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COINCIDENCE OF HYPERBOLIC AND GENERALIZED KOBAYASHI DENSITIES ON PLANE DOMAINS

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Abstract. We show that under certain condition, the hyperbolic density and the generalized Kobayashi density coincide on a hyperbolic plane domain. This coincidence has a variety of results.

1. Introduction and key facts

In [1] and [2], Keen and Lakic defined new densities that generalize the hyperbolic density for a plane domain. One is a generalization of the standard Kobayashi density defined by focusing on source and the other is a generalization of Carathéodory density defined by focusing on target.

We show that if Ω has certain property (see Section 2) and X is an arbitrary hyperbolic plane domain, then the hyperbolic density is equal to the generalized Kobayashi density on X (As we will see later, in order to define the generalized Kobayashi density on X, we need a hyperbolic plane domain Ω).

In this section, we state the definitions, theorems, and the facts that we use in this paper. In Section 2, we state and prove our result, first in a special case and then in the general form. We will have several interesting corollaries.

Definition 1.1. The hyperbolic density on the unit disk Δ is defined as

$$\rho(z) = \frac{1}{1-|z|^2},$$

for z in Δ. In addition, the hyperbolic distance between two points z and w in Δ is defined as

$$\rho(z,w) = \inf \int_\gamma \rho(t)\,|dt|, \tag{1}$$

where the infimum is over all paths γ in Δ joining z to w.

Definition 1.2. The hyperbolic density on a hyperbolic domain Ω is defined as

$$\rho_\Omega(w) = \frac{\rho(t)}{|\pi'(t)|},$$

where ρ is the hyperbolic density on the unit disk and π is a holomorphic covering map from the unit disk onto Ω with $\pi(t) = w$. In addition, the hyperbolic distance

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between two points z and w in Ω is defined as

$$\rho_\Omega(z,w) = \inf \int_\gamma \rho_\Omega(t)\,|dt|, \tag{2}$$

where the infimum is over all paths γ in Ω joining z to w.

Theorem 1.1. (Generalized Schwarz–Pick lemma) *Let Ω and X be two arbitrary hyperbolic domains and let $f\colon \Omega \to X$ be a holomorphic map. Then $\rho_X(f(z))|f'(z)| \leq \rho_\Omega(z)$, for every z in Ω and $\rho_X(f(z), f(w)) \leq \rho_\Omega(z,w)$, for every pair z, w in Ω.*

For a proof of Theorem 1.1, see [1], p. 130, or [2]. If f, in the theorem above, is a covering map from a hyperbolic domain Ω onto a hyperbolic domain X, then it is an infinitesimal isometry. That is $\rho_X(\pi(z))|\pi'(z)| = \rho_\Omega(z)$.

Definition 1.3. Let X be a subdomain of a domain Ω. X is called a Lipschitz or ρ-Lip subdomain of Ω if the infinitesimal contraction constant, which is defined as $m(X,\Omega) = \sup_{z\in X} \frac{\rho_\Omega(z)}{\rho_X(z)}$, is strictly less than 1.

It is easy to show that the Lipschitz property is invariant under a conformal homeomorphism. In other words, if f is a conformal homeomorphism from the hyperbolic domain Ω onto $f(\Omega)$, then X is a Lipschitz subdomain of Ω if and only if $f(X)$ is a Lipschitz subdomain of $f(\Omega)$.

Definition 1.4. Let Ω be a hyperbolic plane domain and let X be a plane domain. The generalized Kobayashi density for z in X is defined by

$$\kappa_X^\Omega(z) = \inf \frac{\rho_\Omega(w)}{|f'(w)|},$$

where ρ_Ω is the hyperbolic density on Ω and the infimum is over all holomorphic functions f from Ω to X and all points w in Ω such that $f(w) = z$.

One can check that for a hyperbolic domain X and for every z in X, $\rho_X(z) \leq \kappa_X^\Omega(z)$. In the case that Ω is a covering space of X, then for every z in X we have $\rho_X(z) = \kappa_X^\Omega(z)$.

By using the definition of the generalized Kobayashi density we can deduce that if Ω_1 and Ω_2 are two hyperbolic plane domains which are conformally homoeomorphic, then $\kappa_X^{\Omega_1}(z) = \kappa_X^{\Omega_2}(z)$, for every $z \in X$. It is also easy to verify that if a map $f\colon X \to Y$ is a conformal homeomorphism then it is an infinitesimal isometry with respect to the Kobayashi density. That is, $\kappa_Y^\Omega(f(z))|f'(z)| = \kappa_X^\Omega(z)$, for every z in X.

Definition 1.5. A subdomain X is a Kobayashi–Lipschitz or κ-Lip subdomain of Ω if the Kobayashi contraction constant, which is defined as $m\kappa(X,\Omega) = \sup_{z\in X} \frac{\kappa_\Omega^\Omega(z)}{\kappa_X^\Omega(z)}$, is strictly less than 1.

Theorem 1.2. *X is a κ-Lip subdomain of Ω if and only if there exists $k < 1$ such that for every holomorphic function $f\colon \Omega \to X$ and every pair of points z and w in Ω we have $\rho_\Omega(f(z), f(w)) \leq k\rho_\Omega(z,w)$.*

For a proof of Theorem 1.2, see [1], p. 181. Now, we define the generalized Carathéodory density.

Definition 1.6. Let X be a hyperbolic plane domain. The generalized Carathéodory density for ω in Ω is defined by

$$c_X^\Omega(w) = \sup \rho_X(f(w))|f'(w)|,$$

where ρ_X is the hyperbolic density on X and the supremum is over all the holomorphic functions f from Ω to X.

One can check that if Ω is a hyperbolic domain then $c_X^\Omega(w) \le \rho_\Omega(w)$, for every $w \in \Omega$. In the case that Ω is a covering space for X we have $c_X^\Omega(w) = \rho_\Omega(w)$.

Definition 1.7. A subdomain X is a Carathéodory–Lipschitz or c-Lip subdomain of Ω if the Carathéodory contraction constant, which is defined as $mc(X, \Omega) = \sup_{z \in X} \frac{c_X^\Omega(z)}{c_X^X(z)}$, is strictly less than 1.

Theorem 1.3. *X is a c-Lip subdomain of Ω if and only if there exists $k < 1$ such that for every holomorphic function $f \colon \Omega \to X$ and every pair of points z and w in X we have $\rho_X(f(z), f(w)) \le k\rho_X(z, w)$.*

For a proof of Theorem 1.3, see [1], p. 184.

2. The main theorem

As we mentioned in Section 1, if X is a hyperbolic plane domain then for every z in X we have $\rho_X(z) \le \kappa_X^\Omega(z)$. We state a condition on Ω, under which, for every hyperbolic plane domain X and every z in X we have $\rho_X(z) = \kappa_X^\Omega(z)$. First, we state the theorem in the special case that Ω is a non-Lipschitz subdomain of the unit disk. Then we state the theorem in the general form.

Theorem 2.1. *Suppose Ω is a non-Lipschitz subdomain of the unit disk and X is any hyperbolic plane domain. Then*

$$\rho_X(z) = \kappa_X^\Omega(z),$$

for every z in X.

Proof. Since $\rho_X(z) \le \kappa_X^\Omega(z)$, we need only to show that $\kappa_X^\Omega(z) \le \rho_X(z)$. As X is hyperbolic, there is a holomorphic covering

$$\pi \colon \Delta \to X.$$

Since π is a covering, it is locally an isometry and we have

$$\rho_\Delta(\omega) = \rho_X(z) \cdot |\pi'(\omega)|, \tag{3}$$

where $\pi(\omega) = z$. By precomposing by a Möbius map we can let ω be anywhere in Ω.

Now, let f be the restriction of π to Ω. Therefore,

$$\kappa_X^\Omega(z) \le \frac{\rho_\Omega(\omega)}{|f'(\omega)|} = \frac{\rho_\Omega(\omega)}{|\pi'(\omega)|}.$$

By equation 3,

$$|\pi'(\omega)| = \frac{\rho_\Delta(\omega)}{\rho_X(z)}.$$

Consequently,

$$\kappa_X^\Omega(z) \le \frac{\rho_\Omega(\omega)}{\rho_\Delta(\omega)} \cdot \rho_X(z).$$

Since Ω is a non-Lipschitz subdomain of Δ, $\frac{\rho_\Omega(\omega)}{\rho_\Delta(\omega)}$ can be made as close as we wish to 1 by choosing ω properly. Therefore,

$$\kappa_X^\Omega(z) \le \rho_X(z).$$

This completes the proof. $\square$

Now, we want to generalize Theorem 2.1 for a large class of domains in the plane.

Definition 2.1. A domain Ω in the complex plane is called quasi-bounded if the smallest simply connected plane domain containing Ω is a proper subset of the complex plane $\mathbf{C}$. The smallest simply connected domain containing Ω is denoted by $\hat{\Omega}$.

In fact, $\hat{\Omega}$ is obtained by adding to Ω all the bounded connected components of $\mathbf{C} \setminus \Omega$.

Example. If Ω is the round annulus $\{z : r < |z| < 1\}$, then it is quasi-bounded and $\hat{\Omega} = \Delta$.

Example. If $\Omega = \mathbf{C} \setminus \{-1, 1\}$, then it is not quasi-bounded because the smallest simply connected domain containing Ω is $\mathbf{C}$.

Theorem 2.2. *Suppose Ω is quasi-bounded and is a non-Lipschitz subdomain of $\hat{\Omega}$. Then for any hyperbolic plane domain X we have*

$$\rho_X(z) = \kappa_X^{\Omega}(z),$$

for every z in X.

Proof. Since Ω is quasi-bounded, by the Riemann mapping theorem there is a conformal homeomorphism f from $\hat{\Omega}$ onto Δ. Since the Lipschitz property is invariant under a conformal homeomorphism, $f(\Omega)$ is a non-Lipschitz subdomain of Δ.

By Theorem 2.1

$$\rho_X(z) = \kappa_X^{f(\Omega)}(z), \tag{4}$$

for every z in X.

As Ω and $f(\Omega)$ are conformally homeomorphic, we have

$$\kappa_X^{\Omega}(z) = \kappa_X^{f(\Omega)}(z), \tag{5}$$

for every z in X.

By equations 4 and 5, we conclude that

$$\rho_X(z) = \kappa_X^{\Omega}(z).$$

This completes the proof. □

Example. Let $\Omega = \Delta \setminus \{0\}$. We have $\rho_\Omega(z) = \frac{1}{2|z| \ln(|\frac{1}{z}|)}$ (see [1], p. 135) and $\hat{\Omega} = \Delta$. In fact, Ω is a non-Lipschitz subdomain of $\hat{\Omega}$, because $\frac{\rho_{\hat{\Omega}}(z)}{\rho_\Omega(z)} = \frac{2|z| \ln(|\frac{1}{z}|)}{1-z^2} \to 1$, as $|z| \to 1$. Therefore, Ω satisfies the conditions of the theorem above and we have $\rho_X(z) = \kappa_X^{\Omega}(z)$, for any hyperbolic domain X and any z in X.

Corollary 2.1. *If X is a subdomain of a hyperbolic domain Ω non-Lipschitz in $\hat{\Omega}$ then $m\kappa(X, \Omega) = m(X, \Omega)$. In particular, X is a κ-Lip subdomain of Ω if and only if it is a ρ-Lip subdomain of Ω.*

Proof. By Theorem 2.2 we have

$$\rho_X(z) = \kappa_X^{\Omega}(z). \tag{6}$$

As any domain is a covering for itself we have

$$\kappa_\Omega^{\Omega} = \rho_\Omega. \tag{7}$$

So we have,

$$mκ(X,Ω) = \sup_{z∈X} \frac{κ_Ω^Ω(z)}{κ_X^Ω(z)} = \sup_{z∈X} \frac{ρ_Ω(z)}{ρ_X(z)} = m(X,Ω). \tag{8}$$

This completes the proof. □

Corollary 2.2. *If X is a subdomain of a hyperbolic domain $Ω$ then $mc(X,Ω) \le m(X,Ω)$. In particular, a $ρ$-Lip subdoamin of $Ω$ is always a c-Lip subdomain of $Ω$, and, moreover, if $Ω$ is non-Lipschitz in $\hat{Ω}$ then any $κ$-Lip subdomain of $Ω$ is a c-Lip subdomain of $Ω$.*

Proof. Since X is a covering for itself we have

$$c_X^X(z) = ρ_X(z). \tag{9}$$

In addition, as we mentioned in Section 2

$$c_X^Ω(z) \le ρ_Ω(z). \tag{10}$$

Therefore, we have

$$mc(X,Ω) = \sup_{z∈X} \frac{c_X^Ω(z)}{c_X^X(z)} \le \sup_{z∈X} \frac{ρ_Ω(z)}{ρ_X(z)} = m(X,Ω). \tag{11}$$

This completes the proof. □

Remark. For an example that X is c-Lip in $Ω$ but not $κ$-Lip, see Example 3 of Chapter 10 in [1].

Corollary 2.3. *Suppose $Ω$ is a non-Lipschitz subdomain of $\hat{Ω}$. If there is $k_1 < 1$ such that for every holomorphic function $f\colon Ω \to X$ and every pair of points z and w in $Ω$ we have $ρ_Ω(f(z), f(w)) \le k_1 ρ_Ω(z,w)$ then there is $k_2 < 1$ such that for every holomorphic function $f\colon Ω \to X$ and every pair of points z and w in X we have $ρ_X(f(z), f(w)) \le k_2 ρ_X(z,w)$.*

Proof. It clearly follows from Theorems 1.2, 1.3 and Corollary 2.2. □

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