle a_k', f_*(\chi_j) \rangle,$$

we may write $f^*a_k' = \sum_{j=1}^{2g} m_{kj} a_j$. Thus the induced map $f^*\colon H^1(Y,\mathbf{R}) \to H^1(X,\mathbf{R})$ is represented by the transpose ${}^tM$ with respect to the dual bases and it implies that f^* maps $H^1(Y,\mathbf{Z})$ into $H^1(X,\mathbf{Z})$. Thus we may re-write Theorem 1 in terms of $H^1(X,\mathbf{Z})$ as

Theorem 1'. *Let $\widetilde{X}$, X_1, X_2 be closed Riemann surfaces of genera ≥ 1 and let $f_i\colon \widetilde{X} \to X_i$ $(i=1,2)$ be non-constant holomorphic maps. Assume that there exists a homomorphism $H'\colon H^1(X_2,\mathbf{Z}) \to H^1(X_1,\mathbf{Z})$ which commutes with the induced homomorpisms $f_i^*\colon H^1(X_i,\mathbf{Z}) \to H^1(\widetilde{X},\mathbf{Z})$ $(i=1,2)$, i.e. $f_2^* = f_1^* \circ H'$. Then there* *exists a unique (modulo a translation in genus 1) holomorphic map* $h\colon X_1 \to X_2$ *with* $f_2 = h \circ f_1$.

Let χ_1, χ_2, χ_3, χ_4 and T be as in Theorem 2. Then choosing $\chi_5, \ldots, \chi_{2g}$ properly and renumbering χ_2, χ_3 and χ_4 as χ_{g+1}, χ_2 and χ_{g+2}, respectively, we get a canonical homology basis $\chi_1, \ldots, \chi_{2g}$. Let $a_1, \ldots, a_{2g} \in H^1(X, \mathbf{Z})$ be the dual basis. Denote by L the matrix representation of the $T \in \mathrm{Aut}(X)$ with respect to the basis $\chi_1, \ldots, \chi_{2g}$. We denote by $\mathbf{e}_k$ the g-tuple column vector whose k-th entry is 1 and others are 0, as usual. Then the j-th column of L is $\mathbf{e}_j$ for $j = 1,\ 2,\ g+1,\ g+2$. Since L is symplectic, writing $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $g \times g$ blocks, we have $L^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$. Hence the j-th row of L^{-1} is ${}^t\mathbf{e}_j$ for $j = 1,\ 2,\ g+1,\ g+2$ and this means that $T^{-1*}a_j = a_j$ for $j = 1,\ 2,\ g+1,\ g+2$ (equivalently $T^*a_j = a_j$ for $j = 1,\ 2,\ g+1,\ g+2$) since ${}^tL^{-1}$ is the matrix representation of T^{-1*} with respect to $a_1, \ldots, a_{2g}$. Conversely, suppose that a_1, a_2, a_3, $a_4 \in H^1(X, \mathbf{Z})$ satisfy $\iint_X a_1 \wedge a_3 = 1$, $\iint_X a_2 \wedge a_4 = 1$, otherwise $\iint_X a_i \wedge a_j = 0$. Suppose that $T \in \mathrm{Aut}(X)$ satisfies $T^*a_i = a_i$ for $i = 1, 2, 3, 4$. Then taking the dual and a little modification of the argument above leads us to the conclusion that there exist χ_1, χ_2, χ_3, $\chi_4 \in H_1(X, \mathbf{Z})$ so that $\chi_1 \cdot \chi_3 = 1$, $\chi_2 \cdot \chi_4 = 1$, otherwise $\chi_i \cdot \chi_j = 0$ and that $T(\chi_i) = \chi_i$ for $i = 1, 2, 3, 4$.

From this observation, we have Theorem 2 in terms of $H^1(X, \mathbf{Z})$ as

Theorem 2'. *Let X be a closed Riemann surface of genus greater than one. Let T be an automorphism of X. Suppose that there exist a_1, a_2, a_3, $a_4 \in H^1(X, \mathbf{Z})$ so that $\iint_X a_1 \wedge a_3 = 1$, $\iint_X a_2 \wedge a_4 = 1$, otherwise $\iint_X a_i \wedge a_j = 0$ and that $T \in \mathrm{Aut}(X)$ satisfies $T^*a_i = a_i$ for $i = 1, 2, 3, 4$. Then T is the identity.*

In Theorem 3, if $\widetilde{X} = X_1 = X_2$, then the conformal map h is just $f_2 \circ f_1^{-1}$ which may not be the identity. Thus Theorem 3 does not contain Theorem 2 strictly. However, in holomorphic mapping cases (i.e. if $\widetilde{X} \neq X_1$), it is natural to identify $f_1\colon \widetilde{X} \to X_1$ with $f_2\colon \widetilde{X} \to X_2$ if they are *isomorphic*, i.e. there exists a conformal map $h\colon X_1 \to X_2$ which satisfies $f_2 = h \circ f_1$. Indeed, if we observe the function fields $\widetilde{K}$, K_1 and K_2 of $\widetilde{X}$, X_1 and X_2, respectively, non-constant holomorphic maps $f_i\colon \widetilde{X} \to X_i$ induce injective homomorphisms $F_i\colon K_i \to \widetilde{K}$ $(i = 1, 2)$ and they determine the same subfield of $\widetilde{K}$ if and only if f_1 and f_2 are isomorphic. Thus Theorem 3 can be viewed as a generalization of Theorem 2 for holomorphic maps. If the homomorphism H' in the hypotheses of Theorem 1' is an isomorphism, then f_1 and f_2 are isomorphic. The hypotheses of Theorem 3 is weaker than those of Theorem 1' (= Theorem 1) in this case.

3. Lemmata

Let a be a differential 1-form given by $a = f\,dx + g\,dy$ using a local coordinate $z = x + yi$. The *conjugation* operator $*$ is defined by

$$^*a = -g\,dx + f\,dy.$$

Recall that for each cohomology class, we can choose a harmonic differential 1-form as a representative. If a is harmonic, then $a + i\,{}^*a$ is a holomorphic differential and conversely every holomorphic differential can be written in the form $a + i\,{}^*a$ where a is some harmonic differential. Thus choosing a harmonic representative a, we can create a holomorphic differential $a + i\,{}^*a$ from each cohomology class. Now we define a relation for points on X. We denote by U_p a neighborhood of a point p. For a holomorphic differential α and a coordinated neighborhood (U_p, z), we denote by $\alpha_p(z)dz$ the expression for α in terms of the local coordinate.

Definition 1. Let $a,\, b \in H^1(X, \mathbf{Z})$ so that $\iint_X a \wedge b = 1$. Let $p,\, q \in X$. We denote by α and β the holomorphic differentials created from a and b, respectively. We say that p is (a,b)-equivalent to q and write $p \sim_{ab} q$ if the following two conditions are satisfied.

(I)
$$\begin{pmatrix} \int_p^q a \\ \int_p^q b \end{pmatrix} \equiv \mathbf{0} \quad (\mathrm{mod.}\mathbf{Z})$$

(II) There are coordinated neighborhoods (U_p, z) and (U_q, ζ) for p and q, respectively, and there exists a conformal map $g\colon U_p \to U_q$ such that $\alpha_q(\zeta)\, d\zeta = \alpha_q(g(z))g'(z)\, dz = \alpha_p(z)\, dz$ and $\beta_q(\zeta)\, d\zeta = \beta_q(g(z))g'(z)\, dz = \beta_p(z)\, dz$ hold.

It is easy to see that (a,b)-equivalence defines a equivalent relation. We want to show that the quotient $X/\sim_{ab}$ is a Riemann surface. In order to simplify the situation, we first remove a finite number of points from $X/\sim_{ab}$ and show that the punctured quotient is a Riemann surface. We subsequently fill in the removed points and show that we obtain a compact Riemann surface conformally equivalent to the original surface X. Put

$$\phi = \alpha/\beta$$

where α and β are holomorphic differentials created from a and from b, respectively. Then ϕ is a non-constant meromorphic function on X. To see ϕ is non-constant, we will recall some basic facts about period matrices of Riemann surfaces (cf. [2, Ch. 3]). Let $\{\chi_1, \ldots, \chi_{2g}\}$ be a canonical homology basis on a Riemann surface X and $\{a_1, \ldots, a_{2g}\}$ be the dual basis for $H^1(X, \mathbf{Z})$. We denote by

$$G = (\lambda_{kj}), \quad k, j = 1,\, 2,\, \ldots, 2g,$$

the matrix representation of the conjugation operator $*$ with respect to the basis $\{a_1, \ldots, a_{2g}\}$. Thus

$${}^*a_k = \sum_{j=1}^{2g} \lambda_{kj} a_j, \quad k = 1,\, 2,\, \ldots, 2g.$$

If we write

$$G = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$$

in $g \times g$ blocks, we have

$$\lambda_4 = -{}^t\lambda_1, \quad \lambda_2 = {}^t\lambda_2, \quad \lambda_3 = {}^t\lambda_3, \quad \lambda_2 > 0, \quad -\lambda_3 > 0. \tag{1}$$

Then there exists a unique basis $\{\omega_1,\, \omega_2,\, \ldots, \omega_g\}$ for the space of holomorphic differentials on X such that the period matrix $(\int_{\chi_k} \omega_j)$ is of the form (E, Π) where E is the $g \times g$ identity matrix. Furthermore, Π must be of the form

$$\Pi = (-\lambda_3)^{-1}\,{}^t\lambda_1 + i(-\lambda_3)^{-1}. \tag{2}$$

Without loss of generality, we may assume that $a = a_1$ and $b = a_{g+1}$. If $\phi = \alpha/\beta = (a_1 + i\,{}^*a_1)/(a_{g+1} + i\,{}^*a_{g+1})$ is constant then every λ_j is of the form

$$\begin{pmatrix} c_j & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

because of equations (1). Then from the equation (2), Π must be of the form

$$\begin{pmatrix} \pi_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

But this contradicts a theorem of Martens (cf. [5]) which states that if (E, Π) is a period matrix with respect to a canonical homology basis on a closed Riemann surface of genus $g > 1$ then Π is *not* of the form

$$\Pi = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}$$

where A is an $n \times n$ matrix, $0 < n < g$. We have established the following.

Lemma 1. *Let a, $b \in H^1(X, \mathbf{Z})$ so that $\iint_X a \wedge b = 1$. Put*

$$\phi = \alpha/\beta$$

where α and β are holomorphic differentials created from a and from b, respectively. Then ϕ is a non-constant meromorphic function on X.

Let B be the set of inverse images of all the branch points on $\widehat{\mathbf{C}}$ via ϕ. Set

$$S = X'/\sim_{ab}$$

where $X' = X - B$.

Lemma 2. *The quotient S is a Hausdorff space.*

Proof. Let

$$\pi \colon X' \to S$$

be the projection. We induce the quotient topology on S. Then it is easy to see that the projection π is an open mapping. Suppose that there exists two points $p', q' \in S$ such that for any neighborhoods $U'_{p'}$ and $U'_{q'}$ of p' and q', respectively, $U'_{p'} \cap U'_{q'} \neq \emptyset$ holds. We will show it implies that $p' = q'$ in the following. Let $p \in \pi^{-1}(p')$ and $q \in \pi^{-1}(q')$. We take sequences of neighborhoods

$$U_{p1} \supset U_{p2} \supset \cdots \supset U_{pj} \ldots$$

with

$$\bigcap_{j=1}^{\infty} U_{pj} = \{p\}$$

and

$$U_{q1} \supset U_{q2} \supset \cdots \supset U_{qj} \ldots$$

with

$$\bigcap_{j=1}^{\infty} U_{qj} = \{q\}.$$

Since π is an open mapping, $\pi(U_{pj})$ and $\pi(U_{qj})$ are neighborhoods of p' and q', respectively. By the assumption, there exist points $r_{pj} \in U_{pj}$ and $r_{qj} \in U_{qj}$ such that

$$\pi(r_{pj}) = \pi(r_{qj}) \in \pi(U_{pj}) \cap \pi(U_{qj})$$

for each j. Then by the condition (I) of Definition 1,

$$\begin{pmatrix} \int_{r_{pj}}^{r_{qj}} a \\ \int_{r_{pj}}^{r_{qj}} b \end{pmatrix} \equiv \mathbf{0} \quad (\mathrm{mod.}\mathbf{Z})$$

for all j. Taking the limit, we see that

$$\begin{pmatrix} \int_{p}^{q} a \\ \int_{p}^{q} b \end{pmatrix} \equiv \mathbf{0} \quad (\mathrm{mod.}\mathbf{Z}) \tag{3}$$

holds.

By the condition (II) of Definition 1, $\pi(r_{pj}) = \pi(r_{qj})$ implies that $\phi(r_{pj}) = \phi(r_{qj}) \in \widehat{\mathbf{C}}$ where $\phi = \alpha/\beta$. Taking the limit again, we have $\phi(p) = \phi(q)$. Recalling the definition of X', we see that ϕ is locally conformal. Thus we can take small coordinated neighborhoods $(U_p, \tilde{z})$ and $(U_q, \tilde{\zeta})$ of p and q, respectively, such that $\phi(U_p) = \phi(U_q)$ and

$$\psi = \phi|_{U_q}{}^{-1} \circ \phi|_{U_p} \colon U_p \to U_q$$

is a conformal map. On the other hand, for each point r_{pj} in the U_p, there exist coordinated open neighborhoods $(U_{r_{pj}}, z)$ and $(U_{r_{qj}}, \zeta)$ in the U_p and U_q, respectively, and there exists a conformal map

$$g_j \colon U_{r_{pj}} \to U_{r_{qj}}$$

such that

$$\alpha_{r_{qj}}(g_j(z))g_j'(z)\,dz = \alpha_{r_{pj}}(z)\,dz, \quad \beta_{r_{qj}}(g_j(z))g_j'(z)\,dz = \beta_{r_{pj}}(z)\,dz$$

hold for the condition (II) of Definition 1. Then by the definition of ψ, $\psi|_{U_{r_{pj}}} = g_j$ holds. It implies that

$$\alpha_q(\psi(\tilde{z}))\psi'(\tilde{z})\,d\tilde{z} = \alpha_p(\tilde{z})\,d\tilde{z}, \quad \beta_q(\psi(\tilde{z}))\psi'(\tilde{z})\,d\tilde{z} = \beta_p(\tilde{z})\,d\tilde{z}$$

hold on $U_{r_{pj}} \subset U_p$. By the theorem of identity, this also holds on U_p. Combining this with (3) above, we see $\pi(p) = \pi(q)$ holds and it implies that S is a Hausdorff space. □

Lemma 3. *The Hausdorff space S admits a Riemann surface structure such that the projection $\pi \colon X' \to S$ is holomorphic.*

Proof. By the condition (II) of Definition 1, there exists a map $\sigma \colon S \to \widehat{\mathbf{C}}$ which satisfies $\phi = \sigma \circ \pi$. Since σ is a locally homeomorphism, it induces an complex structure on S via the complex structure on $\widehat{\mathbf{C}}$. Now $\sigma \colon S \to \widehat{\mathbf{C}}$ is holomorphic and thus the projection $\pi \colon X' \to S$ is holomorphic since $\phi = \sigma \circ \pi$. □

Recall that $X' = X - B$ where B is a finite set and that a holomorphic map maps a punctured disk to a punctured disk. Thus we can extend the projection π

to $X \to \overline{S}$ holomorphically where $\overline{S}$ is the compactification of S. We denote the extended projection by the same symbol π.

Lemma 4. *The compactification $\overline{S}$ is conformally equivalent to X.*

Proof. We want to prove that there exist projections of a and of b on $\overline{S}$, but first we will show that for the holomorphic differentials α and β created from a and b, respectively, there are holomorphic differentials α' and β' on $\overline{S}$ such that $\pi^*\alpha' = \alpha$ and $\pi^*\beta' = \beta$. By the construction of S, it is easy to see that there are such projections α' and β' on S. Let $p \in B$. Without loss of generality, we may suppose that π is written as $w = z^n\,(n \in \mathbf{N})$ using local coordinates around p and $\pi(p)$ where p and $\pi(p)$ are corresponding to $z = 0$ and $w = 0$, respectively. Except $w = 0$, the projection $\alpha'(w)\,dw$ is defined, that is $\alpha'(\pi(z))\,dw/dz = \alpha(z)$ holds. Define

$$W(z) = \int_0^z \alpha(z)\,dz$$

in a sufficiently small neighborhood of p. Take an arbitrary point w and put

$$\{z_1, z_2, \ldots, z_n\} = \pi^{-1}(w).$$

Then

$$W(z_j) = \int_0^{z_j} \alpha(z)\,dz = \lim_{z_0\to 0}\int_{z_0}^{z_j} \alpha(z)\,dz = \lim_{w_0\to 0}\int_{w_0}^{w} \alpha'(w)\,dw \quad (j = 1, 2, \ldots, n)$$

where $w_0 = \pi(z_0)$. This implies that

$$W(z_1) = W(z_2) = \cdots = W(z_n)$$

and thus the zero order of W at $z = 0$ is $\geq n$. Therefore the zero order of $\alpha(z) = dW(z)/dz$ is $\geq n-1$ and $\alpha'(\pi(z)) = \alpha(z)(dw/dz)^{-1}$ is bounded around $z = 0$. It implies that $z = 0$ is a removable singularity and we get the holomorphic differential α' on $\overline{S}$ such that $\pi^*\alpha' = \alpha$. By the same consideration as above for β, we get the projection β' on $\overline{S}$.

Recall that $\alpha = a + i\,{}^*a$ and $\beta = b + i\,{}^*b$ where a and b are harmonic representative. Since α' is holomorphic, α' and $\overline{\alpha'}$ are harmonic and so is

$$a' = \frac{\alpha' + \overline{\alpha'}}{2}.$$

Thus we can write

$$\alpha' = a' + i\,{}^*a'$$

and

$$a + i\,{}^*a = \alpha = \pi^*\alpha' = \pi^*(a' + i\,{}^*a') = \pi^*(a') + i\pi^*({}^*a') = \pi^*(a') + i\,{}^*(\pi^*(a')).$$

The last equality comes from the fact that the conjugation operator $*$ is compatible with pull-back via a holomorphic map. Comparing the real part, we get

$$a = \pi^*(a').$$

Similarly, denoting

$$b' = \frac{\beta' + \overline{\beta'}}{2},$$

we get

$$b = \pi^*(b').$$

We see that a', $b' \in H^1(\overline{S}, \mathbf{Z})$ as the following. Let c' be a closed curve on $\overline{S}$ with the base point p'_0. Let $p_0 \in \pi^{-1}(p'_0)$ and lift the closed curve c' via π to a curve c with initial point p_0 and denote by p_1 the end point of the lift c. Then

$$\begin{pmatrix} \int_{c'} a' \\ \int_{c'} b' \end{pmatrix} = \begin{pmatrix} \int_{c} a \\ \int_{c} b \end{pmatrix} \equiv \begin{pmatrix} \int_{p_0}^{p_1} a \\ \int_{p_0}^{p_1} b \end{pmatrix} \equiv \mathbf{0} \pmod{\mathbf{Z}}$$

for $\pi(p_0) = \pi(p_1)$.

Since a', $b' \in H^1(\overline{S}, \mathbf{Z})$, using Riemann bilinear relation, we see

$$\iint_{\overline{S}} a' \wedge b' \in \mathbf{Z}.$$

Denote by d the degree of the map π. Then

$$1 = \iint_X a \wedge b = d \iint_{\overline{S}} a' \wedge b'$$

and d must be 1. Now we see that $\pi\colon X \to \overline{S}$ is a conformal map and the proof is completed. □

4. Proof of Theorem 3

Now we will prove Theorem 3 by applying Lemma 4 as follows. The notation here is the same as in Theorem 3.

We denote by α_i and β_i the holomorphic differentials created from a_i and b_i $(i = 1, 2)$, respectively. Let $B_i = f_i^{-1}(B'_i)$ where B'_i is the set of all the branch points on X_i of f_i $(i = 1, 2)$. We put

$$\widetilde{X}' = \widetilde{X} - (B_1 \cup B_2),$$

and we use the same symbol f_i for the restricted map $f_i|_{\widetilde{X}'}$ $(i = 1, 2)$. Then $f_i\colon \widetilde{X}' \to f_i(\widetilde{X}') \subset X_i$ $(i = 1, 2)$ are locally conformal.

Let p, $q \in \widetilde{X}'$ with $f_1(p) = f_1(q)$. Then

$$\begin{pmatrix} \int_p^q f_1^* a_1 \\ \int_p^q f_1^* b_1 \end{pmatrix} = \begin{pmatrix} \int_c a_1 \\ \int_c b_1 \end{pmatrix} \equiv \mathbf{0} \pmod{\mathbf{Z}}$$

where c is a closed curve on X_1 with the base point $f_1(p) = f_1(q)$. On the other hand, taking suitable paths of integration,

$$\begin{pmatrix} \int_p^q f_1^* a_1 \\ \int_p^q f_1^* b_1 \end{pmatrix} = \begin{pmatrix} \int_p^q f_2^* a_2 \\ \int_p^q f_2^* b_2 \end{pmatrix} = \begin{pmatrix} \int_{f_2(p)}^{f_2(q)} a_2 \\ \int_{f_2(p)}^{f_2(q)} b_2 \end{pmatrix},$$

thus we have

$$\begin{pmatrix} \int_{f_2(p)}^{f_2(q)} a_2 \\ \int_{f_2(p)}^{f_2(q)} b_2 \end{pmatrix} \equiv \mathbf{0} \pmod{\mathbf{Z}}$$

and this means $f_2(p)$ and $f_2(q)$ satisfy the condition (I) of Definition 1 for a_2 and b_2. By the assumption of Theorem 3 and the compatibility of the conjugation operator $*$ with pull-back via a holomorphic map, we see that $f_1^* \alpha_1 = f_2^* \alpha_2$ and $f_1^* \beta_1 = f_2^* \beta_2$ holds. From this and the fact that $f_i|_{\widetilde{X}'}$ $(i = 1, 2)$ are locally conformal, we see $f_2(p)$

and $f_2(q)$ satisfy the condition (II) of Definition 1 for a_2 and b_2. Then $f_2(p) = f_2(q)$ on X_2 by Lemma 4. Thus there exists a holomorphic map

$$h\colon f_1(\widetilde{X}') \to f_2(\widetilde{X}')$$

such that $h \circ f_1|_{\widetilde{X}'} = f_2|_{\widetilde{X}'}$ holds. $B_1 \cup B_2 = \widetilde{X} - \widetilde{X}'$ consisits of removable singular points for h and we can extend h to have $h\colon X_1 \to X_2$ such that $h \circ f_1 = f_2$ holds where we denote the extended map by the same symbol h.

For p, $q \in \widetilde{X}'$ with $f_2(p) = f_2(q)$, a little modification of above argument leads us to the conclusion that the inverse map h^{-1} exists. Thus $h\colon X_1 \to X_2$ is conformal and satisfies $h \circ f_1 = f_2$. □

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