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REDUCED MODULUS WITH FREE BOUNDARY AND ITS APPLICATIONS

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Abstract. We derive an asymptotic formula for the modulus (= reciprocal of capacity) of generalized condenser whose field is an arbitrary multiply-connected domain on the complex sphere and whose plates degenerate into a finite number of inner and/or boundary points of the field. We call the constant term in this asymptotic formula the reduced modulus with free boundary. Our modulus generalizes several previously introduced concepts. The asymptotic formula is given in terms of a generalized version of the classical Neumann function. This generalized Neumann function is introduced in the paper and its properties are studied. The usefulness of the new modulus is illustrated by two applications: a two-point distortion theorem for univalent functions defined in annulus and preserving the unit circle and an inequality for the quadratic form in the difference of the Neumann and Robin functions.

1. Introduction

The idea of the reduced modulus of a domain in the extended complex plane $\overline{\mathbf{C}}$ is rooted in the works of Grötszch and Teichmüller. The classical definition deals with the condenser whose one plate is the complement of the domain of interest while the other plate is a small disk centered at an inner point of this domain. The constant term in the asymptotic expansion for the reciprocal of this condenser's capacity (called the modulus) as the radius of the disk goes to zero gives the classical reduced modulus of the domain. This notion found important applications in the geometric theory of functions of a complex variable [1, 19, 20, 24, 29, 30]. If the center of the disk lies at the point at infinity the reduced modulus is expressed in terms of logarithmic capacity of the domain's complement (see [21, formula (15), page 253] or [30, formula 2.6, page 18]). This fact is sometimes referred to as Pfluger's theorem.

The notion was further extended by Kuzmina [22], Emel'yanov [15] and Solynin [28] who introduced the reduced moduli of digons and triangles using the concept of extremal length. Soon thereafter Dubinin [3] proposed the idea that the notion may be extended to n-gons and suggested how this idea may be realized. In a series of papers Dubinin and his students fulfilled this program and computed generalized reduced moduli based on generalized condensers having more than two plates. When one plate is the complement of the domain of interest as before while a finite number of other plates collapse into inner points of the domain we obtain a generalized reduced modulus whose value can be expressed in terms of the Green function of the

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domain [4, Theorem 1]. When one plate is a fixed closed subset of the boundary of the domain of interest while all other plates degenerate into a finite number of inner and/or boundary points (of course distinct from the first plate) the reduced modulus can be expressed in terms of the Robin function (see [5, Theorem 7], or [8, Theorem 2.5]). The Robin capacity (see [13, 14]) is a special case of this construction. A formula for the reduced modulus of the entire complex sphere was derived in [10, Theorem 1]. Another version of the reduced modulus was earlier introduced by Mityuk in [25].

The applications of the reduced moduli are numerous including distortion theorems for univalent functions in both simply and multiply connected domains [2, 4, 7, 8, 19, 27, 30], coefficient inequalities [7, 8], extremal partition problems [4, 7, 8, 15, 16, 24], polynomial inequalities [7, 8, 9, 10], variational principles for conformal mappings [11] and other similar problems for analytic functions. See detailed account in the survey articles by Kuzmina [24], Solynin [29], Dubinin–Karp [8] and in Vasil'ev's book [30].

In the present paper we consider yet another version of the reduced modulus which complements naturally the variations considered previously. Namely, our reduced modulus is defined as the constant term in the asymptotic expansion for the modulus of the generalized condenser all whose plates degenerate into points. Quite expectedly the Neumann function comes into play as the main ingredient in the expression for this reduced modulus derived in this paper. We call our modulus the reduced modulus with free boundary emphasizing that the values of admissible functions on the boundary of the domain of interest are not prescribed unlike the previous definitions (see precise statements below). The idea to consider this type of modulus was expressed in [5], where it was also suggested that the Neumann function will play a role in such construction. It turned out that the classical Neumann function is insufficient to compute the modulus when some of the plates lie on the domain's boundary. We address this issue in the first part of the paper where the classical definition of the Neumann function is extended in two directions.

Particular cases of the reduced modulus introduced here do occur in the literature. Most notably, the reduced modulus of digon mentioned above represents a special case of our construction here. For simply connected domains Dubinin and Eyrikh found a formula for the reduced modulus with free boundary in terms of the Riemann mapping in [6, Theorem 4]. We present an alternative derivation of their formula from our main theorem in section 7. Emel'yanov obtained an inequality for the weighted sum of the reduced moduli of digons in [16]. Applications of and recent developments around the reduced moduli of digons and triangles can be found in [2, 27, 30].

The paper is organized as follows. The definitions of generalized condenser, its capacity and the reduced modulus with free boundary together with a technical lemma are collected in section 2. Sections 3 and 4 are concerned with the extensions of the standard definition of the Neumann function suited for our needs here. Sections 5 and 6 present the derivation of the formula for the reduced modulus with free boundary for analytic Jordan domains and general domains, respectively. In section 7 we deduce a few explicit formulas for some canonical domains. Finally, section 8 is devoted to applications. They are: a two-point distortion theorem for univalent functions defined in an annulus and preserving the unit circle and an inequality for the quadratic form in the difference of the Neumann and Robin functions.

2. Definitions

Suppose G is a finitely-connected domain in the extended complex plane $\overline{\mathbf{C}}_z$. Let $\overline{G}$ denote its compactification by Carathéodory's prime ends and let the boundary ∂G be the collection of prime ends. A neighbourhood is any open set in $\overline{G}$. When this cannot lead to confusion we will make no distinction between the elements of $\overline{G}$ corresponding to the inner points of G and these points. We will also use single notation for the support of an accessible boundary point and the boundary point itself. If G is a Jordan domain, $\overline{G}$ and ∂G defined above agree with usual closure and boundary.

Definition 1. *Generalized condenser* is the triple $C = (G, \mathscr{E}, \Delta)$, where $\mathscr{E} = \{E_k\}_{k=1}^n$ is a collection of closed in $\overline{G}$ pairwise disjoint sets, $n \geq 2$, and $\Delta = \{\delta_k\}_{k=1}^n$ is a collection of reals containing at least two distinct numbers.

The sets E_k will be called the plates of the condenser C, while $G \setminus \bigcup E_k$ is called its field.

Definition 2. *Capacity of* C denoted by $\operatorname{cap} C$ is the infimum of the Dirichlet integral

$$I(v, G) := \iint\limits_G |\nabla v|^2 \, dx \, dy, \quad z = x + iy,$$

taken over all admissible functions $v \colon \overline{G} \to \mathbf{R}$, i.e., real-valued functions continuous in $\overline{G}$, satisfying the Lipschitz condition in a neighbourhood of every finite point of G possibly excluding a finite number of such points and assuming the value δ_k in a neighbourhood of the plate E_k, $k = 1, \ldots, n$.

Let f be the univalent conformal mapping of G onto a Jordan domain D whose boundary ∂D consists of a finite number of analytic Jordan curves (for brevity such domains will be called analytic Jordan domains).

Definition 3. A point $z_0 \in \overline{G}$ is called *admissible* if $z_0 \in G$ or $z_0 \in \partial G$ is accessible and for some $0 < \beta_G(z_0) \leq 2$

$$f(z) - f(z_0) = \begin{cases} (z - z_0)^{\beta_D(z_0)/\beta_G(z_0)}(c(z_0) + o(1)) & \text{as } z \to z_0 \neq \infty, \\ (1/z)^{\beta_D(z_0)/\beta_G(z_0)}(c(z_0) + o(1)) & \text{as } z \to \infty, \end{cases} \tag{1}$$

where $c(z_0) \neq 0$ and

$$\beta_D(z_0) = \begin{cases} 2, & f(z_0) \in D, \\ 1, & f(z_0) \in \partial D. \end{cases}$$

Here we assumed that $f(z_0) \neq \infty$ which does not entail any loss of generality. Clearly, (1) holds true for $z_0 \in G$ by Taylor expansion with $\beta_G(z_0) = 2$, $c(z_0) = f'(z_0)$, so all inner points of G are admissible. If $z_0 \in \partial G$ is an accessible boundary point, (1) holds true for analytic corners (i.e., intersections of analytic boundary arcs) with $\pi\beta_G(z_0)$ being the angle at such corner by Lewy–Lehman theorem [26, Theorem 3.8]. In fact, even weaker conditions suffice for (1) to hold—see details in [26].

Given a finite $z_0 \in \overline{\mathbf{C}}_z$ and $r > 0$ denote by $D(z_0, r)$ the closed disk of radius r centered at z_0. For the point at infinity set $D(\infty, r) := \{z : |z| \geq 1/r\}$.

Definition 4. A parametric family of closed sets $\{\tilde{D}(z_0, r)\}_{0<r<r_0}$ will be said to comprise *almost disks* if

$$D(z_0, r_1(r)) \subset \tilde{D}(z_0, r) \subset D(z_0, r_2(r)), \quad 0 < r < r_0,$$

for some positive functions $r_1(r)$, $r_2(r)$ such that $\lim_{r\downarrow 0}[r_i(r)/r] = 1$, $i = 1, 2$.

For a given domain $G \subset \overline{\mathbf{C}}_z$ and a point $z_0 \in G$ introduce the notation $E(z_0, r, G) = \tilde{D}(z_0, r)$ with small enough r to get the inclusion $\tilde{D}(z_0, r) \subset G$; if z_0 is an accessible boundary point of G, $E(z_0, r, G)$ will mean the closure in $\overline{G}$ of the connected component of $G \cap \tilde{D}(z_0, r)$ in which z_0 is accessible. We will abbreviate $E(z_0, r, G)$ to $E(z_0, r)$ when the underlying domain G is apparent.

Suppose, $m \geq 2$, $Z = \{z_k\}_{k=1}^m$ is a collection of distinct admissible points of $\overline{G}$ and $\Delta = \{\delta_k\}_{k=1}^m$ is a collection of reals containing at least two different numbers. For a sufficiently small $r > 0$ define the condenser

$$C(r; G, Z, \Delta, \Psi) = (G; \{E(z_1, \psi_1(r)), \quad E(z_2, \psi_2(r)), \dots, E(z_m, \psi_m(r))\}, \Delta), \tag{2}$$

where

$$\Psi = \{\psi_k(r)\}_{k=1}^m, \quad \psi_k(r) = \mu_k r^{\nu_k}, \quad \mu_k, \nu_k > 0, \quad k = 1, \dots, m.$$

Definition 5. *The reduced modulus of the domain G with free boundary with respect to the collections Z, Δ and Ψ* is defined by

$$M(G, Z, \Delta, \Psi) = \lim_{r\downarrow 0} \left(|C(r; G, Z, \Delta, \Psi)| + \frac{\nu}{\pi} \log r \right), \tag{3}$$

if the limit in (3) exists. Here

$$|C(r; G, Z, \Delta, \Psi)| = (\operatorname{cap} C(r; G, Z, \Delta, \Psi))^{-1}, \quad \nu = \left(\sum_{k=1}^m \frac{\delta_k^2 \beta_G(z_k)}{\nu_k} \right)^{-1}. \tag{4}$$

Denote

$$C^*(r; G, Z, \Delta, \Psi)$$
$$= (G; \{\langle D(z_1, \psi_1(r)) \cap G\rangle, \langle D(z_2, \psi_2(r)) \cap G\rangle, \dots, \langle D(z_m, \psi_m(r)) \cap G\rangle\}, \Delta),$$

where $\langle D(z_k, \psi_k(r)) \cap G\rangle$ is the closure in $\overline{G}$ of the connected component of $D(z_k, \psi_k(r)) \cap G$ in which z_k is accessible. For inner points $z_k \in G$, $\langle D(z_k, \psi_k(r)) \cap G\rangle$ simply equals $D(z_k, \psi_k(r)) \cap G$.

Lemma 1. *If*

$$\lim_{r\downarrow 0} \left(|C^*(r; G, Z, \Delta, \Psi)| + \frac{\nu}{\pi} \log r \right) \tag{5}$$

exists then limit (3) *also exists and they are equal. Conversely, if limit* (3) *exists for some choice of almost disks then limit* (5) *also exists and they are equal.*

Proof. Repeats word for word the proof of the similar lemma in [4, Lemma 1]. □

3. Generalized Neumann function

Let G denote an analytic Jordan domain in $\overline{\mathbf{C}}_z$ and $\varphi(z)$ be any continuous real function on ∂G such that

$$\int_{\partial G} \varphi(z)|dz| = -\pi. \tag{6}$$

Define *the generalized Neumann function* $N_{G,\varphi}(z,\zeta)$, $z \in \overline{G}$, $\zeta \in \overline{G}$, of G associated with boundary values $\varphi(z)$ with pole at ζ by the following requirements:

1) $N_{G,\varphi}(z,\zeta)$ is harmonic in $G\backslash\{\zeta\}$ and differentiable in $\overline{G}\backslash\{\zeta\}$ as a function of z.

2) $N_{G,\varphi}(z,\zeta)+\frac{1}{\beta}\log|z-\zeta|$ is harmonic a in neighbourhood of $\zeta \neq \infty$ or $N_{G,\varphi}(z,\zeta)-\frac{1}{\beta}\log|z|$ is harmonic in the neighbourhood of $\zeta = \infty$, where

$$\beta = \begin{cases} 2, & \zeta \in G, \\ 1, & \zeta \in \partial G. \end{cases}$$

3) The (outer) normal derivative satisfies

$$\frac{\partial N_{G,\varphi}(z,\zeta)}{\partial n} = \varphi(z)$$

for all $z \in \partial G$ possibly except $z = \zeta$.

In what follows we will use $N_G(z,\zeta)$ as the generic notation for any of the functions satisfying 1) – 3) for some φ.

Lemma 2. *The set of generalized Neumann functions of an analytic Jordan domain G coincides with the set of functions of the form*

$$N_G(z,\zeta) + h(z) + c(\zeta), \tag{7}$$

where $N_G(z,\zeta)$ is any fixed generalized Neumann function, $h(z)$ is harmonic in G and has continuous normal derivative on ∂G and $c(\zeta)$ is any function of ζ.

Proof. Clearly, any function of the form (7) falls under the definition of the generalized Neumann function. Conversely, let $N_{G,\psi}(z,\zeta)$ and $N_{G,\varphi}(z,\zeta)$ be two generalized Neumann functions. Then the function $u(z,\zeta) = N_{G,\psi}(z,\zeta) - N_{G,\varphi}(z,\zeta)$ is harmonic in G and has continuous values of the normal derivative at the boundary $\partial u/\partial n = \psi(z) - \varphi(z)$. Hence, it is a solution of the Neumann problem with the boundary function $\psi(z) - \varphi(z)$ and thus can be written as [18, page 264]:

$$u(z,\zeta) = h(z) + c(\zeta). \qquad \square$$

Lemma 3. *For any analytic Jordan domain G and any given continuous function φ on ∂G satisfying* (6) *the generalized Neumann function $N_{G,\varphi}(z,\zeta)$ exists.*

Proof. Note first that by Lemma 2 it suffices to prove the lemma for some continuous boundary function ψ, then by adding the solution of the Neumann problem with normal derivative $\varphi - \psi$ on the boundary we obtain the required function $N_{G,\varphi}(z,\zeta)$.

Suppose first that G is simply-connected. Then a generalized Neumann function is given explicitly by

$$N_G(z,\zeta) = -\frac{1}{2}\log|f(z) - f(\zeta)||1 - \overline{f(z)}f(\zeta)|, \tag{8}$$

where f is the Riemann mapping. Properties 1) – 3) can be verified directly taking account of the fact that expansion (1) is valid for every $\zeta \in \overline{G}$.

For a multiply-connected domain G denote by K_i, $i = 1, 2, \ldots, n$, the connected components of $\overline{\mathbf{C}}_z \backslash G$. Without loss of generality we may assume that either $\zeta \in G$ or $\zeta \in \partial K_1$. Since the domain $\overline{\mathbf{C}}_z \backslash K_1$ is simply connected we can construct its generalized Neumann function by (8). Denote by $\gamma_i(s, \zeta)$, $i = 1 \ldots, n$, the normal derivative of $N_{\overline{\mathbf{C}}_z \backslash K_1}$ on ∂K_i. By definition $\gamma_1(s)$ is independent of ζ, and

$$\int\limits_{\partial K_i} \gamma_i(s, \zeta)\, ds = 0, \quad i = 2, \ldots, n, \quad \int\limits_{\partial K_1} \gamma_1(s) ds = -\pi,$$

since $N_{\overline{\mathbf{C}}_z \backslash K_1}$ is harmonic in $\overline{K_i}$, $i = 2 \ldots, n$. Consider the following Neumann problem: find a function u harmonic in G whose normal derivative satisfies

$$\frac{\partial u}{\partial n}(s) = -\gamma_i(s, \zeta), \quad s \in \partial K_i, \quad i = 2, \ldots, n, \quad \frac{\partial u}{\partial n}(s) = 0, \quad s \in \partial K_1,$$

This problem always has a solution and for $N_G = N_{\overline{\mathbf{C}}_z \backslash K_1} + u$ we will have

$$\int\limits_{\partial G} \frac{\partial N_G(z, \zeta)}{\partial n}\, ds = -\pi,$$

i.e., we have built the generalized Neumann function for $\psi(z) = 0$, $z \in \partial K_i$, $i = 2, \ldots, n$, $\psi(z) = \gamma_1(z)$ for $z \in \partial K_1$. $\square$

For an arbitrary finitely connected domain G without degenerate boundary components define the generalized Neumann function by the formula

$$N_G(z, \zeta) := N_{f(G)}(f(z), f(\zeta)),$$

where f is the univalent conformal mapping of G onto an analytic Jordan domain $f(G)$. For an admissible point ζ this definition implies the expansion

$$N_G(z, \zeta) = -\frac{1}{\beta_G} \log|z - \zeta| + N(\zeta) + o(1), \quad z \to \zeta,\ \zeta \neq \infty, \tag{9}$$

$$N_G(z, \infty) = \frac{1}{\beta_G} \log|z| + N(\infty) + o(1), \quad z \to \infty,\ \zeta = \infty, \tag{10}$$

where β_G is taken from expansion (1).

The generalized Neumann functions gives the same boundary representation formula as the classical one allowing to recover a harmonic function from the boundary values of its normal derivative. It is also symmetric in its two variables under additional normalization condition

$$\int\limits_{\partial G} N_{G,\varphi}(z, \zeta) \varphi(z) |dz| = \text{const}$$

and unique for any chosen value of the constant. An advantage of the generalized Neumann function over its classical counterpart is that its conformal transplantation leads to the generalized Neumann function of the transformed domain. Unlike the classical Neumann function it is defined for unbounded as well as for bounded domains. The arbitrariness in choosing φ leads to additional freedom when computing the generalized Neumann function for specific domains.

4. Bipolar Neumann function

In what follows we will need a version of the Neumann function whose normal derivative vanishes on the boundary. Condition (6) prohibits such behavior for the generalized Neumann function and one has to consider a function with two poles instead of one. For an analytic Jordan domain G and two different points z_* and z_0 from $\overline{G}$, $v_G(z, z_0|z_*)$ will be called *the bipolar Neumann function normalized at* z_* if

1) $v_G(z, z_0|z_*)$ is continuous in $\overline{G} \setminus \{z_0, z_*\}$ and harmonic in $G \setminus \{z_0, z_*\}$,

2) in the neighborhoods of z_0 and z_*

$$v_G(z, z_0|z_*) = \begin{cases} -\beta_0^{-1} \log|z - z_0| + R(z_0) + o(1), & z \to z_0,\ z_0 \neq \infty, \\ \beta_0^{-1} \log|z| + R(z_0) + o(1), & z \to z_0,\ z_0 = \infty, \end{cases} \tag{11}$$

$$v_G(z, z_0|z_*) = \begin{cases} \beta_*^{-1} \log|z - z_*| + o(1), & z \to z_*,\ z_* \neq \infty, \\ -\beta_*^{-1} \log|z| + o(1), & z \to z_*,\ z_* = \infty. \end{cases} \tag{12}$$

Here β_0 and β_* equal 2 for inner points of G and 1 for boundary points as before.

3) on the boundary of G

$$\frac{\partial v_G(z, z_0|z_*)}{\partial n_z} = 0, \quad z \in \partial G \backslash \{z_0, z_*\}.$$

Requirements 1) – 3) define the unique function $v(z, z_0|z_*)$. This function can be built by taking the difference of generalized Neumann functions:

$$v_G(z, z_0|z_*) = N_{G,\varphi}(z, z_0) - N_{G,\varphi}(z, z_*) + N(z_*) - N_{G,\varphi}(z_*, z_0), \tag{13}$$

where the constant $N(z_*)$ is taken from (9) or by taking a difference of bipolar Neumann functions with one common pole:

$$v_G(z, z_0|z_1) = v_G(z, z_0|z_*) - v_G(z, z_1|z_*) + R(z_1) - v_G(z_1, z_0|z_*). \tag{14}$$

The bipolar Neumann function with $z_* = \infty$ was used in [13], where it was shown that it plays the same role for the Robin capacity logarithm plays for the logarithmic capacity. Standard computations (see, for instance, [18, formula (15.6-8)]) show that $v_G(z, z_0|z_*)$ is symmetric in first two variables:

$$v_G(z_1, z_2|z_*) = v_G(z_2, z_1|z_*),$$

including the case when any of the points z_1, z_2, z_* belong to ∂G.

For a general multiply connected domain G and admissible points z_0, z_* we can define the bipolar Neumann function by conformal transplantation:

$$v_G(z, z_0|z_*) := v_{f(G)}(f(z), f(z_0)|f(z_*)) - \frac{1}{\beta_*} \log|c_*|, \tag{15}$$

where f is the univalent conformal mapping of G onto an analytic Jordan domain and $c_* = c(z_*)$ is the constant from expansion (1).

5. Computation of the reduced modulus for analytic Jordan domains

To formulate our main theorem we will need the following constants ($k,l = 2,\dots,m$):

$$R_{k,l}=\begin{cases} v_G(z_k,z_l|z_1) & \text{if } k\neq l,\\ \lim\limits_{z\to z_k}[v_G(z,z_k|z_1)+\beta_k^{-1}\log|z-z_k|] & \text{if } k=l \text{ and } z_k\neq\infty,\\ \lim\limits_{z\to z_k}[v_G(z,z_k|z_1)-\beta_k^{-1}\log|z|] & \text{if } k=l \text{ and } z_k=\infty.\end{cases} \tag{16}$$

Theorem 1. *Suppose G is a finitely connected analytic Jordan domain and the collections Z, Δ, Ψ are as defined above. Then the reduced modulus* (3) *exists if*

$$\sum_{l=1}^{m}\frac{\delta_l\beta_l}{\nu_l}=0 \tag{17}$$

and is found from the formula

$$M=-\frac{\nu^2}{\pi}\left(\sum_{k=1}^{m}\frac{\beta_k\delta_k^2\log\mu_k}{\nu_k^2}-\sum_{k,l=2}^{m}\frac{\beta_k\beta_l\delta_k\delta_l R_{k,l}}{\nu_k\nu_l}\right), \tag{18}$$

where ν is defined by (4). *If condition* (17) *is violated the modulus is infinite.*

Proof. Most of the proof follows the line of argument from [10, Theorem 1]. Without loss of generality we may assume that $z_1=0$. Introduce the notation

$$v_k(z)=v_G(z,z_k|z_1).$$

First we make

assumption 1: $\delta_l\neq 0,\ l=1,2,...,m.$

Later we will get rid of this assumption. Consider the function

$$g_r(z)=-\sum_{l=2}^{m}\frac{\delta_l\beta_l}{\log\psi_l(r)}v_l(z)-\sum_{k,l=2}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_l(r)\log\psi_k(r)}v_k(z) \tag{19}$$

defined in G. It is clearly harmonic in $G\backslash Z$ and

$$\frac{\partial g_r(z)}{\partial n_z}=0,\ z\in\partial G\backslash Z.$$

Fix $n\geq 2$. We can rearrange the definition of $g_r(z)$ to get:

$$\begin{aligned} g_r(z)&=\frac{v_n(z)}{-\log\psi_n(r)}\left(\delta_n\beta_n+\sum_{l=2}^{m}\frac{\beta_l\delta_l\beta_n R_{n,l}}{\log\psi_l(r)}\right)\\ &\quad-\sum_{\substack{l=2\\ l\neq n}}^{m}\frac{\beta_l\delta_l v_l(z)}{\log\psi_l(r)}-\sum_{\substack{k,l=2\\ k\neq n}}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}v_k(z)}{\log\psi_l(r)\log\psi_k(r)}. \end{aligned} \tag{20}$$

This representation shows that $g_r(z)$ has the same sign as δ_n in some neighborhood of z_n for sufficiently small r. Indeed, $-\log\psi_n(r)>0$ for sufficiently small r and all quantities under summations cannot affect the sign of $g_r(z)$. Since $g_r(z)\to 0$ as $r\downarrow 0$

for each $z \in G\setminus Z$, then in a neighbourhood of z_n there are points $z(r)$ satisfying $g_r(z) = \delta_n$. For such points we obtain from (20) and the definition of $v_n(z)$:

$$\delta_n = \delta_n \frac{\log|z(r) - z_n|}{\log \psi_n(r)} + \mathscr{O}\left(\frac{1}{\log r}\right), \ r \downarrow 0,$$

or

$$\frac{\log|z(r) - z_n|}{\log \psi_n(r)} = 1 + \mathscr{O}\left(\frac{1}{\log r}\right), \ r \downarrow 0,$$

which implies $z(r) \to z_n$ as $r \downarrow 0$. Substituting the last relation into (20) and taking account of $v_l(z) = R_{n,l} + o(1)$ as $z \to z_n$ we get

$$\begin{aligned}
\delta_n = &\left[\frac{\log|z - z_n|}{\beta_n \log \psi_n(r)} - \frac{R_{n,n}}{\log \psi_n(r)} + o\left(\frac{1}{\log r}\right)\right]\left[\delta_n\beta_n + \frac{\delta_n\beta_n^2 R_{n,n}}{\log \psi_n(r)} + \sum_{\substack{l=2\\ l\neq n}}^{m} \frac{\delta_l\beta_l\beta_n R_{n,l}}{\log \psi_l(r)}\right] \\
&- \sum_{\substack{l=2\\ l\neq n}}^{m} \frac{\delta_l\beta_l R_{n,l}}{\log \psi_l(r)} + o\left(\frac{1}{\log r}\right) = \delta_n \frac{\log|z - z_n|}{\log \psi_n(r)} + \frac{\delta_n\beta_n R_{n,n}}{\log \psi_n(r)}\left(1 + \mathscr{O}\left(\frac{1}{\log r}\right)\right) \\
&+ \frac{1}{\beta_n}\sum_{\substack{l=2\\ l\neq n}}^{m} \frac{\delta_l\beta_l\beta_n R_{n,l}}{\log \psi_l(r)}\left(1 + \mathscr{O}\left(\frac{1}{\log r}\right)\right) - \frac{\delta_n\beta_n R_{n,n}}{\log \psi_n(r)} - \sum_{\substack{l=2\\ l\neq n}}^{m} \frac{\delta_l\beta_l R_{n,l}}{\log \psi_l(r)} + o\left(\frac{1}{\log r}\right) \\
= &\ \delta_n \frac{\log|z - z_n|}{\log \psi_n(r)} + o\left(\frac{1}{\log r}\right).
\end{aligned}$$

This equality yields after rearrangement:

$$\log|z(r) - z_n| - \log \psi_n(r) = o(1) \ \Rightarrow \ |z(r) - z_n| \sim \psi_n(r), \ r \downarrow 0.$$

The last asymptotic formula implies that the parametric family of sets

$$E(z_n, \psi_n(r)) = \{z : g_r(z)/\delta_n \geq 1\}, \quad n = 2, \ldots, m, \tag{21}$$

comprises almost disks as $r \downarrow 0$.

Now we want to prove that the same conclusion is true in a neighborhood of $z_1 = 0$. In contrast with $n \geq 2$ all functions $v_k(z)$ have a pole at z_1. According to (12) and (19) we have as $z \to 0$:

$$g_r(z) = \frac{\log|z|}{\beta_1}\left(-\sum_{l=2}^{m} \frac{\delta_l\beta_l}{\log \psi_l(r)} - \sum_{k,l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\log \psi_l(r) \log \psi_k(r)}\right) + o_z(1)\, \mathscr{O}_r\left(\frac{1}{\log r}\right), \tag{22}$$

where the subscripts in o_z and $\mathscr{O}_r$ are intended to emphasize the underlying asymptotic variable. Clearly,

$$\frac{1}{\log \psi_k(r)} = \frac{1}{\nu_k \log r}\left(1 - \frac{\log \mu_k}{\nu_k \log r} + \mathscr{O}\big([\log r]^{-2}\big)\right), \quad r \downarrow 0. \tag{23}$$

Hence, we have the following expansion for the expression in parentheses in (22):

$$
\begin{aligned}
&-\sum_{l=2}^{m}\frac{\delta_l\beta_l}{\log\psi_l(r)}-\sum_{k,l=2}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_l(r)\log\psi_k(r)}\\
&=-\frac{1}{\log r}\sum_{l=2}^{m}\frac{\delta_l\beta_l}{\nu_l}\left[1-\frac{\log\mu_l}{\nu_l\log r}+\mathscr{O}\big([\log r]^{-2}\big)\right]\\
&\quad-\frac{1}{(\log r)^2}\sum_{k,l=2}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}}{\nu_l\nu_k}\big(1+\mathscr{O}\big([\log r]^{-1}\big)\big)\\
&=\frac{1}{\log r}\left[-\sum_{l=2}^{m}\frac{\delta_l\beta_l}{\nu_l}+\sum_{l=2}^{m}\frac{\delta_l\beta_l\log\mu_l}{\nu_l^2\log r}-\frac{1}{\log r}\sum_{k,l=2}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}}{\nu_l\nu_k}+\mathscr{O}\big([\log r]^{-2}\big)\right].
\end{aligned}
$$

Now we make

$$
\textit{assumption } 2\text{:}\ \sum_{l=1}^{m}\frac{\beta_l\delta_l\log\mu_l}{\nu_l^2}=\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_l R_{k,l}}{\nu_k\nu_l}. \tag{24}
$$

Later we will get rid of this assumption. Under (24) and in view of (17) the expression in brackets becomes

$$
\frac{\delta_1\beta_1}{\nu_1}\left(1-\frac{\log\mu_1}{\nu_1\log r}\right)+\mathscr{O}\big([\log r]^{-2}\big)=\frac{\delta_1\beta_1\log r}{\log\psi_1(r)}+\mathscr{O}\big([\log r]^{-2}\big).
$$

Thus

$$
g_r(z)=\frac{\delta_1\log|z|}{\log\psi_1(r)}\big(1+\mathscr{O}_r\big([\log r]^{-2}\big)\big)+o_z(1)\,\mathscr{O}_r\!\left(\frac{1}{\log r}\right).
$$

This formula shows that solutions $z(r)$ of $g_r(z)=\delta_1$ do exist and $z(r)\to 0$ as $r\downarrow 0$. Then repeating the argument given for the neighborhoods of z_n, $n\geq 2$, we conclude that the family of sets

$$
E(z_1,\psi_1(r))=\{z: g_r(z)/\delta_1\geq 1\} \tag{25}
$$

comprises almost disks as $r\downarrow 0$.

According to the extended Dirichlet principle [17] the function $g_r(z)$ coincides with the potential function of the condenser $C(r;G,Z,\Delta,\Psi)$ defined by (2) in the field of this condenser. Hence, by an application of the second and the first Green's formulas we have

$$
\begin{aligned}
\operatorname{cap}C(r;G,Z,\Delta,\Psi)&=\iint\limits_{G\setminus\cup_{k=1}^{m}E(z_k,\psi_k)}|\nabla g_r(z)|^2\,dx\,dy=-\sum_{k=1}^{m}\int\limits_{\partial E(z_k,\psi_k)}\delta_k\frac{\partial g_r}{\partial n}\,ds\\
&=-\sum_{k=1}^{m}\delta_k\int\limits_{\partial D(z_k,\rho)\cap G}\frac{\partial g_r}{\partial n}\,ds=-\sum_{k=2}^{m}\delta_k\int\limits_{\partial D(z_k,\rho)\cap G}\frac{\partial g_r}{\partial n}\,ds-\delta_1\int\limits_{\partial D(z_1,\rho)\cap G}\frac{\partial g_r}{\partial n}\,ds,
\end{aligned} \tag{26}
$$

where $\rho>0$ is sufficiently small. From (11) and (19) we have in a neighbourhood of $z=z_k$, $k=2,\dots,m$:

$$
g_r(z)=\frac{\log|z-z_k|}{\beta_k}\left(\frac{\delta_k\beta_k}{\log\psi_k(r)}+\sum_{l=2}^{m}\frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_k(r)\log\psi_l(r)}\right)+H(z),
$$

where $H(z)$ is harmonic in a neighbourhood of z_k. Consequently,

$$\int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial g_r}{\partial n}\, ds = \left(\frac{\delta_k\beta_k}{\log\psi_k(r)} + \sum_{l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_k(r)\log\psi_l(r)} \right) \frac{1}{\beta_k} \int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial \log|z-z_k|}{\partial n}\, ds$$

$$+ \int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial H}{\partial n}\, ds = \pi \left(\frac{\beta_k\delta_k}{\log\psi_k(r)} + \sum_{l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_k(r)\log\psi_l(r)} \right) + o(1)$$

as $\rho \to 0$, since by definition of β_k

$$\frac{1}{\beta_k} \int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial \log|z-z_k|}{\partial n}\, ds = \frac{1}{\beta_k} \int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial \log\rho}{\partial\rho}\rho\, d\theta = \pi + o(1) \quad \text{as } \rho \to 0$$

and

$$\int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial H}{\partial n}\, ds = \rho \int\limits_{\partial D(z_k,\rho)\cap G} \frac{\partial H}{\partial n}\, d\theta = \mathscr{O}(\rho) \quad \text{as } \rho \to 0.$$

In a neighbourhood of $z_1 = 0$ we have by (12) and (19):

$$g_r(z) = -\frac{\log|z|}{\beta_1} \left(\sum_{l=2}^{m} \frac{\delta_l\beta_l}{\log\psi_l(r)} + \sum_{k,l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_l(r)\log\psi_k(r)} \right) + H_1(z),$$

where $H_1(z)$ is harmonic. Acting as before we get

$$\int\limits_{\partial D(z_1,\rho)\cap G} \frac{\partial g_r}{\partial n}\, ds = -\pi \left(\sum_{l=2}^{m} \frac{\delta_l\beta_l}{\log\psi_l(r)} + \sum_{k,l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_l(r)\log\psi_k(r)} \right) + o(1) \quad \text{as } \rho \to 0.$$

Substitution of these formulas into (26) yields:

$$\frac{1}{\pi}\operatorname{cap} C(r; G, Z, \Delta, \Psi) = -\sum_{k=2}^{m} \frac{\beta_k\delta_k^2}{\log\psi_k(r)} - \sum_{k,l=2}^{m} \frac{\delta_k\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_k(r)\log\psi_l(r)}$$

$$+ \sum_{l=2}^{m} \frac{\delta_1\delta_l\beta_l}{\log\psi_l(r)} + \sum_{k,l=2}^{m} \frac{\delta_1\delta_l\beta_l\beta_k R_{k,l}}{\log\psi_l(r)\log\psi_k(r)} + o(1) \quad \text{as } \rho \to 0.$$

Since capacity of $C(r; G, Z, \Delta, \Psi)$ is independent of ρ we have $o(1) = 0$ in the last formula. Using (23) we derive for $r \downarrow 0$:

$$\frac{1}{\pi}\operatorname{cap} C(r; G, Z, \Delta, \Psi) = \frac{-1}{\log r} \sum_{k=2}^{m} \frac{\delta_k^2\beta_k}{\nu_k} \left(1 - \frac{\log\mu_k}{\nu_k \log r} \right) + \frac{\delta_1}{\log r} \sum_{l=2}^{m} \frac{\delta_l\beta_l}{\nu_l} \left(1 - \frac{\log\mu_l}{\nu_l \log r} \right)$$

$$- \sum_{k,l=2}^{m} \frac{\delta_k\delta_l\beta_l\beta_k R_{k,l}}{\nu_k\nu_l[\log r]^2} + \sum_{k,l=2}^{m} \frac{\delta_1\delta_l\beta_l\beta_k R_{k,l}}{\nu_k\nu_l[\log r]^2} + \mathscr{O}\big([\log r]^{-3}\big)$$

$$= \frac{-1}{\log r} \sum_{k=2}^{m} \frac{\delta_k^2\beta_k}{\nu_k} - \frac{\delta_1^2\beta_1}{\nu_1 \log r} + \frac{1}{[\log r]^2} \sum_{k=2}^{m} \frac{\delta_k^2\beta_k \log\mu_k}{\nu_k^2} - \frac{\delta_1}{[\log r]^2} \sum_{l=2}^{m} \frac{\delta_l\beta_l \log\mu_l}{\nu_l^2}$$

$$- \frac{1}{[\log r]^2} \sum_{k,l=2}^{m} \frac{\delta_k\delta_l\beta_l\beta_k R_{k,l}}{\nu_k\nu_l} + \frac{\delta_1}{[\log r]^2} \sum_{k,l=2}^{m} \frac{\delta_l\beta_l\beta_k R_{k,l}}{\nu_k\nu_l} + \mathscr{O}\big([\log r]^{-3}\big),$$

where we used (17). An application of (24) gives:

$$\frac{1}{\pi}\operatorname{cap}C(r;G,Z,\Delta,\Psi)=\frac{-1}{\log r}\sum_{k=1}^{m}\frac{\delta_k^2\beta_k}{\nu_k}+\frac{1}{[\log r]^2}\sum_{k=1}^{m}\frac{\delta_k^2\beta_k\log\mu_k}{\nu_k^2}$$
$$-\frac{1}{[\log r]^2}\sum_{k,l=2}^{m}\frac{\delta_k\delta_l\beta_l\beta_kR_{k,l}}{\nu_k\nu_l}+\mathscr{O}\big([\log r]^{-3}\big)$$
$$=-\frac{1}{\nu\log r}+\frac{M_1}{[\log r]^2}+\mathscr{O}\big([\log r]^{-3}\big),$$

where

$$M_1=\sum_{k=1}^{m}\frac{\delta_k^2\beta_k\log\mu_k}{\nu_k^2}-\sum_{k,l=2}^{m}\frac{\delta_k\delta_l\beta_l\beta_kR_{k,l}}{\nu_k\nu_l},\quad \nu=\left(\sum_{k=1}^{m}\frac{\delta_k^2\beta_k}{\nu_k}\right)^{-1}.$$

Taking the reciprocals we get:

$$\pi|C(r;G,Z,\Delta,\Psi)|=-\nu\log r\left[1-\frac{M_1\nu}{\log r}+\mathscr{O}\big([\log r]^{-2}\big)\right]^{-1}$$
$$=-\nu\log r\left[1+\frac{M_1\nu}{\log r}+\mathscr{O}\big([\log r]^{-2}\big)\right]$$
$$=-\nu\log r-M_1\nu^2+\mathscr{O}\big([\log r]^{-1}\big)$$

or

$$|C(r;G,Z,\Delta,\Psi)|+\frac{\nu}{\pi}\log r=-\frac{\nu^2}{\pi}M_1+\mathscr{O}\big([\log r]^{-1}\big).$$

Hence, by definition of the reduced modulus

$$M=-\frac{\nu^2}{\pi}\left(\sum_{k=1}^{m}\frac{\delta_k^2\beta_k\log\mu_k}{\nu_k^2}-\sum_{k,l=2}^{m}\frac{\delta_k\delta_l\beta_k\beta_lR_{k,l}}{\nu_k\nu_l}\right).$$

This proves the theorem under additional assumptions 1 and 2.

We now remove assumption 1. According to formula (33)

$$\sum_{k,l=2}^{m}\frac{\delta_k\delta_l\beta_k\beta_lR_{k,l}}{\nu_k\nu_l}=\sum_{k,l=1}^{m}\frac{\delta_k\delta_l\beta_k\beta_lN_{kl}}{\nu_k\nu_l}$$

which implies that the left-hand side of this formula is invariant with respect to renumbering of points z_k. Hence we may safely assume that $\delta_1\neq 0$. Suppose further that $\delta_l\neq 0$, $l=1,\dots,p$, and $\delta_l=0$, $l=p+1,\dots,m$. Denote $Z'=\{z_1,\dots,z_p\}$, $\Psi'=\{\psi_1,\dots,\psi_p\}$, $\Delta'=\{\delta_1,\dots,\delta_p\}$. Using these collections we can build the function $g_r(z)$ by (19). It will coincide with the potential function of the condenser

$$C'(r)=(G;\{E(z_1,\psi_1(r)),\dots,E(z_p,\psi_p(r)\},\Delta'),$$

inside its field. Here the sets $E(z_k,\psi_k(r))$, $k=1\dots,p$, are defined by (21) and (25). Set as before

$$C(r)=(G;\{E(z_1,\psi_1(r)),\dots,E(z_m,\psi_m(r))\},\Delta),$$

where $E(z_k,\psi_k(r))=D(z_k,\psi_k(r))\cap\overline{G}$ for $k=p+1,\dots,m$. The definition of capacity implies

$$\operatorname{cap}C'(r)\leq\operatorname{cap}C(r)$$

for all sufficiently small $r > 0$. The sets $E(z_k, R) := D(z_k, R) \cap \overline{G}$ do not intersect for sufficiently small $R > 0$ and we put

$$M_k(r) = \max\{|g_r(z)|\colon |z - z_k| = R\}, \quad k = p+1, \dots, m.$$

From (19) we have

$$M_k(r) = \mathscr{O}\left(\frac{1}{\log r}\right), \quad r \downarrow 0.$$

Choose r sufficiently small to have $\psi_k(r) < R$ for $k = p+1, \dots, m$ and define the auxiliary functions

$$f_k(z) = M_k(r) \frac{\log(|z - z_k|/\psi_k)}{\log(R/\psi_k)}$$

for z in the annuli $\psi_k(r) \le |z - z_k| \le R$. The function

$$h_k(z) = \max\{\min\{g_r(z), f_k(z)\}, -f_k(z)\}, \quad k = p+1, \dots, m,$$

is Lipschitz in $K_k(r, R) := G \cap \{\psi_k(r) \le |z - z_k| \le R\}$ (see, for instance, [10, Theorem 1]) and $h_k(z) = g_r(z)$ for $|z - z_k| = R$, since

$$-f_k(z) = -M_k \le g_r(z) \le M_k = f_k(z).$$

For $|z - z_k| = \psi_k(r)$ we have $f_k(z) = 0$ and hence $h_k(z) = 0$ irrespective of the sign of $g_r(z)$. Thus the function

$$h(z) := \begin{cases} g_r(z), & z \notin [\bigcup_{k=1}^{p} E(z_k, \psi_k(r))] \bigcup [\bigcup_{k=p+1}^{m} E(z_k, R)], \\ h_k(z), & z \in K_k(r, R), \ k \ge p+1, \\ \delta_k, & z \in E(z_k, \psi_k(r)), \ k = 1, \dots, m, \end{cases}$$

defined in G is admissible for the condenser $C(r)$. Then,

$$\begin{aligned} \operatorname{cap} C'(r) \le \operatorname{cap} C(r) &\le \iint\limits_{G} |\nabla h|^2 \, dx \, dy \\ &\le \iint\limits_{G \setminus \bigcup_{l=1}^{p} E(z_l, \psi_l(r))} |\nabla g_r|^2 \, dx \, dy + \sum_{k=p+1}^{m} \iint\limits_{K_k(r,R)} |\nabla f_k|^2 \, dx \, dy \\ &= \operatorname{cap} C'(r) + \sum_{k=p+1}^{m} \frac{2\pi M_k(r)^2}{\log(R/\psi_k(r))} = \operatorname{cap} C'(r) + \mathscr{O}\big([\log r]^{-3}\big), \quad r \downarrow 0. \end{aligned}$$

Since

$$\frac{1}{\pi} \operatorname{cap} C(r; D, Z, \Delta, \Psi) = -\frac{1}{\nu \log r} + \frac{M_1}{[\log r]^2} + \mathscr{O}\big([\log r]^{-3}\big),$$

we conclude that

$$M(G, Z, \Delta, \Psi) = M(G, Z', \Delta', \Psi')$$

implying that formula (18) can be applied without changes when some of δ_k are equal to zero.

It is left to remove assumption 2 (recall that $\delta_1 \ne 0$). To this end we shall apply the mapping $f(z) = az$ with positive a. The family of condensers $C(r; G, Z, \Delta, \Psi)$ will transform into the family $C(r; G', Z', \Delta, \Psi')$ with parameters

$$G' = aG, \quad Z' = \{az_k\}_{k=1}^{m}, \quad \Psi' = \{\mu_k' r^{\nu_k}\}, \quad \mu_k' = a\mu_k. \tag{27}$$

The constants $R'_{k,l}$ for the new configuration are related to those from the original one as follows:

$$R'_{k,k}=R_{k,k}+\frac{1}{\beta_1}\log a+\frac{1}{\beta_k}\log a,\quad R'_{k,l}=R_{k,l}+\frac{1}{\beta_1}\log a,\quad k\neq l. \tag{28}$$

Then

$$\sum_{l=1}^{m}\frac{\beta_l\delta'_l\log\mu'_l}{\nu_l'^2}-\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta'_lR'_{k,l}}{\nu'_l\nu'_k}=\sum_{l=1}^{m}\frac{\beta_l\delta_l(\log\mu_l+\log a)}{\nu_l^2}$$
$$-\sum_{\substack{k,l=2\\k\neq l}}^{m}\frac{\beta_l\beta_k\delta_l(R_{k,l}+\beta_1^{-1}\log a)}{\nu_l\nu_k}-\sum_{k=2}^{m}\frac{\beta_k^2\delta_k(R_{k,k}+\beta_1^{-1}\log a+\beta_k^{-1}\log a)}{\nu_k^2}$$
$$=\sum_{l=1}^{m}\frac{\beta_l\delta_l\log\mu_l}{\nu_l^2}-\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_lR_{k,l}}{\nu_l\nu_k}+\log a\left(\frac{\beta_1\delta_1}{\nu_1^2}-\frac{1}{\beta_1}\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_l}{\nu_l\nu_k}\right).$$

Using (17) the expression in parentheses becomes

$$\frac{\beta_1\delta_1}{\nu_1^2}-\frac{1}{\beta_1}\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_l}{\nu_l\nu_k}=\frac{\beta_1\delta_1}{\nu_1^2}-\frac{1}{\beta_1}\sum_{k=2}^{m}\frac{\beta_k}{\nu_k}\left(-\frac{\beta_1\delta_1}{\nu_1}\right)=\frac{\delta_1}{\nu_1}\sum_{k=1}^{m}\frac{\beta_k}{\nu_k}.$$

Hence, choosing a equal to

$$\exp\left\{\left(\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_lR_{k,l}}{\nu_l\nu_k}-\sum_{l=1}^{m}\frac{\beta_l\delta_l\log\mu_l}{\nu_l^2}\right)\left(\frac{\delta_1}{\nu_1}\sum_{k=1}^{m}\frac{\beta_k}{\nu_k}\right)^{-1}\right\}$$

we infer that condition (24) is valid for the new family $C(r;G',Z',\Delta,\Psi')$. It is left to show that the reduced modulus is invariant under the mapping $z'=az$. Indeed, by (18) and in view of (27) and (28) we get ($\nu'=\nu$):

$$-\frac{\pi}{\nu^2}M'=\sum_{k=1}^{m}\frac{\beta_k\delta_k^2\log a\mu_k}{\nu_k^2}-\sum_{\substack{k,l=2\\k\neq l}}^{m}\frac{\beta_l\beta_k\delta_l\delta_k(R_{k,l}+\beta_1^{-1}\log a)}{\nu_k\nu_l}$$
$$-\sum_{k=2}^{m}\frac{\beta_k^2\delta_k^2(R_{k,k}+\beta_1^{-1}\log a+\beta_k^{-1}\log a)}{\nu_k^2}$$
$$=-\frac{\pi}{\nu^2}M+\sum_{k=1}^{m}\frac{\beta_k\delta_k^2\log a}{\nu_k^2}-\frac{1}{\beta_1}\sum_{k,l=2}^{m}\frac{\beta_l\beta_k\delta_k\delta_l\log a}{\nu_k\nu_l}-\sum_{k=2}^{m}\frac{\beta_k\delta_k^2\log a}{\nu_k^2}$$
$$=-\frac{\pi}{\nu^2}M+\frac{\beta_1\delta_1^2\log a}{\nu_1^2}-\frac{\log a}{\beta_1}\left(\sum_{k=2}^{m}\frac{\beta_k\delta_k}{\nu_k}\right)^2=-\frac{\pi}{\nu^2}M,$$

where we used (17) in the last equality.

Finally, let us see what happens with the reduced modulus when condition (17) is violated. In this case we can shift all δ_k to satisfy (17):

$$\tilde{\delta}_l=\delta_l-\gamma,\quad \gamma=\frac{\sum_{l=1}^{m}(\beta_l\delta_l/\nu_l)}{\sum_{l=1}^{m}(\beta_l/\nu_l)},\quad \tilde{\Delta}=\{\tilde{\delta}_1,\ldots,\tilde{\delta}_m\},$$

and $\sum_{l=1}^m(\beta_l\tilde\delta_l/\nu_l)=0$. Define

$$\tilde\nu=\left(\sum_{l=1}^m\frac{\beta_l\tilde\delta_l^2}{\nu_l}\right)^{-1}.$$

Direct computation shows that $\tilde\nu>\nu$. The definition of capacity gives

$$\operatorname{cap}C(r;G,Z,\tilde\Delta,\Psi)=\operatorname{cap}C(r;G,Z,\Delta,\Psi),$$

since for every function v admissible for the second condenser, the function $v-\gamma$ is admissible for the first and $\nabla v=\nabla(v-\gamma)$. By what we have proved so far

$$|C(r;G,Z,\tilde\Delta,\Psi)|+\frac{\tilde\nu}{\pi}\log r=M+o(1),\quad r\downarrow 0,$$

which implies

$$|C(r;G,Z,\Delta,\Psi)|+\frac{\nu}{\pi}\log r\to+\infty,\quad r\downarrow 0. \qquad \square$$

6. Computation of the reduced modulus for general domains

Theorem 2. *Suppose $G\subset\overline{\mathbf{C}}_z$ is a finitely connected domain, $Z=\{z_k\}_{k=1}^m$ comprises admissible points of $\overline{G}$, $\beta_k=\beta_G(z_k)$ ($\beta_k=2$ for inner points) and the collections $\Delta=\{\delta_k\}_{k=1}^m$, $\Psi=\{\mu_k r^{\nu_k}\}$ satisfy* (17). *Then the reduced modulus* (3) *exists and is found from the formula* (18), *where ν is defined by* (4). *If condition* (17) *is violated the modulus is infinite.*

Proof. Let f be the univalent conformal mapping of G onto an analytic Jordan domain G^*. We will use asterisk to denote quantities associated with the domain G^*. According to (1) the mapping f in the neighborhood of z_k has the form

$$f(z)-f(z_k)=(z-z_k)^{\beta_k^*/\beta_k}(c_k+o(1)),\quad z\to z_k, \tag{29}$$

for finite z_k or

$$f(z)-f(z_k)=(1/z)^{\beta_k^*/\beta_k}(c_k+o(1)),\quad z\to z_k, \tag{30}$$

if $z_k=\infty$. These expansions and the definition (15) of bipolar Neumann function for general domains yield

$$\begin{aligned}R_{k,l}^*&=R_{k,l}+\frac{1}{\beta_1^*}\log|c_1|,\quad k\neq l,\\ R_{k,k}^*&=R_{k,k}+\frac{1}{\beta_1^*}\log|c_1|+\frac{1}{\beta_k^*}\log|c_k|,\end{aligned} \tag{31}$$

where the constants $R_{k,l}$ are defined in (16). Indeed, by (15) and (16)

$$R_{k,l}=v_G(z_k,z_l|z_1)=v_{G^*}(f(z_k),f(z_l)|f(z_1))-\frac{1}{\beta_1^*}\log|c_1|=R_{k,l}^*-\frac{1}{\beta_1^*}\log|c_1|$$

for $k \neq l$. Further, for $k = l$, $z_k \neq \infty$, we have by (11), (15), (16) and (29):

$$\begin{aligned}
R_{k,k} &= \lim_{z\to z_k} \left[v_G(z, z_k|z_1) + \beta_k^{-1}\log|z - z_k|\right] \\
&= \lim_{z\to z_k} \left[v_{G^*}(f(z), f(z_k)|f(z_1)) - \frac{1}{\beta_1^*}\log|c_1| + \beta_k^{-1}\log|z - z_k|\right] \\
&= \lim_{z\to z_k} \left[-(\beta_k^*)^{-1}\log|f(z) - f(z_k)| + R^*_{k,k} - \frac{1}{\beta_1^*}\log|c_1| + \beta_k^{-1}\log|z - z_k| + o(1)\right] \\
&= \lim_{z\to z_k} \Big[-(\beta_k^*)^{-1}\log|(z - z_k)^{\beta_k^*/\beta_k}(c_k + o(1))| + R^*_{k,k} \\
&\quad -\frac{1}{\beta_1^*}\log|c_1| + \beta_k^{-1}\log|z - z_k| + o(1)\Big] = R^*_{k,k} - \frac{1}{\beta_k^*}\log|c_k| - \frac{1}{\beta_1^*}\log|c_1|.
\end{aligned}$$

The same relation is obtained for $z_k = \infty$.

The set $E(z_k, \psi_k(r); G)$ is mapped onto the set $E(f(z_k), \psi_k^*(r); G^*)$ where

$$\psi_k^*(r) = \mu_k^{\beta_k^*/\beta_k}|c_k|r^{\nu_k\beta_k^*/\beta_k}.$$

Hence, the image $C(r; G^*; Z^*, \Delta^*, \Psi^*)$ of the condenser $C(r; G; Z, \Delta, \Psi)$ under f consists of the following collections:

$$Z^* = \{f(z_k)\}_{k=1}^m, \quad \Delta^* = \Delta, \quad \Psi^* = \{\mu_k^* r^{\nu_k^*}\}_{k=1}^m,$$

where $\mu_k^* = \mu_k^{\beta_k^*/\beta_k}|c_k|$, $\nu_k^* = \nu_k\beta_k^*/\beta_k$. A straightforward computation shows that $\nu^* = \nu$. Conformal invariance of capacity implies

$$\begin{aligned}
M(G, Z, \Delta, \Psi) &= \lim_{r\downarrow 0}\left(|C(r; G, Z, \Delta, \Psi)| + \frac{\nu}{\pi}\log r\right) \\
&= \lim_{r\downarrow 0}\left(|C(r; G^*, Z^*, \Delta^*, \Psi^*)| + \frac{\nu^*}{\pi}\log r\right) = M(G^*, Z^*, \Delta^*, \Psi^*).
\end{aligned}$$

Thus by Theorem 1

$$\begin{aligned}
-\frac{\pi}{\nu^2}M(G, Z, \Delta, \Psi) &= \sum_{k=1}^m \frac{\beta_k^*\delta_k^2\log\mu_k^*}{(\nu_k^*)^2} - \sum_{k,l=2}^m \frac{\beta_k^*\beta_l^*\delta_k\delta_l R^*_{k,l}}{\nu_k^*\nu_l^*} \\
= \sum_{k=1}^m &\frac{\beta_k^*\delta_k^2((\beta_k^*/\beta_k)\log\mu_k + \log|c_k|)}{(\nu_k\beta_k^*/\beta_k)^2} - \sum_{k,l=2}^m \frac{\beta_k^*\beta_l^*\delta_k\delta_l(R_{k,l} + \frac{1}{\beta_1^*}\log|c_1|)}{\nu_k\nu_l\beta_k^*\beta_l^*/(\beta_k\beta_l)} \\
-\sum_{k=2}^m &\frac{(\beta_k^*)^2\delta_k^2(\frac{1}{\beta_k^*}\log|c_k|)}{\nu_k^2(\beta_k^*)^2/\beta_k^2} \\
= \sum_{k=1}^m &\frac{\beta_k\delta_k^2\log\mu_k}{\nu_k^2} - \sum_{k,l=2}^m \frac{\beta_k\beta_l\delta_k\delta_l R_{k,l}}{\nu_k\nu_l} + \frac{\log|c_1|}{\beta_1^*}\left(\frac{\beta_1^2\delta_1^2}{\nu_1^2} - \sum_{k,l=2}^m \frac{\beta_k\beta_l\delta_k\delta_l}{\nu_k\nu_l}\right).
\end{aligned}$$

The expression in parentheses vanishes due to (17). □

Formula (18) is not symmetric in $z_1, z_2, \ldots, z_m$ due to a special role played by z_1 in the definition of the bipolar Neumann function normalized at z_1. On the other hand the definition of the reduced modulus (3) is symmetric in all points $z_1, z_2, \ldots, z_m$. Hence it seems desirable to find a symmetric form of formula (18). This can be achieved by employing representations (13) or (14). For the generalized Neumann function N_G defined in section 2 denote

$$N_{kl} = N_G(z_k, z_l),\ k \neq l; \quad N_{kk} = N(z_k),$$

where the constant $N(z_k)$ is taken form expansion (9) or (10). Then according to (13) and (16):

$$R_{k,l} = N_{kl} - N_{k1} - N_{1l} + N_{11}, \quad k, l = 2, \dots, m. \tag{32}$$

Substituting this relation into (18) and repeatedly using (17) we obtain:

$$\begin{aligned}
\frac{\pi}{\nu^2} M + \sum_{k=1}^{m} \frac{\beta_k \delta_k^2 \log \mu_k}{\nu_k^2} &= \sum_{k,l=2}^{m} \frac{\beta_k \beta_l \delta_k \delta_l (N_{kl} - N_{k1} - N_{1l} + N_{11})}{\nu_k \nu_l} \\
&= \sum_{k,l=2}^{m} \frac{\beta_k \beta_l \delta_k \delta_l N_{kl}}{\nu_k \nu_l} + \frac{\beta_1 \delta_1}{\nu_1} \sum_{k=2}^{m} \frac{\beta_k \delta_k N_{k1}}{\nu_k} + \frac{\beta_1 \delta_1}{\nu_1} \sum_{l=2}^{m} \frac{\beta_l \delta_l N_{1l}}{\nu_l} + \frac{\beta_1^2 \delta_1^2}{\nu_1^2} N_{11} \\
&= \sum_{k,l=1}^{m} \frac{\beta_k \beta_l \delta_k \delta_l N_{kl}}{\nu_k \nu_l}
\end{aligned}$$

or

$$M = -\frac{\nu^2}{\pi} \left(\sum_{k=1}^{m} \frac{\beta_k \delta_k^2 \log \mu_k}{\nu_k^2} - \sum_{k,l=1}^{m} \frac{\beta_k \beta_l \delta_k \delta_l N_{kl}}{\nu_k \nu_l} \right) \tag{33}$$

which is the desired representation.

7. The reduced moduli of some canonical domains

1) Consider the unit disk $U = \{z \colon |z| < 1\}$ and a collection of points z_k, $k = 1, \dots, m$, located either in U or on ∂U. The classical Neumann function of U is given by (see [18, page 272]):

$$N_U(z, \zeta) = \frac{1}{2} \log \frac{1}{|z - \zeta||1 - \overline{z}\zeta|}. \tag{34}$$

This definition extends to the case $|\zeta| = 1$ without changes. Hence, we have

$$N_{kl} = -\frac{1}{2} \log |z_k - z_l||1 - \overline{z_k} z_l|,\ k \neq l, \quad N_{kk} = \begin{cases} -\frac{1}{2} \log(1 - |z_k|^2), & z_k \in U, \\ 0, & z_k \in \partial U. \end{cases}$$

So by (33):

$$\begin{aligned}
M = -\frac{\nu^2}{\pi} \sum_{k=1}^{m} \frac{\beta_k \delta_k^2 \log \mu_k}{\nu_k^2} &- \frac{\nu^2}{2\pi} \sum_{\substack{k,l=1 \\ k \neq l}}^{m} \frac{\beta_k \beta_l \delta_k \delta_l}{\nu_k \nu_l} \log |z_k - z_l||1 - \overline{z_k} z_l| \\
&- \frac{\nu^2}{2\pi} {\sum_{k=1}^{m}}' \frac{\beta_k^2 \delta_k^2}{\nu_k^2} \log(1 - |z_k|^2),
\end{aligned}$$

where the prime at the summation sign here and henceforth means that the infinite terms are omitted. As before $\beta_k = 2$ if $z_k \in U$ and $\beta_k = 1$ if $z_k \in \partial U$.

2) Consider the upper half-plane $\Pi^+ = \{z\colon \Im z > 0\}$ and a collection of points z_k, $k = 1, \dots, m$, lying either inside Π^+ or on its boundary. The function $f(z) = (z-i)/(z+i)$ maps Π^+ onto U. It follows that

$$\begin{aligned} N_{\Pi^+}(z,\zeta) &= -\frac{1}{2}\log|f(z)-f(\zeta)||1-\overline{f(z)}f(\zeta)| \\ &= -\frac{1}{2}\log|z-\zeta||z-\overline{\zeta}| + \log|z+i||\zeta+i| - \log 2 \end{aligned}$$

is the generalized Neumann function of Π^+. As shown in section 3 we may omit the constant and put

$$N_{\Pi^+}(z,\zeta) = -\frac{1}{2}\log|z-\zeta||z-\overline{\zeta}| + \log|z+i||\zeta+i|.$$

Hence, the constants N_{kl} equal

$$\begin{aligned} N_{kl} &= -\frac{1}{2}\log|z_k-z_l||z_k-\overline{z_l}| + \log|z_k+i||z_l+i|, \quad k \neq l, \\ N_{kk} &= \begin{cases} -\frac{1}{2}\log|z_k-\overline{z_k}| + 2\log|z_k+i|, & z_k \in \Pi^+, \\ 2\log|z_k+i|, & z_k \in \partial\Pi^+. \end{cases} \end{aligned} \tag{35}$$

Then by (33) and using (17) we get:

$$M = -\frac{\nu^2}{\pi}\left(\sum_{k=1}^{m}\frac{\beta_k\delta_k^2\log\mu_k}{\nu_k^2} + \frac{1}{2}\sum_{\substack{k,l=1\\k\neq l}}^{m}\frac{\beta_k\beta_l\delta_k\delta_l}{\nu_k\nu_l}\log|z_k-z_l||z_k-\overline{z_l}| \right.$$
$$\left. + \frac{1}{2}{\sum_{k=1}^{m}}'\frac{\beta_k^2\delta_k^2}{\nu_k^2}\log|z_k-\overline{z_k}|\right),$$

where $\beta_k = 2$ if $z_k \in \Pi^+$ and $\beta_k = 1$ if $z_k \in \partial\Pi^+$.

3) The reduced modulus of a simply-connected domain. For a finite point z and an arbitrary point $\zeta \in \overline{\mathbf{C}}_z$ define $d(z,\zeta) := |z-\zeta|$ if $\zeta \neq \infty$ and $d(z,\zeta) := 1/|z|$ if $\zeta = \infty$. Let B be a simply connected hyperbolic domain and suppose that Z is a collection of admissible points while Δ and Ψ are as above. Denote by $f(z)$ the univalent conformal mapping of B onto the upper half-plane Π^+, $w_k = f(z_k)$, $W = \{w_k\}$, $k = 1, 2, ..., m$. We assume that for boundary points $z_k \in Z$

$$d(f(z), f(z_k)) \sim c_k d(z, z_k)^{\beta_k^*/\beta_k} \text{ as } z \to z_k \text{ in } B,$$

where $\beta_k\pi > 0$ is the inner angle of B at z_k, $\beta_k^* = 1$. For inner points $z_k \in Z$ we have $\beta_k = \beta_k^* = 2$, $c_k = |f'(z_k)|$ as before. For the reduced modulus to be finite we must assume that condition (17) holds. Put $\nu_k^* = \nu_k\beta_k^*/\beta_k$, $\mu_k^* = \mu_k^{\beta_k^*/\beta_k}|c_k|$, $\Psi^* = \mu_k^* r^{\nu_k^*}$. Similarly to the proof of Theorem 2 we check that

$$M(B, Z, \Delta, \Psi) = M(\Pi^+, W, \Delta, \Psi^*).$$

Hence,

$$M(B, Z, \Delta, \Psi) = -\frac{\nu^2}{\pi}\left(\sum_{k=1}^{m} \frac{\beta_k \delta_k^2 \log \mu_k}{\nu_k^2} + \sum_{k=1}^{m} \frac{\beta_k^2 \delta_k^2}{\nu_k^2 \beta_k^*} \log |c_k| \right.$$

$$(36) \qquad + \frac{1}{2} \sum_{\substack{k,l=1 \\ k \neq l}}^{m} \frac{\beta_k \beta_l \delta_k \delta_l}{\nu_k \nu_l} \log |w_k - w_l||w_k - \overline{w}_l|$$

$$\left. + \frac{1}{2} \sum_{k=1}^{m}{}' \frac{\beta_k^2 \delta_k^2}{\nu_k^2} \log |w_k - \overline{w_k}| \right).$$

This formula was earlier derived in [6, Theorem 4] using reflection principle and reduced modulus of the complex sphere.

4) Using formula (36) Dubinin and Eyrikh found in [6, Examples 4.2, 4.3, page 159] explicit expressions for the reduced moduli of triangles and rectangles with respect to their corners. The reduced modulus of a triangle T is given by

$$M(T, Z, \Delta, \Psi) = -\frac{1}{\pi}\left(\sum_{k=1}^{3} \beta_k \delta_k^2 / \nu_k\right)^{-2} \log \prod_{k=1}^{3} (\mu_k \beta_k B(\beta_1, \beta_2))^{\beta_k \delta_k^2 / \nu_k^2},$$

where $Z = \{z_k\}_{k=1}^{3}$, $z_1 = 0$, $z_2 = 1$, $\Im z_3 > 0$, T is the triangle with vertices Z and inner angles $\beta_k \pi > 0$, $k = 1, 2, 3$. The collections Δ and Ψ are from the definition of the reduced modulus and such that condition (17) is satisfied. $B(\beta_1, \beta_2)$ is Euler's beta function.

The reduced modulus of the rectangle R (described below) is found from

$$M(R, Z, \Delta, \Psi) = -\frac{2}{\pi}\left(\sum_{k=1}^{4} \frac{\delta_k^2}{\nu_k}\right)^{-2} \log \left\{ 2^{\frac{\delta_1 \delta_4}{\nu_1 \nu_4}} \left(\frac{2}{\lambda}\right)^{\frac{\delta_2 \delta_3}{\nu_2 \nu_3}} \left(\frac{1}{\lambda} - 1\right)^{\frac{\delta_1 \delta_2}{\nu_1 \nu_2} + \frac{\delta_3 \delta_4}{\nu_3 \nu_4}} \right.$$

$$\left. \cdot \left(\frac{1}{\lambda} + 1\right)^{\frac{\delta_1 \delta_3}{\nu_1 \nu_3} + \frac{\delta_2 \delta_4}{\nu_2 \nu_4}} \prod_{k=1}^{4} (\mu_k \sqrt{c_k})^{(\delta_k / \nu_k)^2} \right\}.$$

Here the vertices of R are located at the points $z_1 = K(\lambda)$, $z_2 = K(\lambda) + iK'(\lambda)$, $z_3 = -K(\lambda) + iK'(\lambda)$, $z_4 = -K(\lambda)$, where $K(\lambda)$ is the complete elliptic integral of the first kind, $K'(\lambda) = K(\sqrt{1-\lambda^2})$. The constants are defined by $c_1 = c_4 = (1-\lambda^2)/2$, $c_2 = c_3 = (1-\lambda^2)/(2\lambda)$.

5) Consider the annulus $A = \{z : \mu < |z| < 1\}$ and a collection of inner points $z_k \in A$, $k = 1, \ldots, m$. We need to compute a generalized Neumann function of A. Let us begin with the Neumann function of the unit disk (34). It is rotation invariant:

$$(37) \qquad N_U(ze^{i\theta}, \zeta e^{i\theta}) = N_U(z, \zeta).$$

Hence, we may assume $\zeta = \tau \in (0, 1)$ without loss of generality. When $\rho < \tau$ we get by expanding logarithms:

$$2N_U(\rho e^{i\alpha}, \tau) = -\log |\rho e^{i\alpha} - \tau| - \log |1 - \rho e^{-i\alpha} \tau|$$

$$= \log(1/\tau) + \sum_{n=1}^{\infty} \rho^n (\tau^n + \tau^{-n}) \cos(\alpha n)/n.$$

The normal derivative of this function at the points $|z| = \mu$ is given by

$$\frac{\partial N_U(\rho e^{i\alpha}, \tau)}{\partial \rho} = \frac{1}{2}\sum_{n=1}^{\infty} \rho^{n-1}\cos(\alpha n)(\tau^n + \tau^{-n})|_{\rho=\mu}. \tag{38}$$

Suppose from here on that $\tau \in (\mu, 1)$. To build a generalized Neumann function of A we need a harmonic function $M(\rho e^{i\alpha}, \tau)$ having the following normal derivatives at the boundary of A:

$$\frac{\partial M(\rho e^{i\alpha}, \tau)}{\partial \rho} = -\frac{\partial N_U(\rho e^{i\alpha}, \tau)}{\partial \rho}$$

for $\rho = \mu$ and

$$\frac{\partial M(\rho e^{i\alpha}, \tau)}{\partial \rho} = 0$$

for $\rho = 1$. Then $M(z, \tau) + N_U(z, \tau)$ gives a generalized Neumann function of A. Direct computation shows that the function

$$M(\rho e^{i\alpha}, \tau) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{\mu^{n-1}(\tau^n + \tau^{-n})}{n(\mu^{n-1} - \mu^{-n-1})}(\rho^n + \rho^{-n})\cos(\alpha n), \ \mu \le \rho \le 1, \ 0 \le \alpha \le 2\pi,$$

has the required properties. Hence,

$$\begin{aligned} 2N_A(\rho e^{i\alpha}, \tau) = &- \log|\rho e^{i\alpha} - \tau||1 - \rho e^{-i\alpha}\tau| \\ &+ \sum_{n=1}^{\infty} \frac{\mu^{2n}(\tau^n + \tau^{-n})}{n(1 - \mu^{2n})}(\rho^n + \rho^{-n})\cos(\alpha n). \end{aligned} \tag{39}$$

When $\rho < \tau$ this function can be written as the series

$$2N_A(\rho e^{i\alpha}, \tau) = \log(1/\tau) - \sum_{n=1}^{\infty} \frac{\cos(\alpha n)(\tau^n + \tau^{-n})}{n(\mu^n - \mu^{-n})}[(\rho/\mu)^n + (\rho/\mu)^{-n}].$$

When $\rho > \tau$ the variables ρ and τ swap roles:

$$2N_A(\rho e^{i\alpha}, \tau) = \log(1/\rho) - \sum_{n=1}^{\infty} \frac{\cos(\alpha n)(\rho^n + \rho^{-n})}{n(\mu^n - \mu^{-n})}[(\tau/\mu)^n + (\tau/\mu)^{-n}].$$

We can express $N_A(\rho e^{i\alpha}, \tau)$ in terms of Jacobi's theta function

$$\begin{aligned} \vartheta_1(z; q) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{i(2n+1)z} \\ &= 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z \\ &= 2q^{1/4}\sin z \prod_{n=1}^{\infty}(1 - q^{2n})(1 - q^{2n}e^{2iz})(1 - q^{2n}e^{-2iz}). \end{aligned}$$

To this end we compute

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{2n}(x^n e^{i\alpha n}+x^{-n}e^{-i\alpha n})}{n(1-q^{2n})} &= \sum_{n=1}^{\infty}\frac{q^{2n}(x^n e^{i\alpha n}+x^{-n}e^{-i\alpha n})}{n}\sum_{k=0}^{\infty}q^{2nk}\\ &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(q^{2k+2})^n(x^n e^{i\alpha n}+x^{-n}e^{-i\alpha n})}{n}\\ &= -\sum_{k=0}^{\infty}\log(1-q^{2k+2}xe^{i\alpha})(1-q^{2k+2}x^{-1}e^{-i\alpha})\\ &= -\log\prod_{m=1}^{\infty}(1-q^{2m}xe^{i\alpha})(1-q^{2m}x^{-1}e^{-i\alpha})\\ &= \log\frac{2q^{1/4}\sin\left(\frac{\alpha-i\log x}{2}\right)\prod_{k=1}^{\infty}(1-q^{2k})}{\vartheta_1\left(\frac{\alpha-i\log x}{2}\right)}.\end{aligned}$$

Using this formula we obtain after some calculations:

$$2N_A(\rho e^{i\alpha},\tau)=\log\frac{4\mu^{1/2}\prod_{k=1}^{\infty}(1-\mu^{2k})^2\left|\sin\left(\frac{\alpha-i\log(\rho\tau)}{2}\right)\sin\left(\frac{\alpha-i\log(\rho/\tau)}{2}\right)\right|}{\left|\vartheta_1\left(\frac{\alpha-i\log(\rho\tau)}{2};\mu\right)\vartheta_1\left(\frac{\alpha-i\log(\rho/\tau)}{2};\mu\right)\right|}.$$

Applying

$$\begin{aligned}\sin\left(\frac{-i}{2}\log(z\overline{\zeta})\right) &= \frac{z\overline{\zeta}-1}{2i\sqrt{z\overline{\zeta}}},\\ \sin\left(\frac{-i}{2}\log(z/\zeta)\right) &= \frac{z-\zeta}{2i\sqrt{z\zeta}}\end{aligned}\tag{40}$$

we get:

$$\begin{aligned}2N_A(z,\zeta) &= \log\frac{\mu^{1/2}\prod\limits_{k=1}^{\infty}(1-\mu^{2k})^2|(z\overline{\zeta})^{1/2}-(z\overline{\zeta})^{-1/2}||(z/\zeta)^{1/2}-(z/\zeta)^{-1/2}|}{|z-\zeta||1-z\overline{\zeta}|}\\ &\quad-\log\left|\vartheta_1(-i\log(z\overline{\zeta})/2;\mu)\,\vartheta_1(-i\log(z/\zeta)/2;\mu)\right|\\ &= \log\frac{\mu^{1/2}\prod\limits_{k=1}^{\infty}(1-\mu^{2k})^2}{|z\overline{\zeta}|}-\log\left|\vartheta_1(-i\log(z\overline{\zeta})/2;\mu)\,\vartheta_1(-i\log(z/\zeta)/2;\mu)\right|.\end{aligned}$$

The first term here is a sum of harmonic function (in z) and a constant and therefore it can be omitted by Lemma 2. Thus, finally in view of the formula $\vartheta_1(-w)=-\vartheta_1(w)$ we get:

$$N_A(z,\zeta)=-\frac{1}{2}\log\left|\vartheta_1(i\log(z\overline{\zeta})/2;\mu)\,\vartheta_1(i\log(z/\zeta)/2;\mu)\right|.\tag{41}$$

Using (40) this formula can be cast into the form

$$N_A(z,\zeta) = -\frac{1}{2}\log|z-\zeta| + \frac{1}{2}\log\left|\frac{z\overline{\zeta}}{1-z\overline{\zeta}}\right| - \frac{1}{2}\log\left|\frac{\vartheta_1\big(\frac{i}{2}\log(z\overline{\zeta});\mu\big)\,\vartheta_1\big(\frac{i}{2}\log(z/\zeta);\mu\big)}{4\sin\left(\frac{i}{2}\log(z\overline{\zeta})\right)\sin\left(\frac{i}{2}\log(z/\zeta)\right)}\right|. \tag{42}$$

This yields the following expansion in the neighborhood of $z=\zeta$:

$$N_A(z,\zeta) = -\frac{1}{2}\log|z-\zeta| + N(\zeta) + o(1), \quad z\to\zeta,$$
$$N(\zeta) = \frac{1}{2}\log\left|\frac{4|\zeta|^2\sin(i\log|\zeta|)}{(1-|\zeta|^2)\vartheta_1(i\log|\zeta|;\mu)\,\vartheta_1'(0;\mu)}\right|.$$

Hence, finally the reduced modulus of A with respect to inner points in found from (33) with

$$N_{kl} = \begin{cases} -\frac{1}{2}\log|\vartheta_1(i\log(z_k\overline{z_l})/2;\mu)\,\vartheta_1(i\log(z_k/z_l)/2;\mu)|\,, & k\neq l \\ \frac{1}{2}\log|4|z_k|^2\sin(i\log|z_k|)\,/[(1-|z_k|^2)\vartheta_1(i\log|z_k|;\mu)\,\vartheta_1'(0;\mu)]|\,, & k=l. \end{cases} \tag{43}$$

8. Applications

The applications below and most other potential applications hinge on some sort of monotonic behavior of the reduced modulus under certain transformations of the underlying domain. Two types of such monotonic behavior will be required here. First, the reduced modulus $M(G,Z,\Delta,\Psi)$ is non-increasing under the expansion of G. Indeed, consider $M(G',Z,\Psi,\Delta)$, where $G\subset G'$ and all boundary points $z_k\in Z$ together with their respective angles $\beta_k\pi>0$ remain fixed and belong to the boundary of G'. Restriction to G of any function admissible for the condenser $C(r;G',Z,\Delta,\Psi)$ yields a function admissible for $C(r;G,Z,\Delta,\Psi)$. This implies that the class of admissible functions for $C(r;G,Z,\Delta,\Psi)$ in not smaller than that for $C(r;G',Z,\Delta,\Psi)$ and hence

$$\operatorname{cap}C(r;G,Z,\Delta,\Psi) \le \operatorname{cap}C(r;G',Z,\Delta,\Psi)$$
$$\Leftrightarrow\ |C(r;G,Z,\Delta,\Psi)| \ge |C(r;G',Z,\Delta,\Psi)|.$$

In view of the definition (3) of the reduced modulus passing to the limit as $r\downarrow 0$ in this inequality gives

$$M(G',Z,\Delta,\Psi) \le M(G,Z,\Delta,\Psi). \tag{44}$$

Another type of monotonicity appears when our reduced modulus is compared with the reduced modulus $M(G,\Gamma,Z,\Delta,\Psi)$ introduced by Dubinin in [5, Theorem 7]. It is defined by the same formula (3) with condenser (2) replaced by the condenser

$$\begin{aligned} &C(r;\Gamma;G,Z,\Delta,\Psi) \\ &= (G;\{\Gamma,E(z_1,\psi_1(r)),E(z_2,\psi_2(r)),\dots,E(z_m,\psi_m(r))\},\{0,\delta_1,\dots,\delta_m\}), \end{aligned} \tag{45}$$

where Γ is a closed non-empty subset of ∂G. Now if $G\subset G'$ and $G'\cap\partial G\subset\Gamma$ then mimicking the reasoning above shows that

$$M(G',Z,\Delta,\Psi) \ge M(G,\Gamma,Z,\Delta,\Psi). \tag{46}$$

Denote by $\mathscr{M}(R)$ the class of functions meromorphic and univalent in the ring $K := \{z : 1 < |z| < R\}$, $1 < R < \infty$, such that $f(K) \subset D$, where $D := \{z : |z| > 1\}$ and $f(\partial D) = \partial D$.

Theorem 3. *Let $f \in \mathscr{M}(R)$ and $a, b \in K$ are arbitrary. Then*

$$\begin{aligned}(47)\qquad &\frac{|f'(a)f'(b)|(|f(a)|^2-1)(|f(b)|^2-1)}{|f(b)-f(a)|^2|\overline{f(a)}f(b)-1|^2}\\ &\geq \frac{\vartheta_1'(0;\mu)^2(1-|a|^2)(1-|b|^2)|\vartheta_1(i\log|a|;\mu)\,\vartheta_1(i\log|b|;\mu)\,|}{16|a|^2|b|^2|\sinh(\log|a|)\sinh(\log|b|)\,||\vartheta_1\big(i\log(a\overline{b})/2;\mu\big)\,\vartheta_1(i\log(a/b)/2;\mu)\,|^2},\end{aligned}$$

where $\mu = 1/R$ and ϑ_1 is Jacobi's theta-function.

Proof. Suppose $Z = \{a, b\}$, $\Delta = \{+1, -1\}$, $\Psi = \{r, r\}$, $W = \{f(a), f(b)\}$. By a straightforward computation $\nu = 1/4$ and

$$\begin{aligned}M(K, Z, \Delta, \Psi) &= M(f(K), W, \Delta, \{r|f'(a)|, r|f'(b)|\})\\ &= M(f(K), W, \Delta, \{r, r\}) - \frac{1}{8\pi}\log|f'(a)f'(b)|\end{aligned}$$

according to (33). Further by inequality (44):

$$\begin{aligned}&M(f(K), W, \Delta, \Psi) \geq M(D, W, \Delta, \Psi) = \frac{1}{4\pi}\sum_{k,l=1}^{2}\delta_k\delta_l N_{lk}^D\\ &= \frac{1}{8\pi}\Big(-\log(|f(a)|^2-1)(|f(b)|^2-1) + 2\log|(f(b)-f(a))(\overline{f(a)}f(b)-1)|\Big)\\ &= \frac{1}{8\pi}\log\frac{|f(b)-f(a)|^2|\overline{f(a)}f(b)-1|^2}{(|f(a)|^2-1)(|f(b)|^2-1)}.\end{aligned}$$

According to (33) we have

$$M(K, Z, \Delta, \Psi) = \frac{1}{4\pi}(N_{11}^K + N_{22}^K - 2N_{12}^K).$$

Combining the above inequalities with the last formula we obtain:

$$\begin{aligned}&2(N_{11}^K + N_{22}^K - 2N_{12}^K) + \log|f'(a)f'(b)| \geq \log\frac{|f(b)-f(a)|^2|\overline{f(a)}f(b)-1|^2}{(|f(a)|^2-1)(|f(b)|^2-1)}\\ &\Rightarrow \frac{|f'(a)f'(b)|(|f(a)|^2-1)(|f(b)|^2-1)}{|f(b)-f(a)|^2|\overline{f(a)}f(b)-1|^2} \geq \exp(-2N_{11}^K - 2N_{22}^K + 4N_{12}^K).\end{aligned}$$

The constants N_{ij}^K can be computed from formula (43). Indeed, the generalized Neumann functions of the annuli A and K are equal for $\mu = 1/R$, since, in view of $\vartheta_1(-w) = -\vartheta_1(w)$, we have

$$\begin{aligned}(48)\qquad N_K(z,\zeta) = N_A(1/z, 1/\zeta) &= -\frac{1}{2}\log\left|\vartheta_1\big(-i\log(z\overline{\zeta})/2;\mu\big)\,\vartheta_1(-i\log(z/\zeta)/2;\mu)\right|\\ &= -\frac{1}{2}\log\left|\vartheta_1\big(i\log(z\overline{\zeta})/2;\mu\big)\,\vartheta_1(i\log(z/\zeta)/2;\mu)\right|.\end{aligned}$$

Hence, we get by (43):

$$\exp(-2N_{11}^K - 2N_{22}^K + 4N_{12}^K)$$
$$= \frac{\vartheta_1'(0;\mu)^2(1-|a|^2)(1-|b|^2)|\vartheta_1(i\log|a|;\mu)\,\vartheta_1(i\log|b|;\mu)\,|}{16|a|^2|b|^2|\sinh(\log|a|)\sinh(\log|b|)\,||\vartheta_1\big(i\log(a\bar{b})/2;\mu\big)\,\vartheta_1(i\log(a/b)/2;\mu)\,|^2}. \quad \square$$

The upper bound for the product $|f'(a)f'(b)|$ is given in [8, Theorem 3.2].

Duren and Schiffer showed in [14, p. 194] that for a domain B containing the point at infinity and bounded by a finite number of smooth Jordan curves, any distinct points z_1, z_2, …, z_k of B and arbitrary real parameters δ_1, δ_2, …, δ_k the inequality

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\delta_k\delta_l\,[g_{B'}(z_k,z_l) - g_B(z_k,z_l;\Gamma)] \le 0 \tag{49}$$

holds. Here Γ is a non-empty closed subset of ∂B comprising a finite number of non-degenerate connected components and B' is the unbounded component of $\overline{\mathbf{C}}\backslash\Gamma$. The functions $g_{B'}(z,\zeta)$ and $g_B(z,\zeta;\Gamma)$ are Green and Robin functions of their corresponding domains, respectively [14]. For $k=l$ both functions in brackets are infinite but their difference is to be interpreted as the appropriate limit. Inequality (49) is sharp if we allow domains with slits along analytic arcs. This inequality is a generalization of the inequality between Robin and logarithmic capacities. Relation (49) can also be obtained using the reduced moduli [12, Proposition 1, Section 4].

Here we compare quadratic forms in the Neumann and Robin functions. Suppose a domain $B \subset \overline{\mathbf{C}}$, $\infty \in B$ is bounded by a finite number of piecewise analytic curves. Let Γ be a non-empty closed subset of ∂B comprising a finite number of non-degenerate connected components. Here ∂B is understood as a collection accessible boundary points of B. Suppose $\Gamma_1 = \overline{\partial B\backslash\Gamma}$ and B_1 is the unbounded component of $\overline{\mathbf{C}}\backslash\hat{\Gamma}_1$ (here $\hat{\Gamma}_1$ is the collection of the supports of the points of Γ_1). Assume in addition that $n \ge 2$ and $\sum_{k=1}^n \delta_k = 0$. Then the following sharp inequality is true:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\delta_k\delta_l\,[2N_{B_1}(z_k,z_l) - g_B(z_k,z_l;\Gamma)] \ge 0. \tag{50}$$

To prove this inequality note that $B \subset B_1$ and $\partial B_1 \cap B \subset \Gamma$. Hence we are in the position to apply (46). Writing formula (33) for $M(B_1,Z,\Delta,\Psi)$ and [5, formula (7)] (or [8, formula (2.6)]) for $M(B,\Gamma,Z,\Delta,\Psi)$ we immediately obtain the required inequality. In both of the above moduli $Z=\{z_1,\dots,z_n\}$, $\Delta=\{\delta_1,\dots,\delta_n\}$ and $\Psi=\{r,\dots,r\}$.

The following example shows that (50) is sharp. Take $B=\{z:|z|>1\}\setminus\Gamma$, $\Gamma=[2,3]$, $z_1=2+i$, $z_2=2-i$, $z_3=-2+i$, $z_4=-2-i$, $\delta_1=\delta_3=1$, $\delta_2=\delta_4=-1$. We have $B_1=\{z:|z|>1\}$. Denote by u the function harmonic in $B_1\setminus\bigcup_{i=1}^4 D(z_i,r)$, having $\partial u/\partial n=0$ on ∂B_1 and $u=\delta_i$ on $D(z_i,r)$. This function is a solution of mixed boundary value problem with continuous boundary data and hence it clearly exists [18, Proposition 15.7b]. By symmetry $u=0$ on Γ. Hence according to the extended Dirichlet principle u is the potential function for both condenser $C(r;\Gamma;B,Z,\Delta,\Psi)$ defined by (45) and $C(r;B_1,Z,\Delta,\Psi)$ defined by (2) which implies that

$$|C(r;\Gamma;B,Z,\Delta,\Psi)| = |C(r;B_1,Z,\Delta,\Psi)| \;\Rightarrow\; M(B,\Gamma,Z,\Delta,\Psi) = M(B_1,Z,\Delta,\Psi),$$

and we have equality in (50).

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