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A SEWING PROBLEM IN METRIC SPACES

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Abstract. This note is devoted to the solution of a sewing problem between metric spaces sharing quasisymmetric copies of a given metric space. It is proved that the sewing yields a well-defined conformal gauge, and we study properties inherited by the new space. It follows from the construction that if Y is a closed uniformly perfect subset of a proper metric space X, then, for any $\varepsilon > 0$, one can find a metric d in the conformal gauge of X so that the Hausdorff dimensions of both (X, d) and (Y, d) are ε-close to their conformal dimension.

The classical sewing problem, as was explained to me by A. Douady, asks for the following: given an orientation preserving homeomorphism of the unit circle $f\colon \mathbf{S}^1 \to \mathbf{S}^1$, do there exist univalent maps $F_1\colon \mathbf{D} \to \widehat{\mathbf{C}}$ and $F_2 : \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}} \to \widehat{\mathbf{C}}$ which extend continuously to the closures as homeomorphisms such that $F_1(\mathbf{D}) \cap F_2(\widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}) = \emptyset$, $F_1(\overline{\mathbf{D}}) \cup F_2(\widehat{\mathbf{C}} \setminus \mathbf{D}) = \widehat{\mathbf{C}}$ and $F_1(t) = F_2(f(t))$ for $t \in \mathbf{S}^1$?

A standard solution says that if f is quasisymmetric, then such maps exist and are essentially unique. Actually, a stronger conclusion is established: the maps F_1 and F_2 are both quasisymmetric. One ingredient of the proof consists in extending f as a quasiconformal map (Beurling–Ahlfors theorem [7]).

In the context of metric spaces, gluing metric spaces together usually takes place over isometric subspaces. In this paper, we propose to address the following problem:

Sewing Problem. *Let X_1, X_2 be two metric spaces, and let us consider two closed subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$, and a homeomorphism $f\colon Y_1 \to Y_2$. Does there exist a metric $\widehat{d}$ on $\widehat{X} = X_1 \cup X_2/(f)$ such that* $\mathrm{Id}\colon X_j \to \widehat{X}$ *is quasisymmetric and* $\mathrm{Id}\colon X_j \setminus Y_j \to \widehat{X}$ *is locally quasisimilar for $j = 1, 2$?*

Here, a homeomorphism $f\colon X \to Y$ between metric spaces is *quasisymmetric* if there is an increasing homeomorphism $\eta\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that, for any distinct points x, y, z, the following 3-point condition holds:

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x - y|}{|x - z|}\right). \tag{1}$$

Here and in the sequel, $|\cdot|$ stands for the distance between two points of a metric space.

An embedding $f\colon X \to Y$ between metric spaces is a *quasisimilarity* if constants $\lambda > 0$ and $C \ge 1$ exist such that

$$(1/C) \le \frac{|f(x) - f(y)|}{\lambda |x - y|} \le C$$

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for all $x \neq y \in X$. Such a map is of course bi-Lipschitz.

A map will be said *locally quasisimilar* if every point admits a neighborhood on which the map is a quasisimilarity and the constant C can be chosen independently from the point.

Conformal maps between general metric spaces do not bear as much significance as in the plane. The notion of local quasisimilarity is much more natural in this context. On the one hand, it implies not only a local bi-Lipschitz condition but also local quasisymmetry (with a uniform distortion function), and, on the other hand, this propery holds as well for univalent maps by the Koebe distortion theorem.

We observe that it is necessary that f be quasisymmetric for a solution to our sewing problem to exist. In this paper, we give a positive answer when f is assumed to be quasisymmetric and when X_1 and X_2 are proper, and Y_1 and Y_2 are uniformly perfect. A metric space X is *proper* if closed balls (of finite radius) are compact and is *uniformly perfect* if X contains at least two points and if there is some constant $\lambda \in (0,1)$ such that, for any $x \in X$ and any $r \leq \operatorname{diam} X$, there is some point $y \in X$ such that $\lambda \cdot r \leq |x-y| \leq r$.

A connected set is always uniformly perfect, so this ensures that these sets are rather "thick" in the sense that the diameter of a ball can be controlled by its radius.

There are of course many possible metrics which answer the problem. But our procedure defines a unique *conformal gauge* which depends only on the gauges of X_1 and X_2. Given a metric space (X,d), its conformal gauge is the family of all metrics on X which are quasisymmetric equivalent to d. Thus, our metric space $(\widehat{X},\widehat{d})$ is well defined up to quasisymmetric homeomorphisms.

Our main result says

Theorem 1. *Let X_1, X_2 be proper metric spaces containing at least two points. Let us assume that $Y_1 \subset X_1$, $Y_2 \subset X_2$ are two closed uniformly perfect subsets such that there is a quasisymmetric homeomorphism $f\colon Y_1 \to Y_2$. We also suppose that X_1 is bounded if Y_1 is.*

There are a metric $\widehat{d}$ on $\widehat{X} = X_1 \cup X_2/(f)$ and a constant $c > 0$ such that

(1) *For any $(x_1,x_2) \in X_1 \times X_2$, $\widehat{d}(x_1,x_2) \geq c \cdot \inf_{y\in Y_1}\{\widehat{d}(x_1,y) + \widehat{d}(f(y),x_2)\}$;*

(2) $\mathrm{Id}\colon X_j \to \widehat{X}$ *is quasisymmetric for $j = 1,2$, and*

(3) $\mathrm{Id}\colon X_j \setminus Y_j \to \widehat{X}$ *is locally quasisimilar for $j = 1,2$.*

Furthermore, the conformal gauge of $\widehat{d}$ depends only on the conformal gauges of X_1 and X_2.

The pattern of the proof is the same as the original one. It relies on a metric version of the Beurling–Ahlfors extension theorem of quasisymmetric maps which we state now.

Theorem 2. *Let (X,d_X) be a proper metric space containing at least two points and (Y,d_Y) a proper uniformly perfect space. Let us assume that there is a quasisymmetric embedding $f\colon Y \to X$. Then there is a metric $\widehat{d}$ on X such that*

(1) $\mathrm{Id}\colon (X,d_X) \to (X,\widehat{d})$ *is quasisymmetric;*

(2) $\mathrm{Id}\colon (X\setminus f(Y),d_X) \to (X\setminus f(Y),\widehat{d})$ *is locally quasisimilar;*

(3) $f\colon (Y,d_Y) \to (X,\widehat{d})$ *is bi-Lipschitz onto its image.*

Remark. The proof will show that there is a finite constant $C \geq 1$ such that, for any $x \in X \setminus f(Y)$, the restriction of $\mathrm{Id}\colon (X, d_X) \to (X, \widehat{d})$ to $B_X(x, \mathrm{dist}(x, f(Y))/2)$ is C-quasisimilar.

This result can also be interpreted in terms of conformal gauges, which may have its own interest since their structure remains quite mysterious. For instance, there are very few known ways to deform a metric in a gauge: a metric d can snow-flaked [13, Chap. 10], meaning, that we consider d^α, for some $\alpha > 0$ hoping that it still defines a metric (which is the case if $\alpha < 1$); if the space carries a doubling measure, then Semmes has a procedure which deforms the metric within the gauge so that the measure becomes Ahlfors regular [13, Chap. 14]; if the space is the boundary at infinity of a hyperbolic group, then one may change the set of generators to define a new metric [5, § 3]. Thus, Theorem 2 provides another deformation based on the deformation of a subset. A usually interesting numerical characteristic of the conformal gauge of a metric space X is its *conformal dimension* $\dim_c X$ defined as the infimum of the Hausdorff dimensions over all the metrics in the conformal gauge of X. Theorem 2 has the following corollary concerning the conformal dimensions of pairs $Y \subset X$ which might be interesting in its own right:

Corollaire 3. *Let X be a proper metric space and $Y \subset X$ a closed uniformly perfect subset of X. Then, for all $\varepsilon > 0$, there exists a metric $\widehat{d}$ in the conformal gauge of X such that*

$$\begin{cases} \dim(X, \widehat{d}) \leq \dim_c X + \varepsilon, \\ \dim(Y, \widehat{d}) \leq \dim_c Y + \varepsilon. \end{cases}$$

Quasisymmetric maps preserve many properties such as completeness, uniform perfectness, etc. These notions thus apply to gauges (see [13, Chap. 15] for details). It also follows from our construction that finer properties are preserved. In particular, we have the following.

Theorem 4. *Let (X_1, d_1) and (X_2, d_2) be Q-Loewner and Q-Ahlfors regular metric spaces for some $Q > 1$. Assume that $Y_1 \subset X_1$ and $Y_2 \subset X_2$ are both uniformly perfect and porous, and that they are quasisymmetric equivalent. We also suppose that X_1 is bounded if Y_1 is. Then the gauge provided by Theorem 1 contains a Q-Loewner and Q-Ahlfors regular metric.*

The notions used in this statement are defined in the first section.

Another point of view from Beurling–Ahlfors theorem has already been studied on metric spaces: quasisymmetric maps between boundaries at infinity of Gromov hyperbolic spaces extend as quasi-isometries of the hyperbolic spaces involved (see [19, 10]).

Conformal deformations of metric measure spaces have been introduced by Bonk, Heinonen and Koskela in [9] for other purposes (in relation with the Gromov hyperbolicity of the quasihyperbolic metric). The basic assumptions are the following: the space Y is a closed subset of a proper metric space X such that $X \setminus Y$ is uniform. One then considers a positive continuous function ρ on $X \setminus Y$ which satisfies a Harnack-type inequality of the form $1/C \leq \rho(x)/\rho(x') \leq C$ if $|x - x'| \leq \mathrm{dist}(x, Y)/2$. Then one can define a new metric $d_\rho(x, x') = \inf \int_\gamma \rho$ over all curves $\gamma \subset X \setminus Y$ joining $x, x' \in X \setminus Y$. The uniformity of $X \setminus Y$ is a key assumption in the study of the quasisymmetry of $(X, d_X) \xrightarrow{\mathrm{Id}} (X, d_\rho)$ [9, Chap. 4] and of deformations of Loewner spaces [9, Chap. 6]. In our setting, we do have a Harnack-type inequality but not the uniform assumption of $X \setminus Y$, so our method will be quite different.

It is hoped that the results in the present paper will be used to construct interesting examples of conformal dynamical systems on metric spaces. For instance, if H is a non-elementary quasiconvex subgroup of a hyperbolic group G (see [5] for definitions and references therein), then its boundary ∂H is uniformly perfect and quasisymmetric to its limit set $\Lambda(H)$ in ∂G which is also porous. Furthermore, if H is a malnormal quasiconvex subgroup of two hyperbolic groups G_1 and G_2, then the amalgamated product $\widehat{G} = G_1 *_H G_2$ is known to be also hyperbolic, and its boundary contains quasisymmetric copies of ∂G_1 and ∂G_2 which intersect over quasisymmetric copies of ∂H (cf. [6]).

Relationship with the classical sewing problem. Theorems 1 and 4 enable us to recover the solution to the classical sewing problem in a very indirect and non-trivial way. Indeed, starting from a quasisymmetric homeomorphism $f\colon \mathbf{S}^1 \to \mathbf{S}^1$, one obtains by Theorem 1 a metric 2-sphere $\widehat{X}$. This sphere is 2-Loewner and 2-Ahlfors-regular according to Theorem 4. A theorem of Bonk and Kleiner then implies that there is a quasymmetric homeomorphism $\varphi\colon \widehat{X} \to \widehat{\mathbf{C}}$ [8]. This solves the quasiconformal sewing problem; the univalent maps are obtained by using the measurable Riemann mapping theorem. We omit the details.

Outline of the paper. In § 1, we provide the necessary background for the paper. In § 2, we prove Theorem 2 and Corollary 3. We first define the metric $\widehat{d}$, and then prove it satisfies the right properties. In § 3, we study the problem of sewing quasisymmetric maps together for the uniqueness statement of Theorem 1: they are well-behaved if the sewing has a positive angle, the seam is thick and large enough. In § 4, Theorem 1 and Theorem 4 are proved. The ideas used to define the new metrics are given in the beginning of the sections. The remaining parts consist essentially of the verifications case by case that the procedure goes through.

Background. Quasiconformal maps in Euclidean space have been extensively studied. References include [2, 17] in the plane and [22] in space. An abstract theory of quasiconformal maps in metric spaces has been initiated in the early eighties motivated from Gehring's seminal paper [12]. One can consult [20, 23, 14, 15, 4] and the references therein for its developments. An introduction to analysis in metric spaces can be found in Heinonen's monograph [13] to which we will refer for most of the material used here.

Notation. The non-negative real numbers are denoted by $\mathbf{R}_+$; the positive reals by $\mathbf{R}_+^*$. Throughout the paper, if a, b are positive, we will write $a \lesssim b$ or $b \gtrsim a$ if there is some universal constant u, independent from a and b, such that $a \le ub$. Similarly, $a \asymp b$ means $a \lesssim b$ and $a \gtrsim b$. A ball in a space X centered at x of radius $r > 0$ will be denoted by $B(x, r)$ or $B_X(x, r)$. The diameter of a set E will be written $\mathrm{diam} E$ or $\mathrm{diam}_X E$.

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1. Preliminaries

1.1. Facts on sets. We provide in this section some (very limited) background on geometric measure theory. For more information, we refer to [18].

If X is a metric space and $\alpha > 0$, let us define the α-Hausdorff measure of X as follows: for any $\delta > 0$, let

$$\Lambda_\alpha^\delta(X) = \inf \left\{ \sum (\mathrm{diam} E_j)^\alpha \right\}$$

where the infimum and sum are taken over all coverings of X by sets of diameter at most δ; for $\delta = \infty$, $\Lambda_\alpha^\infty = \mathscr{H}_\alpha$ is called the *Hausdorff content* in dimension α. Let $\Lambda_\alpha(X) = \lim_{\delta \to 0} \Lambda_\alpha^\delta(X)$ be the Hausdorff measure of X of dimension α. The *Hausdorff dimension* $\dim X$ of X is then the infimum over α such that $\Lambda_\alpha(X) = 0$.

A metric space X is said to be *Ahlfors regular of dimension* $\alpha > 0$ if the Hausdorff measure in dimension α of a closed ball of radius $r \le \mathrm{diam} X$ is approximately r^α.

A subset Y of a set X is called *porous* if a constant $c > 0$ exists such that any ball of radius r centered in Y contains a ball of radius cr disjoint from Y. This condition implies in particular, when X is α-Ahlfors regular, that the box dimension of Y is strictly less than α and that some number $\theta \in (0,1)$ exists such that $Y \cap B(y,r)$ can be covered by at most $e^{\alpha\beta}\theta^\beta$ balls (up to some universal factor) of radius $e^{-\beta} r$ for any $y \in Y$ and $\beta \ge 1$. To see this, one may use a dyadic decomposition of the space (which is known to exist in an Ahlfors regular metric space [11]) and count the cubes of a given size which can intersect the porous subset.

A contrario, a uniformly perfect set cannot have a dimension too small. We have the following theorem of Järvi and Vuorinen [16, Thm. 4.1].

Theorem 1.1. *If X is a proper Q-Ahlfors regular metric space and Y is a closed uniformly perfect subset of X, then there are constants $s \in (0,Q)$ and $c > 0$ which depend only on the data above such that for any $y \in Y$ and $r \le \mathrm{diam} Y$,*

$$\mathscr{H}_s(Y \cap B(y,r)) \ge c r^s.$$

The original proof takes place in $\mathbf{R}^n$ but is local: the only properties used are its Ahlfors regularity and the fact that it is proper.

We now introduce a property of metric spaces which is particularly relevant to study quasiconformal geometry. It has been introduced by Heinonen and Koskela [14].

Definition. Given $Q > 1$, a proper, rectifiably arcwise connected metric space X is a *Q-Loewner space* if a decreasing homeomorphism $\psi\colon \mathbf{R}_+^* \to \mathbf{R}_+^*$ exists such that, for any disjoint continua E, F such that $\mathrm{dist}(E,F) \le t \min\{\mathrm{diam} E, \mathrm{diam} F\}$ then

$$\mathrm{mod}_Q(\Gamma(E,F)) \ge \psi(t),$$

where mod_Q denotes the Q-modulus and $\Gamma(E,F)$ denotes the set of curves which join E to F.

Proper Q-Ahlfors regular Q-Loewner spaces carry many interesting properties: they are for instance quasiconvex (meaning that there is some constant $C > 1$ such that any two points x, y can be joined by a curve of length at most $C|x-y|$).

Furthermore, when X is proper Q-Ahlfors regular and quasiconvex, then the Q-Loewner property is characterised by the existence of a so-called $(1,Q)$-Poincaré inequality for bounded continuous functions (cf. Corollary 5.13 in [14]).

In the sequel, moduli of curves and Poincaré inequalities won't be used explicitely, so they will not be defined here. We refer to [14], [13] and the references therein for their definition and main properties.

1.2. Facts on maps. We provide some properties of quasisymmetric maps which will be used in the sequel without further reference. We refer to [13] for the proofs.

Proposition 1.2. *If $f\colon X \to Y$ is η-quasisymmetric, then for all $A, B \subset X$ with $A \subset B$ and B bounded,*

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \leq \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)} \leq \eta\left(2\frac{\operatorname{diam} A}{\operatorname{diam} B}\right).$$

See [13, Prop. 10.8].

When X is uniformly perfect, then the 3-point condition (1) can be well exploited. In particular, quasisymmetric maps can always be controlled by homeomorphisms η of the form

$$\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$$

where $C \geq 1$ and $\alpha \in (0,1]$ (see Theorem 11.3 in [13]). Such maps are called *power quasisymmetric* maps. The proof of Theorem 2 will show that $\mathrm{Id}\colon (X, d_X) \to (X, \widehat{d})$ is power quasisymmetric.

Such a specific form will be used in the sequel to obtain the following estimates: for fixed $\theta > 0$, if $t \geq \theta$ then $t^\alpha \leq \max\{1, t^{1/\alpha}\}$ and $t^{1/\alpha} \geq \theta^{1/\alpha}$ so that $t^\alpha \leq \max\{1, \theta^{-1/\alpha}\} t^{1/\alpha}$.

Since $t \leq \eta(t)$, it follows that if $t, t' \geq \theta$ then

$$\begin{cases} \eta(t)\eta(t') \lesssim \eta(tt'), \\ t\eta(t') \lesssim \eta(tt'), \end{cases} \tag{2}$$

where the implicit constants depend on θ.

Similarly,

$$\begin{cases} \eta(1/t)\eta(1/t') \lesssim \eta(1/(tt')), \\ (1/t)\eta(1/t') \lesssim \eta(1/(tt')). \end{cases} \tag{3}$$

2. Extension of metrics

In this section, we provide a proof of Theorem 2. We first define a prototype of the metric we are looking for: it is a symmetric function $\widehat{q}\colon X \times X \to \mathbf{R}_+$ which vanishes exactly on the diagonal, and which satisfies the right properties besides the triangle inequality.

There are two main ideas which yield to the definition of $\widehat{q}$.

First, for points close to Y, the quasisymmetry condition implies that $\widehat{q}(x,y) \asymp \widehat{q}(y,y') \asymp |y-y'|_Y$ must hold when $y, y' \in Y$, $x \notin Y$ and $|x-y|_X \asymp |y-y'|_X$. Here the uniform perfectness is used to make sure that we have enough points in Y to use this strategy.

The second idea is that if $f\colon U \to V$ is a conformal map between two strict simply connected domains of the complex plane then $|f'(z)|$ behaves like $\operatorname{dist}(f(z), \partial V)/$

$\mathrm{dist}(z, \partial U)$. This means that if $\delta(x)$ and $\widehat{\delta}(x)$ denote the "distance" of x to Y with respect to d_X and $\widehat{q}$, then we wish that, for points x' close enough to x,

$$\widehat{q}(x, x') \asymp \frac{\widehat{\delta}(x)}{\delta(x)} |x - x'|_X.$$

These two conditions essentially define $\widehat{q}$. The defect for $\widehat{q}$ to be a distance is that the triangle inequality might not hold. A metric can be obtained by using a chain argument: if $x, y \in X$ are given, a chain is a finite ordered set of points $x_0, \dots, x_n$ such that $x_0 = x$ and $x_n = y$. Let us define

$$\widehat{d}(x, y) = \inf \sum \widehat{q}(x_j, x_{j-1})$$

where the infimum is taken over all chains joining x to y. If, for all distinct x, y, $\widehat{d}(x, y) \neq 0$, then we have obtained a metric. If, furthermore, there is a universal constant $u > 0$ such that $\widehat{d}(x, y) \geq u\widehat{q}(x, y)$, then $\widehat{d}$ is a genuine distance bi-Lipschitz equivalent to $\widehat{q}$. The idea to show the existence of the constant u above is to insert many points from Y in any given chain, and, ultimately, replace all the points but the extremities by points in Y, so that the triangle inequality of the metric d_Y enables us to conclude.

It then remains to check the properties announced in Theorem 2 are indeed fulfilled.

2.1. Construction of a metric. Let X and Y be proper metric spaces and $f\colon Y \to X$ be an η-quasisymmetric embedding. Let us observe that $f^{-1}\colon f(Y) \to Y$ is also quasisymmetric with distortion function $\eta_{-1}(t) = 1/\eta^{-1}(1/t)$. Replacing both functions by $\max\{\eta, \eta_{-1}\}$, we may – and will – assume that both f and f^{-1} are quasisymmetric with the same distortion function η.

We denote by d_X and d_Y the initial metrics on X and Y as a whole. But, when considering distances between two given points, we will prefer the Polish notation $|\cdot|_X$ and $|\cdot|_Y$. In the sequel, we see Y as a closed subset of X (since f is quasisymmetric and Y complete, $f(Y)$ is a complete subspace of X). We assume that $\mathrm{diam}_Y Y = \mathrm{diam}_X Y = \mathrm{diam}\, Y$ and that Y is λ-uniformly perfect.

A word regarding the construction of the metric $\widehat{d}$. We will define a first function q following the ideas described above. Even though it is very crude, it will be enough to define a "predistance" to Y. The section § 2.1.1 is devoted to obtain some estimates on this predistance. Then, we will define the "premetric" $\widehat{q}$ and state the main result Theorem 2.3 of this section. In § 2.1.2, we will prove several lemmata which will enable us to express $\widehat{q}$ in simpler forms. Finally, in § 2.1.3, we will first establish several approximate triangle inequalities which enable us to simplify the chain sums and to insert points from Y, following the strategy described above. The existence of the metric will then follow.

Definition. (Preliminary function) Let us define a function $q\colon X \times X \to \mathbf{R}_+$ as follows.

- If $y, y' \in Y$, let $q(y, y') = |y - y'|_Y$.
- If $x \in X \setminus Y$ and $y \in Y$, then, let $q(x, y) = q(y, x) = |x - y|_X$ if $|x - y|_X \geq \mathrm{diam} Y/2$; otherwise, set $q(x, y) = q(y, x) = \inf\{|x' - y|_Y\}$ where the infimum is taken over the set of points $x' \in Y$ such that $\lambda|x - y|_X \leq |x' - y|_X \leq |x - y|_X$. We use this second expression for q when Y is unbounded.

If $x \in X \setminus Y$, $y \in Y$, and $|x-y|_X < \mathrm{diam}Y/2$, then we define $A(x,y)$ as the set of points $x' \in Y$ such that $\lambda|x-y|_X \le |x'-y|_X \le |x-y|_X$ and $q(x,y) = |x'-y|_Y$. We note that since Y is closed and X proper, this set is never empty (as soon as $|x-y|_X \le \mathrm{diam}Y$).

Remark. When $|x-y|_X \le \mathrm{diam}Y/2$ and x_1, x_2 are points of Y satisfying both $\lambda|x-y|_X \le |x_j - y|_X \le |x-y|_X$, $j=1,2$, then

$$\frac{|x_1-y|_Y}{|x_2-y|_Y} \le \eta\left(\frac{|x_1-y|_X}{|x_2-y|_X}\right) \le \eta(1/\lambda),$$

so the choice of the infimum is essentially irrelevant for our considerations.

2.1.1. Predistance to Y. We first define and study the "predistance" to Y.

Definition. (Distance and predistance to the boundary) We define $\delta(x) = \mathrm{dist}(x,Y)$. Let us recall that $|\delta(x)-\delta(y)| \le |x-y|_X$. Let $\widehat{\delta}(x) = \inf_{y\in Y}\{q(x,y)\}$ denote the predistance of x to the boundary. We will see later that it will coincide with $\inf_{y\in Y}\{\widehat{q}(x,y)\}$.

If $x \notin X$, we will often denote by w_x a point in Y such that $\delta(x) = |x - w_x|_X$. When there are no ambiguity, we will just write $w = w_x$.

Most of the time, it will not be necessary to specify whether Y is bounded or not. Hence, in the sequel, we will treat the case of Y bounded and unbounded simultaneously. The condition $\delta(x) \ge \mathrm{diam}Y/2$ will only be used when Y is bounded, and may be skipped otherwise. The important case is the other one, $\delta(x) < \mathrm{diam}Y/2$, which always applies as soon as Y is unbounded.

The next two lemmata provide us with estimates of $\widehat{\delta}$.

Lemma 2.1. *Let $x \in X$ and $w \in Y$ be such that $\delta(x) = |x-w|_X$. Then $q(x,w) \lesssim \widehat{\delta}(x)$.*

Proof. We first assume that Y is bounded. If $\delta(x) \ge \mathrm{diam}Y/2$, then for all $y \in Y$, $q(x,y) = |x-y|_X$ so that $q(x,w) = \delta(x) = \widehat{\delta}(x)$.

Otherwise, let $w' \in A(x,w)$. Let us note that from Proposition 1.2 applied to $B = Y$ and $A = \{w, w'\}$, it follows that

$$q(x,w) \le \eta\left(\frac{2|w'-w|_X}{\mathrm{diam}Y}\right)\mathrm{diam}Y \le \eta\left(\frac{2|w-x|_X}{\mathrm{diam}Y}\right)\mathrm{diam}Y \le \eta(1)\mathrm{diam}Y;$$

thus, for any $y \in Y$ such that $|x-y|_X \ge \mathrm{diam}Y/2$, $q(x,w) \lesssim q(x,y)$ holds. We are now left with the case of points $y \in Y$ with $|x-y|_X < \mathrm{diam}Y/2$.

We may now drop the assumption on the boundedness of Y, and consider a point $y \in Y$ such that $|y-x|_X < \mathrm{diam}Y/2$. Let $y' \in A(x,y)$.

If $|y-w|_X \le \lambda\delta(x)/2$ then $|y-w'|_X \ge |w-w'|_X - |w-y|_X \ge \lambda\delta(x)/2$ and $|y-w'|_X \le |y-w|_X + |w'-w|_X \le (1+\lambda/2)\delta(x)$. Therefore, since

$$\frac{q(x,w)}{q(x,y)} = \frac{|w-w'|_Y}{|y'-y|_Y} = \frac{|w'-w|_Y}{|w'-y|_Y}\frac{|w'-y|_Y}{|y'-y|_Y},$$

it follows that

$$\frac{q(x,w)}{q(x,y)} \le \eta\left(\frac{|w'-w|_X}{|w'-y|_X}\right)\cdot\eta\left(\frac{|w'-y|_X}{|y'-y|_X}\right) \le \eta(2/\lambda)\eta(1/\lambda+1/2).$$

If $|y-w|_X \ge \lambda\delta(x)/2$, then

$$q(x,w) \le \eta\left(\frac{|w-w'|_X}{|w-y|_X}\right)\eta\left(\frac{|w-y|_X}{\lambda|x-y|_X}\right)q(x,y).$$

But $|w-y|_X \le |w-x|_X + |x-y|_X \le 2|x-y|_X$ so that

$$q(x,w) \le \eta(2/\lambda)^2 q(x,y).$$

All in all, we see that $q(x,w) \lesssim q(x,y)$ for any $y \in Y$, and so $q(x,w) \lesssim \widehat{\delta}(x)$. □

Lemma 2.2. *Let $x, z \in X \setminus Y$.*

(1) *If Y is bounded, then for any $\theta > 0$, there is some $\theta' > 0$ such that, if $\delta(x) \ge \theta\,\mathrm{diam}Y$ then $\widehat{\delta}(x) \ge \theta'\,\mathrm{diam}Y$.*
(2) *If $|x-z|_X \le \delta(x)/2$, then $\widehat{\delta}(x) \asymp \widehat{\delta}(z)$.*

Proof. Let $w_x \in Y$ be such that $\delta(x) = |x-w_x|_X$. Then Lemma 2.1 implies that $q(x,w_x) \lesssim \widehat{\delta}(x)$.

(1) If $\theta \ge 1/2$, then one may choose $\theta' = \theta$. Let us assume that $\delta(x) < \mathrm{diam}Y/2$. Let $w'_x \in A(x,w_x)$ and pick also $y \in Y$ such that $|w_x - y|_X \ge \mathrm{diam}Y/2$. It follows that $|w_x - y|_Y \gtrsim \mathrm{diam}Y$ and that

$$|w_x - y|_Y \le \eta\left(\frac{|w_x-y|_X}{|w_x-w'_x|_X}\right)|w_x-w'_x|_Y \lesssim \eta\left(\frac{\mathrm{diam}Y}{\lambda\theta\mathrm{diam}Y}\right)\widehat{\delta}(x).$$

Hence $\widehat{\delta}(x) \gtrsim \mathrm{diam}Y$.

(2) The assumptions imply that $|w_x - z|_X \le (3/2)|w_x - x|_X$. Using that $\widehat{\delta}(z) \le q(z,w_x)$, it follows that

$$\frac{\widehat{\delta}(z)}{\widehat{\delta}(x)} \lesssim \frac{q(w_x,z)}{q(w_x,x)} \le \eta\left(\frac{|w_x-z|_X}{\lambda|w_x-x|_X}\right) \le \eta\left(\frac{3}{2\lambda}\right).$$

The second inequality follows from a similar argument noting that $\delta(z) \ge \delta(x)/2$ and using a point $w_z \in Y$ such that $|w_z - z|_X = \delta(z)$. □

2.1.2. The premetric. We define $\widehat{q}$ and establish several estimates which enable us to express $\widehat{q}$ in a simpler way.

Definition. (Premetric) We define $\widehat{q}\colon X \times X \to \mathbf{R}_+$ as follows:

- If $y, y' \in Y$, let $\widehat{q}(y,y') = q(y,y') = |y-y'|_Y$.
- If $x \in X\setminus Y$ and $y \in Y$, then, let $\widehat{q}(x,y) = \widehat{q}(y,x) = \inf_{y'\in Y}\{q(x,y')+q(y,y')\}$.
- If $x, x' \in X \setminus Y$, assume that $\delta(x) \ge \delta(x')$. If $|x-x'|_X \le \delta(x)/2$, then define

$$\widehat{q}(x,x') = \widehat{q}(x',x) = \frac{\widehat{\delta}(x)}{\delta(x)}|x-x'|_X,$$

and otherwise, let

$$\widehat{q}(x,x') = \widehat{q}(x',x) = \inf_{y\in Y}\{q(x,y)+q(y,x')\}.$$

Remark. One may easily check that if $x \in X \setminus Y$, then $\widehat{\delta}(x) = \inf_Y \widehat{q}(x,y)$.

The main result of this section is

Theorem 2.3. *There is a metric bi-Lipschitz equivalent to $\widehat{q}$.*

From Lemma 2.4 to Lemma 2.7, we provide estimates on $\widehat{q}$ for points out of Y and in terms of points which realise the distance to Y.

Lemma 2.4. *If $x \in X \setminus Y$, $y, w \in Y$ with $\delta(x) = |x-w|_X$, then*

$$\widehat{q}(x,y) \asymp q(x,w) + q(w,y).$$

Proof. We already know that $\widehat{q}(x,y) \le q(x,w)+q(w,y)$ from the definition of $\widehat{q}$. If $\delta(x) \ge \mathrm{diam} Y/2$, then $\widehat{q}(x,y) \ge \widehat{\delta}(x) = \delta(x)$ and $q(x,w)+q(w,y) \le \delta(x) + \mathrm{diam} Y \le 3\delta(x)$. We assume from now on that $\delta(x) < \mathrm{diam} Y/2$. Let $w' \in A(x,w)$.

• If $|x-y|_X \le 2\delta(x)$, then $|y-w|_X \le |y-x|_X + |x-w|_X \le 3\delta(x)$ so that

$$q(y,w) \le \eta(3/\lambda) q(x,w).$$

Thus

$$q(x,w) + q(w,y) \lesssim q(x,w) \lesssim \widehat{\delta}(x)$$

by Lemma 2.1 and $\widehat{\delta}(x) \lesssim \widehat{q}(x,y)$ by definition. Hence $\widehat{q}(x,y) \gtrsim q(x,w) + q(w,y)$ holds.

• If $|x-y|_X \ge 2\delta(x)$, then let $z \in Y$. If $|x-z|_X \ge \mathrm{diam} Y/2$, then, on the one hand,

$$q(x,z) + q(z,y) \ge \mathrm{diam} Y/2$$

and on the other hand,

$$q(x,w) + q(w,y) \lesssim \widehat{\delta}(x) + q(w,y) \lesssim \mathrm{diam} Y.$$

Hence

$$q(x,z) + q(z,y) \gtrsim q(x,w) + q(w,y).$$

We assume now that $|x-z|_X \le \mathrm{diam} Y/2$. Let $z' \in A(x,z)$. If $|z-y|_X \le |w-y|_X/2$, then $|w-y|_X/2 \le |z-w|_X \le 2|z-x|_X$ so that

$$\frac{|w-y|_Y}{|z-z'|_Y} = \frac{|w-y|_Y}{|z-w|_Y}\frac{|w-z|_Y}{|z-z'|_Y} \le \eta(2)\eta(2/\lambda).$$

Thus $q(w,y) \lesssim q(x,z)$ and since $q(x,w) \lesssim \widehat{\delta}(x) \le q(x,z)$ by Lemma 2.1, it follows that

$$q(x,w) + q(w,y) \lesssim q(x,z) \le q(x,z) + q(z,y)$$

for any $z \in Y \cap B(y, |w-y|_X/2)$.

On the other hand, if $|z-y|_X \ge |w-y|_X/2$, then $q(w,y) \le \eta(2) q(y,z)$ so that

$$q(x,w) + q(w,y) \lesssim q(x,z) + q(z,y).$$

Hence

$$q(x,w) + q(w,y) \asymp \widehat{q}(x,y).$$

□

We give a more precise version of Lemma 2.4:

Lemma 2.5. *Let $x \in X \setminus Y$ and $y \in Y$. Let $w \in Y$ be such that $|w-x|_X = \delta(x)$.*

- *If $|x-y|_X \le 4\delta(x)$, then $\widehat{q}(x,y) \asymp q(x,y) \asymp q(x,w)$.*
- *If $|x-y|_X \ge 4\delta(x)$, then $\widehat{q}(x,y) \asymp q(w,y)$.*

Proof. We first notice that

$$\max\{q(x,w), q(w,y)\} \lesssim \widehat{q}(x,y) \le q(x,y)$$

by Lemma 2.4 and the definition of $\widehat{q}$. We distinguish four different cases depending on the relative positions of x and y.

• If $\delta(x) \geq \operatorname{diam} Y/2$, then $q(x,y) \leq \delta(x) + \operatorname{diam} Y \leq 3\delta(x) \lesssim q(x,w)$.

• If $\delta(x) \leq \operatorname{diam} Y/2 \leq |x-y|_X \leq 4\delta(x)$, then $\widehat{\delta}(x) \gtrsim \operatorname{diam} Y$ by Lemma 2.2. Thus

$$\widehat{q}(x,y) \leq q(x,y) = |x-y|_X \asymp \delta(x) \lesssim \widehat{\delta}(x).$$

• If $|x-y|_X \leq \min\{4\delta(x), \operatorname{diam} Y/2\}$, then we consider $w' \in A(x,w)$. If $|w-y|_X \geq |w-w'|_X/2$ then, using $y' \in A(x,y)$,

$$\begin{aligned} q(x,y) &\leq \eta\left(\frac{|y'-y|_X}{|y-w|_X}\right) q(y,w) \\ &\leq \eta\left(\frac{|x-y|_X}{|y-w|_X}\right) \eta\left(\frac{|y-w|_X}{\lambda|x-w|_X}\right) q(x,w) \\ &\leq \eta(8/\lambda)\eta(5/\lambda) q(x,w), \end{aligned}$$

where we have used that $|w-y|_X \geq (\lambda/2)\delta(x)$ and $|y-w|_X \leq |y-x|_X + \delta(x) \leq 5\delta(x)$.

Similarly, if $|w-y|_X < |w-w'|_X/2$, then $|y-w'|_X \geq (\lambda/2)\delta(x)$ and $|y-w'|_X \leq |y-w|_X + |w-w'|_X \leq 6\delta(x)$ so that

$$q(x,y) \leq \eta\left(\frac{|x-y|_X}{|y-w'|_X}\right) \eta\left(\frac{|y-w'|_X}{\lambda|x-w|_X}\right) q(x,w) \leq \eta(8/\lambda)\eta(6/\lambda) q(x,w).$$

It follows that $\widehat{q}(x,y) \leq q(x,y) \lesssim q(x,w)$ and $\widehat{q}(x,y) \asymp q(x,y) \asymp q(x,w)$ holds.

• If $|x-y|_X \geq 4\delta(x)$, then $3\delta(x) \leq |w-y|_X \leq \operatorname{diam} Y$ and $|w-y|_X \geq (3/4)|x-y|_X$ so that $\delta(x) < \operatorname{diam} Y/2$ and

$$\frac{q(w,x)}{q(y,w)} \leq \eta\left(\frac{\delta(x)}{|w-y|_X}\right) \leq \eta\left(\frac{4}{3}\frac{\delta(x)}{|x-y|_X}\right) \leq \eta(1/3).$$

Thus, $\widehat{q}(x,y) \asymp \widehat{q}(x,w) + \widehat{q}(w,y) \asymp q(w,y)$. □

Lemma 2.6. *Let $x,z \in X \setminus Y$, $w_x, w_z \in Y$ such that $\delta(x) = |x-w_x|_X$ and $\delta(z) = |z-w_z|_X$. Assume $\delta(x) \geq \delta(z)$.*

(i) *If $\delta(x) \geq \operatorname{diam} Y/2$, then $\widehat{q}(x,z) \asymp |x-z|_X$.*

(ii) *If $|x-z|_X \leq \delta(x)/2$, then $\widehat{q}(x,z) \asymp (\widehat{\delta}(x)/\delta(x))|x-z|_X$.*

(iii) *If $|x-z|_X \geq \delta(x)/2$, then $\widehat{q}(x,z) \asymp q(x,w_x) + q(w_x,w_z) + q(w_z,z)$.*

Proof. Observe first that (ii) holds by definition. We first establish $\widehat{q}(x,z) \gtrsim q(x,w_x) + q(w_x,w_z) + q(w_z,z)$. For any $y \in Y$, Lemma 2.4 implies

$$\begin{cases} q(x,y) \gtrsim q(x,w_x) + q(w_x,y) \\ q(z,y) \gtrsim q(z,w_z) + q(w_z,y) \end{cases}$$

so that

$$\begin{aligned} q(x,y) + q(y,z) &\gtrsim q(x,w_x) + (q(w_x,y) + q(w_z,y)) + q(z,w_z) \\ &\gtrsim q(x,w_x) + q(w_x,w_z) + q(w_z,z) \end{aligned}$$

since $q(w_x,y) + q(w_z,y) = |w_x-y|_Y + |w_z-y|_Y \geq |w_x-w_z|_Y$. Taking the infimum over all $y \in Y$ yields

$$\widehat{q}(x,z) \gtrsim q(x,w_x) + q(w_x,w_z) + q(w_z,z).$$

For the other inequality implying (iii), we distinguish two cases (the first case will also deal with (i)).

• If $\delta(x) \geq \text{diam}Y/2$, then
$$|x-z|_X \leq |x-w_x|_X + |w_x - w_z|_X + |w_z - z|_X \leq 2\delta(x) + \text{diam}Y \leq 4\delta(x).$$
But $\delta(x) = \widehat{\delta}(x) \leq q(x,y) \leq q(x,y) + q(y,z)$ for any $y \in Y$, so $|x-z|_X \lesssim \widehat{q}(x,z)$.

If $|x-z|_X \leq \delta(x)/2$, then since $\widehat{\delta}(x) = \delta(x)$, it follows that $\widehat{q}(x,z) = |x-z|_X$. Otherwise $|x-z|_X \geq \delta(x)/2$ holds, and
$$\widehat{q}(x,z) \leq \widehat{\delta}(x) + \text{diam}Y + \widehat{\delta}(z) \lesssim \widehat{\delta}(x) = \delta(x)$$
since $\delta(z) \leq \delta(x)$. Hence, it follows that $\widehat{q}(x,z) \lesssim |x-z|_X$.

Since we established $|x-z|_X \lesssim \widehat{\delta}(x)$, we obtain from (i) that
$$\widehat{q}(x,z) \lesssim |x-z|_X \lesssim q(x,w_x) \lesssim q(x,w_x) + q(w_x,w_z) + q(w_z,z).$$

• Let us assume that $\delta(x) < \text{diam}Y/2$; we establish
$$\widehat{q}(x,z) \lesssim q(x,w_x) + q(w_x,w_z) + q(w_z,z).$$
By definition,
$$\widehat{q}(x,z) \leq q(x,w_x) + q(w_x,z)$$
holds. In any case, $|w_x - z|_X \leq (3/2)\text{diam}Y$, so that, if $|w_x - z|_X \geq (1/2)\text{diam}Y$, then Proposition 1.2 or Lemma 2.2 ensures that
$$q(w_x,w_z) + q(w_z,z) \gtrsim \text{diam}Y \gtrsim |w_x - z|_X = q(w_x,z).$$
So we might as well assume that $|w_x - z|_X < (1/2)\text{diam}Y$. If $|w_x - z|_X \leq 2|w_x - w_z|_X$ then $q(w_x,z) \leq \eta(2) q(w_x,w_z)$ and we are done; otherwise,
$$|w_x - z|_X \leq |w_x - w_z|_X + |w_z - z|_X \leq |w_x - z|_X/2 + \delta(z)$$
and we note that $|w_x - z|_X \leq 2\delta(z)$ so that Lemma 2.5 implies that
$$q(w_x,z) \lesssim \widehat{\delta}(z) \asymp q(z,w_z).$$
This establishes $\widehat{q}(x,z) \lesssim q(x,w_x) + q(w_x,w_z) + q(w_z,z)$. □

Lemma 2.7 give further estimates on $\widehat{q}$ refining Lemma 2.6 (iii).

Lemma 2.7. *Let $x,z \in X \setminus Y$. Let us assume that $\delta(x) \geq \delta(z)$ and that $|x-z|_X \geq \delta(x)/2$. Let $w_x \in Y$ be such that $|w_x - x|_X = \delta(x)$ and let $w_z \in Y$ be such that $|w_z - z|_X = \delta(z)$.*

- *If $|w_x - w_z|_X \geq |x-z|_X/4$, then $|w_x - w_z|_X \asymp |x-z|_X$ and $\widehat{q}(x,z) \asymp q(w_x,w_z)$.*
- *If $|w_x - w_z|_X \leq |x-z|_X/4$, then $|x-z|_X \asymp \delta(x)$ and $\widehat{q}(x,z) \asymp q(x,w_x) \asymp q(x,w_z)$.*

Proof. It follows from the assumptions that $|w_x - w_z|_X \leq 5|x-z|_X$.

• If $|w_x - w_z|_X \geq |x-z|_X/4$ then $|w_x - w_z|_X \asymp |x-z|_X$ and $\delta(z) \leq \delta(x) \leq 8|w_x - w_z|_X$. It follows from Lemma 2.6 that $q(w_x,w_z) \lesssim \widehat{q}(x,z)$.

– If $\delta(x) \geq \text{diam}Y/2$, then $|w_x - w_z|_X \gtrsim \text{diam}Y$ so that $|w_x - w_z|_Y \gtrsim \text{diam}Y$ holds also. But $\delta(x) \lesssim |w_x - w_z|_X$ so
$$q(x,w_x) \lesssim \text{diam}Y \lesssim q(w_x,w_z).$$

– If $\delta(x) \leq \text{diam}Y/2$, then
$$q(x,w_x) \leq \eta\left(\frac{1}{8\lambda}\right) q(w_x,w_z).$$
Similarly, $q(w_z,z) \lesssim q(w_x,w_z)$ so that $\widehat{q}(x,z) \asymp q(w_x,w_z)$.

• If $|w_x - w_z|_X \le |x-z|_X/4$ then $\delta(x)+\delta(z) \ge (3/4)|x-z|_X$, and so $|x-z|_X \asymp \delta(x)$ holds. We deduce also that $q(w_x, w_z) \lesssim q(x, w_x)$.

By Lemma 2.4,

$$q(x, w_x) \lesssim \widehat{q}(x, w_z) \lesssim q(x, w_x)$$

always holds.

It follows that, since $|x-z|_X \le (8/3)\delta(x)$, then $|x-w_z|_X \le |x-w_x|_X + |w_x - w_z|_X \le 2\delta(x)$ and Lemma 2.5 implies that $\widehat{q}(x, w_z) \asymp q(x, w_z)$.

– If $\delta(z) \ge \operatorname{diam} Y/2$, then since $|x-w_z|_X \ge \delta(x) \ge \delta(z)$, one has $q(z, w_z) \le q(x, w_z) \lesssim q(x, w_x)$. Therefore

$$\widehat{q}(x, z) \asymp q(x, w_x) \asymp q(x, w_z).$$

– If $\delta(z) < \operatorname{diam} Y/2$, then, either $\delta(x) \ge \operatorname{diam} Y/2$, and it follows that

$$q(z, w_z) \lesssim \operatorname{diam} Y \lesssim \widehat{\delta}(x),$$

whence $\widehat{q}(x, z) \lesssim q(x, w_x)$; or $\delta(x) < \operatorname{diam} Y/2$, but then $\delta(z) \le \delta(x) \le |x-w_z|_X$ so that

$$q(z, w_z) \lesssim q(x, w_z)$$

which means that $\widehat{q}(x, z) \asymp q(x, w_z)$ holds also. □

2.1.3. Approximate triangle inequalities. From Lemma 2.8 to Lemma 2.11, we prove building blocks for the proof of Theorem 2.3 which is given afterwards.

Lemma 2.8. *If $x \in X \setminus Y$ and $y, z \in Y$, then $\widehat{q}(y, x) + \widehat{q}(x, z) \gtrsim \widehat{q}(y, z)$.*

Proof. If $\delta(x) \ge \operatorname{diam} Y/2$, then $\widehat{q}(y, x) + \widehat{q}(x, z) \ge \operatorname{diam} Y \ge \widehat{q}(y, z)$. Let us assume that $\delta(x) \le \operatorname{diam} Y/2$ and let $w \in Y$ be such that $|w-x|_X = \delta(x)$. Then Lemma 2.4 implies

$$\widehat{q}(y, x) + \widehat{q}(x, z) \gtrsim 2q(x, w) + |y-w|_Y + |z-w|_Y \ge \widehat{q}(y, z). \qquad \square$$

Lemma 2.9. *If $x, z \in X \setminus Y$ and $y \in Y$, then $\widehat{q}(x, z) + \widehat{q}(z, y) \gtrsim \widehat{q}(x, y)$.*

Proof. If $\delta(z) \ge \operatorname{diam} Y/2$, then Lemma 2.6 implies that

$$\widehat{q}(x, z) + \widehat{q}(z, y) \gtrsim |x-z|_X + |z-y|_X \ge \max\{|x-y|_X, \operatorname{diam} Y/2\} \gtrsim \widehat{q}(x, y)\,.$$

Assume from now on that $\delta(z) \le \operatorname{diam} Y/2$. Let $w_z \in Y$ be such that $\delta(z) = |z - w_z|_X$.

• If $\delta(x) \ge \operatorname{diam} Y/2$, then by Lemma 2.6,

$$\widehat{q}(x, z) + \widehat{q}(z, y) \gtrsim |x-z|_X + \widehat{\delta}(z) + |w_z - y|_Y. \tag{4}$$

Either $|x-z|_X \ge (1/2)|x-w_z|_X$ so (4) becomes

$$\widehat{q}(x, z) + \widehat{q}(z, y) \gtrsim |x-w_z|_X + |w_z - y|_Y \ge \widehat{q}(x, y).$$

Or $|x-z|_X \le (1/2)|x-w_z|_X$, and so

$$|z-w_z|_X \ge |x-w_z|_X - |z-x|_X \ge \delta(x)/2 \ge \operatorname{diam} Y/4.$$

This implies by Lemma 2.2 that $\widehat{\delta}(z) \gtrsim \operatorname{diam} Y$. Moreover,

$$|x-w_z|_X \le |x-z|_X + |z-w_z|_X \le (1/2)|x-w_z|_X + \operatorname{diam} Y/2$$

so that $|x-w_z|_X \le \operatorname{diam} Y$. Thus, $\delta(x) \lesssim \operatorname{diam} Y$ and (4) yields

$$\widehat{q}(x, z) + \widehat{q}(z, y) \gtrsim \operatorname{diam} Y \gtrsim \widehat{q}(x, w_z) + \widehat{q}(w_z, y) \ge \widehat{q}(x, y).$$

• We now assume that $\delta(x) < \operatorname{diam} Y/2$. If $|x-z|_X \ge \delta(x)/2$, then

$$\widehat{q}(x,z)+\widehat{q}(z,y) \gtrsim \widehat{\delta}(x)+q(w_x,w_z)+2\widehat{\delta}(z)+q(w_z,y) \ge \widehat{\delta}(x)+q(w_x,y) \ge \widehat{q}(x,y).$$

And if $|x-z|_X \le \delta(x)/2$, then $\delta(x) \le |x-w_z|_X \le |x-z|_X+\delta(z)$ so $\delta(x) \le 2\delta(z)$ and $|x-z|_X \le \delta(z)$. Therefore, $q(x,w_z) \le \eta(1/\lambda)q(z,w_z)$, so that $\widehat{\delta}(z) \gtrsim q(x,w_z)$ and

$$\widehat{q}(x,z)+\widehat{q}(z,y) \gtrsim \widehat{\delta}(z)+q(w_z,y) \gtrsim q(x,w_z)+q(w_z,y) \ge \widehat{q}(x,y). \qquad \square$$

Lemma 2.10. *Assume that $x,z \in X \setminus Y$ are such that $\delta(z)/2 \le |z-x|_X \le \delta(x)/2$, and let $w_x \in Y$ be such that $\delta(x)=|x-w_x|_X$. Then*

$$\widehat{q}(x,w_x)+\widehat{q}(w_x,z) \asymp \widehat{q}(x,z)\,.$$

Proof. It follows from the assumptions that $\delta(x) \asymp \delta(z) \asymp |x-z|_X$ and by Lemma 2.2 that $\widehat{\delta}(x) \asymp \widehat{\delta}(z)$. By definition,

$$\widehat{q}(x,z)=\frac{|x-z|_X}{\delta(x)}\widehat{\delta}(x) \asymp \widehat{\delta}(x).$$

Thus,

$$\widehat{q}(x,z) \lesssim \widehat{q}(x,w_x)+\widehat{q}(w_x,z).$$

On the other hand, $|z-w_x|_X \le (3/2)\delta(x) \le 3\delta(z)$, so $\widehat{q}(z,w_x) \asymp \widehat{\delta}(x) \asymp \widehat{\delta}(z)$. Hence

$$\widehat{q}(x,z) \asymp \widehat{q}(x,w_x)+\widehat{q}(w_x,z). \qquad \square$$

Lemma 2.11. *If $z_1,z_2,x \in X \setminus Y$ are such that $|x-z_j|_X \le \max\{\delta(x),\delta(z_j)\}/2$, $j=1,2$ and $|z_1-z_2|_X \ge \max\{\delta(z_1),\delta(z_2)\}/2$. Let $w \in Y$ be such that $\delta(x)=|x-w|_X$, then*

$$\widehat{q}(z_1,x)+\widehat{q}(x,z_2) \gtrsim \widehat{q}(z_1,w)+\widehat{q}(w,z_2).$$

Proof. It follows that $\delta(x) \asymp \delta(z_1) \asymp \delta(z_2)$ and so $\widehat{\delta}(x) \asymp \widehat{\delta}(z_1) \asymp \widehat{\delta}(z_2)$ holds too by Lemma 2.2. Therefore,

$$\widehat{q}(z_1,x)+\widehat{q}(x,z_2) \gtrsim \frac{\widehat{\delta}(x)}{\delta(x)}|z_1-z_2|_X \gtrsim \widehat{\delta}(x).$$

Besides, if $\max\{|z_1-w|_X,|z_2-w|_X,\delta(x)\} \le \operatorname{diam} Y/2$, then, for $j=1,2$,

$$\widehat{q}(w,z_j) \le \eta\left(\frac{|w-z_j|}{\lambda|w-x|}\right)\widehat{q}(x,w) \lesssim \eta\left(\frac{3}{\lambda}\right)\widehat{\delta}(x).$$

Hence $\widehat{q}(z_1,w)+\widehat{q}(w,z_2) \lesssim \widehat{\delta}(x) \le \widehat{q}(z_1,x)+\widehat{q}(x,z_2)$.

Otherwise, it follows that $\delta(x) \gtrsim \operatorname{diam} Y$, so that all the distances are comparable to $\delta(x)$. $\square$

Proof of Theorem 2.3. Let us define

$$\widehat{d}(x,y)=\inf\sum_{1\le j\le n}\widehat{q}(x_{j-1},x_j)$$

where the sum is taken over any finite chains $(x_0,\dots,x_n)$ such that $x_0=x$ and $x_n=y$. We will prove that $\widehat{d}(x,y) \gtrsim \widehat{q}(x,y)$ for any $x,y \in X$.

Let $x_0, \ldots, x_n$ be a chain. Let us consider the following partition. Let

$$\begin{cases} K(1) = \{x_j \in Y\}, \\ K(2) = \{x_j \notin Y,\ \delta(x_j) < \mathrm{diam} Y/2\}, \\ K(3) = \{x_j \notin Y,\ \delta(x_j) \geq \mathrm{diam} Y/2\}. \end{cases}$$

The idea of the proof is to replace as many x_j as possible by points in Y, so that we may use the triangle inequality for d_Y.

For the sake of convenience, we will write in this proof

$$\widehat{q}(x_1, x_2) + \ldots + \widehat{q}(x_{n-1}, x_n) = \widehat{q}(x_1, x_2, \ldots, x_n).$$

First step. Let $j_0 = 0$. We first construct inductively a subchain $\widehat{x}_0, \ldots, \widehat{x}_m$ until x_n is reached. Assume $j_0, \ldots, j_k$ are constructed, and let $x_{j_\ell} = \widehat{x}_\ell$ for $0 \leq \ell \leq k$.

• If $\widehat{x}_k \in K(1)$, either $x_{j_k+1} \in K(1)$ and then let $j_{k+1} = \min\{j \geq j_k,\ x_{j+1} \notin K(1)\}$, or set $j_{k+1} = j_k + 1$.

• If $\widehat{x}_k \in K(2)$, either $|x_{j_k+1} - \widehat{x}_k|_X \leq \delta(\widehat{x}_k)/2$ and then let

$$\begin{cases} j_{k+1} = \min\{j \geq j_k,\ |x_{j+1} - \widehat{x}_k| \geq \delta(\widehat{x}_k)/2\} \\ j_{k+2} = j_{k+1} + 1 \end{cases}$$

so that $|\widehat{x}_k - \widehat{x}_{k+2}|_X \geq \delta(\widehat{x}_k)/2$; or let $j_{k+1} = j_k + 1$ otherwise.

• If $\widehat{x}_k \in K(3)$, let $j_{k+1} = \min\{j \geq j_k,\ x_j \notin K(3) \text{ and } |\widehat{x}_k - x_j|_X > \delta(\widehat{x}_k)/2\}$.

In either case, it follows easily that

$$\sum_{j_k \leq j \leq j_{k+1}-1} \widehat{q}(x_j, x_{j+1}) \gtrsim \widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}).$$

Therefore

$$\sum_{0 \leq j \leq n-1} \widehat{q}(x_j, x_{j+1}) \gtrsim \sum_{0 \leq k \leq m-1} \widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}).$$

Second step. We show how to insert points from Y in between points out of Y. We define inductively on k a chain $(\widehat{y}_j)_{0 \leq j \leq p}$ with $\widehat{y}_0 = x_0$ and $\widehat{y}_p = \widehat{x}_m = x_n$. We assume that we have already constructed $\widehat{y}_1, \ldots, \widehat{y}_\ell$ with $\widehat{x}_k = \widehat{y}_\ell$ and such that, for all $j < \ell$, if $\widehat{y}_j \notin K(1)$ then $\widehat{y}_{j+1} \in K(1)$.

• If $\widehat{x}_k \in K(1)$ or $\widehat{x}_{k+1} \in K(1)$, then we set $\widehat{y}_{\ell+1} = \widehat{x}_{k+1}$.

• If $k = m-1$ then we set $\widehat{y}_{\ell+1} = \widehat{y}_p = \widehat{x}_m$. If $\widehat{x}_{m-1}, \widehat{x}_m \notin K(1)$, then $\widehat{y}_{\ell-1} \in K(1)$, so Lemma 2.9 implies that

$$\widehat{q}(\widehat{y}_{p-2}, \widehat{y}_{p-1}, \widehat{y}_p) \gtrsim \widehat{q}(\widehat{y}_{p-2}, \widehat{y}_p).$$

We assume now that $k < m - 1$.

• Let us assume that $\widehat{x}_k \in K(3)$ and $\widehat{x}_{k+1} \in K(2)$. We know from Lemma 2.6 (iii) the existence of some $y \in Y$ such that

$$\widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}) \gtrsim \widehat{q}(\widehat{x}_k, y, \widehat{x}_{k+1}).$$

We set $\widehat{y}_{\ell+1} = y$ and $\widehat{y}_{\ell+2} = \widehat{x}_{k+1}$.

• Let us assume that $\widehat{x}_k \in K(2)$ and $\widehat{x}_{k+1} \in K(2)$. If $|\widehat{x}_k - \widehat{x}_{k+1}|_X > \delta(\widehat{x}_k)/2$, then the definition of $\widehat{q}$ or Lemma 2.10 implies the existence of some point $y \in Y$ such that

$$\widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}) \gtrsim \widehat{q}(\widehat{x}_k, y, \widehat{x}_{k+1}).$$

We set $\widehat{y}_{\ell+1} = y$ and $\widehat{y}_{\ell+2} = \widehat{x}_{k+1}$.

Otherwise, $|\widehat{x}_k - \widehat{x}_{k+1}|_X \le \delta(\widehat{x}_k)/2$ but then, by construction $|\widehat{x}_k - \widehat{x}_{k+2}|_X > \delta(\widehat{x}_k)/2$, so, either $|\widehat{x}_{k+1} - \widehat{x}_{k+2}|_X \le \delta(\widehat{x}_{k+1})/2$ and then Lemma 2.11 shows the existence of some $y \in Y$ such that $\widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_k, y, \widehat{x}_{k+2})$; or, as in the previous case, there is some $y \in Y$ such that $\widehat{q}(\widehat{x}_{k+1}, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_{k+1}, y, \widehat{x}_{k+2})$, and, with Lemma 2.9, one obtains

$$\widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}, y, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_k, y, \widehat{x}_{k+2}).$$

We set $\widehat{y}_{\ell+1} = y$ and $\widehat{y}_{\ell+2} = \widehat{x}_{k+2}$.

• Let us assume that $\widehat{x}_k \in K(2)$ and $\widehat{x}_{k+1} \in K(3)$. From above, $\widehat{q}(\widehat{x}_{k+1}, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_{k+1}, y, \widehat{x}_{k+2})$ holds for some $y \in Y$. Applying Lemma 2.9 again, it follows that

$$\widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_k, \widehat{x}_{k+1}, y, \widehat{x}_{k+2}) \gtrsim \widehat{q}(\widehat{x}_k, y, \widehat{x}_{k+2}).$$

We set $\widehat{y}_{\ell+1} = y$ and $\widehat{y}_{\ell+2} = \widehat{x}_{k+2}$.

In conclusion, we have constructed a new chain $\widehat{y}_0, \dots, \widehat{y}_p$ such that, if $\widehat{y}_j \notin Y$, then $\widehat{y}_{j+1} \in Y$. Moreover, it follows that

$$\widehat{q}(x_0, \dots, x_n) \gtrsim \widehat{q}(\widehat{y}_0, \dots, \widehat{y}_p).$$

Third step. Applying Lemma 2.8, we may take out the points not in Y (besides x_0 and x_n), then use the triangle inequality on Y, and Lemma 2.6 to conclude. □

Remark. It follows from the definitions of $\widehat{q}$ and $\widehat{d}$ that, for all $y, y' \in Y$,

$$\widehat{d}(f(y), f(y')) \le d_Y(y, y'). \tag{5}$$

2.2. Quasisymmetry of the metric. We prove that the metric which has been constructed in the previous section is power quasisymmetric equivalent to d_X.

If $f\colon Z \to Z'$ is a homeomorphism between doubling and connected (or quasiconvex) metric spaces, then the 3-point condition (1) follows from an a priori weaker condition (cf. [13, Thm. 10.19] and [24, Thm. 6.6]): there is a finite constant H such that, for any $z_1, z_2, z_3 \in Z$,

$$|z_1 - z_2| \le |z_1 - z_3| \Longrightarrow |f(z_1) - f(z_2)| \le H \cdot |f(z_1) - f(z_3)|.$$

In our setting however, the proof of the weak quasisymmetry would also require many case by case verifications which do not greatly simplify the arguments. Hence we prefer to prove directly the strong quasisymmetry condition (1) of the identity map in full generality, even when d_X and $\widehat{d}$ enjoy stronger properties. We also note that [24] contains a discussion on relative quasisymmetry; unfortunately, it barely simplifies the proof given below.

Since $\widehat{d}$ and $\widehat{q}$ is bi-Lipschitz, it is enough to establish (1) for $\widehat{q}$, namely: there exists an increasing homeomorphism $\widehat{\eta}\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $x_1, x_2, x_3 \in X$,

$$\frac{\widehat{q}(x_1, x_2)}{\widehat{q}(x_1, x_3)} \le \widehat{\eta}\left(\frac{|x_1 - x_2|_X}{|x_1 - x_3|_X}\right).$$

The proof is cut into six lemmata, each of which deals with a specific situation provided by our estimates for $\widehat{q}$. Since Y is uniformly perfect, we know that η may be chosen of the form $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ with $C \ge 1$ and $\alpha \in (0,1)$. In each lemma, the existence of some increasing homeomorphism $\widehat{\eta}_*$ of the positive reals will be established: it will always be of the same form with the same exponent α.

Lemma 2.12. *A homeomorphism $\widehat{\eta}_1\colon \mathbf{R}_+ \to \mathbf{R}_+$ exists such that, for any $x \in X \setminus Y$ and any $y, z \in Y$,*

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \le \widehat{\eta}_1\left(\frac{|y-x|_X}{|y-z|_X}\right)$$

and

$$\frac{\widehat{q}(y,z)}{\widehat{q}(y,x)} \le \widehat{\eta}_1\left(\frac{|y-z|_X}{|y-x|_X}\right).$$

Proof. Let us first note that if $\delta(x) \ge \mathrm{diam}Y/8$, then $\widehat{q}(x,y) \gtrsim \mathrm{diam}Y$. Therefore, if $y' \in Y$ satisfies $|y-y'|_X \ge \mathrm{diam}Y/2$, then

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} = \frac{\widehat{q}(y,x)}{\widehat{q}(y,y')}\frac{\widehat{q}(y,y')}{\widehat{q}(y,z)} \lesssim \frac{|y-x|_X}{\mathrm{diam}Y}\eta\left(\frac{\mathrm{diam}Y}{|y-z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right),$$

where we have used (2) for the last inequality knowing that $|y-x|_X \gtrsim \mathrm{diam}Y$ and $\mathrm{diam}Y \ge |y-z|_X$.

We proceed similarly for the other inequality.

We now assume that $\delta(x) \le \mathrm{diam}Y/8$. It follows from Lemma 2.5 that if $|x-y|_X \le 4\delta(x)$, then

$$\widehat{q}(x,y) \asymp q(x,y) \le \eta\left(\frac{|x-y|_X}{|y-z|_X}\right)\widehat{q}(y,z).$$

Similarly,

$$\widehat{q}(x,z) \le \eta\left(\frac{|y-z|_X}{\lambda|x-y|_X}\right) q(x,y) \lesssim \widehat{q}(x,y).$$

Otherwise, $\widehat{q}(x,y) \asymp q(w,y)$ where $w \in Y$ satisfies $|w-x|_X = \delta(x)$. Therefore

$$\widehat{q}(y,x) \asymp q(y,w) \le \eta\left(\frac{|y-w|_X}{|y-z|_X}\right)\widehat{q}(y,z) \le \eta\left(\frac{5|y-x|_X}{4|y-z|_X}\right)\widehat{q}(y,z),$$

and

$$\widehat{q}(y,z) \le \eta\left(\frac{|y-z|_X}{|y-w|_X}\right)\widehat{q}(y,w) \lesssim \eta\left(\frac{4|y-z|_X}{3|y-x|_X}\right) q(y,w). \qquad \square$$

Lemma 2.13. *A homeomorphism $\widehat{\eta}_2 : \mathbf{R}_+ \to \mathbf{R}_+$ exists such that, for any $x \in X \setminus Y$ and any $y, z \in Y$,*

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \le \widehat{\eta}_2\left(\frac{|x-z|_X}{|x-y|_X}\right).$$

Proof. If $\delta(x) \ge \mathrm{diam}Y/8$, then the lemma is trivially true. Let $w \in Y$ with $\delta(x) = |x-w|_X$.

We will make a repeated use of Lemma 2.5.

• If $|x-z|_X \le 4\delta(x)$ and $|x-y|_X \le 4\delta(x)$, then $|x-y|_X \asymp |x-z|_X \asymp \delta(x)$ and $\widehat{q}(x,y) \asymp \widehat{q}(x,z) \asymp \widehat{\delta}(x)$.

• If $|x-z|_X \ge 4\delta(x)$ and $|x-y|_X \le 4\delta(x)$, then

$$\widehat{q}(x,y) \asymp q(x,w) \le \eta\left(\frac{\delta(x)}{|w-z|_X}\right) q(w,z) \lesssim \eta\left(\frac{4|x-y|_X}{3|x-z|_X}\right)\widehat{q}(x,z).$$

• If $|x-z|_X \geq 4\delta(x)$ and $|x-y|_X \geq 4\delta(x)$, then $\widehat{q}(x,y) \asymp q(w,y)$ and $\widehat{q}(x,z) \asymp q(w,z)$. Thus

$$\widehat{q}(x,y) \asymp q(y,w) \leq \eta\left(\frac{|w-y|_X}{|w-z|_X}\right) q(w,z) \lesssim \eta\left(\frac{5|x-y|_X}{3|x-z|_X}\right)\widehat{q}(x,z).$$

• If $|x-z|_X \leq 4\delta(x)$ and $|x-y|_X \geq 4\delta(x)$, then

$$\widehat{q}(x,y) \asymp q(w,y) \leq \eta\left(\frac{|w-y|_X}{\lambda|w-x|_X}\right) q(w,x) \lesssim \eta\left(\frac{5|x-y|_X}{\lambda|x-z|_X}\right)\widehat{q}(x,z). \qquad \square$$

Lemma 2.14. *A homeomorphism* $\widehat{\eta}_3\colon \mathbf{R}_+ \to \mathbf{R}_+$ *exists such that, if* $x,z \in X\backslash Y$, *and if* $y \in Y$, *then*

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \leq \widehat{\eta}_3\left(\frac{|y-x|_X}{|y-z|_X}\right).$$

Proof. We distinguish several cases according to Lemma 2.5.

• Assume that $|y-x|_X \leq 4\delta(x)$ and $|y-z|_X \leq 4\delta(z)$, then

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \asymp \frac{q(y,x)}{q(y,z)}.$$

– If $\max\{\delta(x),\delta(z)\} \leq \mathrm{diam}Y/2$, then

$$\frac{q(y,x)}{q(y,z)} \leq \eta\left(\frac{|y-x|_X}{\lambda|y-z|_X}\right).$$

– If $\min\{\delta(x),\delta(z)\} \geq \mathrm{diam}Y/2$, then we have equality.

– If $\delta(x) \leq \mathrm{diam}Y/2 \leq \delta(z)$ then let $y' \in Y$ be such that $|y-y'|_X \geq \mathrm{diam}Y/2$. It follows that

$$\frac{q(y,x)}{q(y,z)} \asymp \frac{|y-y'|_X}{|y-z|_X}\frac{q(y,x)}{|y-y'|_Y} \lesssim \frac{\mathrm{diam}Y}{|y-z|_X}\cdot\eta\left(\frac{|y-x|_X}{\mathrm{diam}Y}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right),$$

where (3) has been used to conclude.

– If $\delta(z) \leq \mathrm{diam}Y/2 \leq \delta(x)$ then let also $y' \in Y$ be such that $|y-y'|_X \geq \mathrm{diam}Y/2$. We proceed as above. One gets

$$\frac{q(y,x)}{q(y,z)} \asymp \frac{|y-x|_X}{|y-y'|_X}\frac{|y-y'|_Y}{q(y,z)} \lesssim \frac{|y-x|_X}{\mathrm{diam}Y}\cdot\eta\left(\frac{\mathrm{diam}Y}{|y-z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right),$$

applying (2) to conclude.

• Assume that $|y-x|_X \leq 4\delta(x)$ and $|y-z|_X \geq 4\delta(z)$. Then

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \asymp \frac{q(y,x)}{q(y,w_z)},$$

where $w_z \in Y$ satisfies $|z-w_z|_X = \delta(z)$. Furthermore, $\delta(z) \leq \mathrm{diam}Y/3$.

– If $\delta(x) \leq \mathrm{diam}Y/2$, then

$$\frac{q(y,x)}{q(y,w_z)} \leq \eta\left(\frac{|y-x|_X}{|y-w_z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right).$$

– If $\delta(x) \geq \mathrm{diam}Y/2$, then let $y' \in Y$ be such that $|y-y'|_X \geq \mathrm{diam}Y/2$. One obtains

$$\frac{q(y,x)}{q(y,w_z)} \asymp \frac{|y-x|_X}{|y-y'|_X}\frac{|y-y'|_Y}{q(y,w_z)} \lesssim \frac{|y-x|_X}{\mathrm{diam}Y}\cdot\eta\left(\frac{\mathrm{diam}Y}{|y-w_z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right),$$

where (2) has also been used to conclude.

• Assume that $|y-x|_X \ge 4\delta(x)$ and $|y-z|_X \le 4\delta(z)$. Then

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \asymp \frac{q(y,w_x)}{q(y,z)},$$

where $w_x \in Y$ satisfies $|x-w_x|_X = \delta(x)$. Furthermore, $\delta(x) \le \mathrm{diam} Y/3$.

– If $\delta(z) \le \mathrm{diam} Y/2$, then

$$\frac{q(y,w_x)}{q(y,z)} \lesssim \eta\left(\frac{|y-w_x|_X}{|y-z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right).$$

– If $\delta(z) \ge \mathrm{diam} Y/2$, then let $y' \in Y$ be such that $|y-y'|_X \ge \mathrm{diam} Y/2$. One obtains

$$\frac{q(y,w_x)}{q(y,z)} \asymp \frac{|y-y'|_X}{|y-z|_X}\frac{q(y,w_x)}{|y-y'|_Y} \lesssim \frac{\mathrm{diam} Y}{|y-z|_X}\cdot\eta\left(\frac{|y-w_x|_X}{\mathrm{diam} Y}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right)$$

by (3).

• Assume that $|y-x|_X \ge 4\delta(x)$ and $|y-z|_X \ge 4\delta(z)$. Then

$$\frac{\widehat{q}(y,x)}{\widehat{q}(y,z)} \asymp \frac{q(y,w_x)}{q(y,w_z)}.$$

Furthermore, $\max\{\delta(x),\delta(z)\} \le \mathrm{diam} Y/3$. Thus

$$\frac{q(y,x)}{q(y,z)} \asymp \frac{q(y,w_x)}{q(y,w_z)} \lesssim \eta\left(\frac{|y-w_x|_X}{|y-w_z|_X}\right) \lesssim \eta\left(\frac{|y-x|_X}{|y-z|_X}\right). \qquad \square$$

Lemma 2.15. *A homeomorphism $\widehat{\eta}_4\colon \mathbf{R}_+ \to \mathbf{R}_+$ exists such that, if $x,z \in X\setminus Y$ are such that $|z-x| \le \delta(x)/2$, and if $y \in Y$, then*

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \le \widehat{\eta}_4\left(\frac{|x-z|_X}{|x-y|_X}\right)$$

and

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \le \widehat{\eta}_4\left(\frac{|x-y|_X}{|x-z|_X}\right).$$

Proof. We pick $w \in Y$ such that $\delta(x) = |x-w|_X$.

• If $|x-y|_X \le 4\delta(x)$, then Lemma 2.5 implies that $\widehat{q}(x,y) \asymp q(x,w)$.

$$\begin{cases} \dfrac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \dfrac{|x-z|_X}{\delta(x)}\dfrac{\widehat{\delta}(x)}{q(x,w)} \lesssim \dfrac{|x-z|_X}{|x-y|_X}, \\ \dfrac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \asymp \dfrac{q(x,w)}{\widehat{\delta}(x)}\dfrac{\delta(x)}{|x-z|_X} \lesssim \dfrac{|x-y|_X}{|x-z|_X}. \end{cases}$$

• Otherwise, $|w-y|_X \ge 3\delta(x)$, and $|x-y|_X \ge |w-y|_X - |x-w|_X \ge (2/3)|w-y|_X$. Thus,

$$\widehat{q}(w,y) \le \widehat{q}(x,y) \asymp \widehat{q}(w,x) + \widehat{q}(w,y) \lesssim \widehat{q}(w,y)$$

so that

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{|x-z|_X}{\delta(x)}\frac{\widehat{\delta}(x)}{\widehat{q}(w,y)} \lesssim \frac{|x-z|_X}{\delta(x)}\eta\left(\frac{\delta(x)}{|x-y|_X}\right) \lesssim \eta\left(\frac{|x-z|_X}{|x-y|_X}\right).$$

Similarly,

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \asymp \frac{\widehat{q}(w,y)}{\widehat{\delta}(x)} \frac{\delta(x)}{|x-z|_X} \lesssim \frac{\delta(x)}{|x-z|_X} \eta\left(\frac{|w-y|_X}{\delta(x)}\right) \lesssim \eta\left(\frac{|x-y|_X}{|x-z|_X}\right).$$

where we have used (3) and (2), respectively. □

Lemma 2.16. *A homeomorphism* $\widehat{\eta}_5 \colon \mathbf{R}_+ \to \mathbf{R}_+$ *exists such that, if* $x, z \in X \setminus Y$ *are such that* $|z-x|_X \geq \delta(x)/2$*, and if* $y \in Y$*, then*

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \leq \widehat{\eta}_5\left(\frac{|x-z|_X}{|x-y|_X}\right)$$

and

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \leq \widehat{\eta}_5\left(\frac{|x-y|_X}{|x-z|_X}\right).$$

Proof. We remark that if $\delta(x) \geq \mathrm{diam} Y/8$, then the lemma follows easily, so let us assume that $\delta(x) < \mathrm{diam} Y/8$. Also, if $|x-z|_X \leq \delta(z)/2$, then the lemma follows essentially as in the proof of Lemma 2.15 since then $\delta(x) \asymp \delta(z)$. So we assume $|x-z|_X \geq \delta(z)/2$. In particular, $|w_x - w_z|_X \leq 5|x-z|_X$ holds.

We will use Lemma 2.5 and Lemma 2.7. So let $w_x \in Y$ be such that $|w_x - x|_X = \delta(x)$ and let $w_z \in Y$ be such that $|w_z - z|_X = \delta(z)$.

• If $|x-y|_X \leq 4\delta(x)$ and if $|w_x - w_z|_X \geq |x-z|_X/4$, then

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{\widehat{q}(w_x,w_z)}{\widehat{q}(w_x,x)} \leq \widehat{\eta}_1\left(\frac{|w_x-w_z|_X}{\delta(x)}\right).$$

Similarly,

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \asymp \frac{\widehat{q}(w_x,x)}{\widehat{q}(w_x,w_z)} \leq \widehat{\eta}_1\left(\frac{\delta(x)}{|w_x-w_z|_X}\right)$$

which provides the result since $|w_x - w_z|_X \gtrsim |x-z|_X$.

• If $|x-y|_X \leq 4\delta(x)$ and if $|w_x - w_z|_X \leq |x-z|_X/4$, then we should distinguish two cases. If $|x-z|_X \asymp \delta(x)$, then all the ratios are essentially constant, so it is fine. Otherwise, $|x-z|_X \asymp \delta(z)$, and Lemma 2.7 implies that $\widehat{q}(x,z) \asymp q(w_x,z)$. Therefore,

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{\widehat{q}(w_x,z)}{\widehat{q}(w_x,x)} \leq \widehat{\eta}_3\left(\frac{|w_x-z|_X}{\delta(x)}\right).$$

Similarly,

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \asymp \frac{\widehat{q}(w_x,x)}{\widehat{q}(w_x,z)} \leq \widehat{\eta}_3\left(\frac{\delta(x)}{|w_x-z|_X}\right).$$

• If $|x-y|_X \geq 4\delta(x)$ and if $|w_x - w_z|_X \geq |x-z|_X/4$, then

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{\widehat{q}(w_x,w_z)}{\widehat{q}(w_x,y)} \leq \widehat{\eta}_1\left(\frac{|w_x-w_z|_X}{|w_x-y|_X}\right).$$

Similarly,

$$\frac{\widehat{q}(x,y)}{\widehat{q}(x,z)} \asymp \frac{\widehat{q}(w_x,y)}{q(w_x,w_z)} \leq \widehat{\eta}_1\left(\frac{|w_x-y|_X}{|w_x-w_z|_X}\right).$$

• If $|x-y|_X \geq 4\delta(x)$ and if $|w_x - w_z|_X \leq |x-z|_X/4$, then we should also distinguish two cases. Either $|z-x|_X \asymp \delta(x)$, and then

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{\widehat{q}(w_x,x)}{\widehat{q}(w_x,y)} \leq \widehat{\eta}_1\left(\frac{|w_x-x|_X}{|w_x-y|_X}\right),$$

and similarly for the inverse.

Or $|z-x|_X \asymp \delta(z)$. In this case,

$$\frac{\widehat{q}(x,z)}{\widehat{q}(x,y)} \asymp \frac{q(w_x,z)}{\widehat{q}(w_x,y)} \leq \widehat{\eta}_1\left(\frac{|w_x-z|_X}{|w_x-y|_X}\right),$$

which also enables us to conclude. We leave the last estimate to the reader's attention. □

Lemma 2.17. *A homeomorphism $\widehat{\eta}_6\colon \mathbf{R}_+ \to \mathbf{R}_+$ exists such that, if $x, z_1, z_2 \in X \setminus Y$, then*

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \leq \widehat{\eta}_6\left(\frac{|x-z_1|_X}{|x-z_2|_X}\right).$$

Proof. It follows from Lemma 2.7 that it is enough to consider the case $\delta(x) < \operatorname{diam} Y/2$.

• If $|x-z_j|_X \leq \delta(x)/2$ for $j=1,2$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{|x-z_1|_X}{|x-z_2|_X}.$$

• If $|x-z_1|_X \leq \delta(x)/2$ and $|x-z_2|_X \geq \delta(x)/2$ then we discuss according to the value of $\widehat{q}(x,z_2)$ provided by Lemma 2.7. We let $w \in Y$ be such that $|w-x|_X = \delta(x)$ and $w_2 \in Y$ be such that $|w_2 - z_2|_X = \delta(z_2)$.

– If $|w-w_2|_X \geq |x-z_2|_X/4$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{|x-z_1|_X}{\delta(x)}\frac{q(x,w)}{q(w,w_2)} \lesssim \frac{|x-z_1|_X}{\delta(x)}\widehat{\eta}_1\left(\frac{\delta(x)}{|w-w_2|_X}\right) \lesssim \widehat{\eta}_1\left(\frac{|x-z_1|_X}{|x-z_2|_X}\right),$$

where the special form of $\widehat{\eta}_1$ and (3) have been used.

– If $|w-w_2|_X \leq |x-z_2|_X/4$ and $q(x,z_2) \asymp q(x,w)$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{|x-z_1|_X}{\delta(x)}\frac{\widehat{\delta}(x)}{\widehat{\delta}(x)} \lesssim \frac{|x-z_1|_X}{|x-z_2|_X}.$$

– If $|w-w_2|_X \leq |x-z_2|_X/4$ and $q(x,z_2) \asymp q(z_2,w)$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{|x-z_1|_X}{\delta(x)}\frac{q(w,x)}{q(w,z_2)} \lesssim \frac{|x-z_1|_X}{\delta(x)}\widehat{\eta}_3\left(\frac{\delta(x)}{|w-z_2|_X}\right) \lesssim \widehat{\eta}_3\left(\frac{|x-z_1|_X}{|x-z_2|_X}\right)$$

using (3) as well for $\widehat{\eta}_3$.

• If $|x-z_1|_X \geq \delta(x)/2$ and $|x-z_2|_X \leq \delta(x)/2$, then we discuss according to the value of $\widehat{q}(x,z_1)$ as above. We let $w \in Y$ be such that $|w-x|_X = \delta(x)$ and $w_1 \in Y$ be such that $|w_1 - z_1|_X = \delta(z_1)$.

– If $|w-w_1|_X \geq |x-z_1|_X/4$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{\delta(x)}{|x-z_2|_X}\frac{q(w,w_1)}{q(x,w)} \lesssim \frac{\delta(x)}{|x-z_2|_X}\widehat{\eta}_1\left(\frac{|w-w_1|_X}{\delta(x)}\right) \lesssim \widehat{\eta}_1\left(\frac{|x-z_1|_X}{|x-z_2|_X}\right)$$

using (2).

– If $|w-w_1|_X \le |x-z_1|_X/4$ and $q(x,z_1) \asymp q(x,w)$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{\delta(x)}{|x-z_2|_X}\frac{\widehat{\delta}(x)}{\widehat{\delta}(x)} \lesssim \frac{|x-z_1|_X}{|x-z_2|_X}.$$

– If $|w-w_1|_X \le |x-z_1|_X/4$ and $q(x,z_1) \asymp q(z_1,w)$, then

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)} \asymp \frac{\delta(x)}{|x-z_2|_X}\frac{q(w,z_1)}{q(w,x)} \lesssim \frac{\delta(x)}{|x-z_2|_X}\widehat{\eta}_3\left(\frac{|w-z_1|_X}{\delta(x)}\right) \lesssim \widehat{\eta}_3\left(\frac{|x-z_1|_X}{|x-z_2|_X}\right)$$

by (2).

• If $|x-z_j|_X \ge \delta(x)/2$ for $j=1,2$, then we discuss according to the values of $\widehat{q}(x,z_1)$ and $\widehat{q}(x,z_2)$ given by Lemma 2.7. It follows that $\widehat{q}(x,z_j)$ can be approximated by either $q(w,w_j)$, $q(w,x)$ or $q(w,z_j)$. In each case, we get similar approximations for the distance in X. So, looking at the ratio

$$\frac{\widehat{q}(x,z_1)}{\widehat{q}(x,z_2)}$$

always yields us to compare two distances concerning w, for which we may apply the quasisymmetry assumption.

When $\delta(z_j) \ge \operatorname{diam} Y/2$, then one is led to use an intermediate point $w' \in Y$ such that $|w-w'|_X \ge \operatorname{diam} Y/2$ and to use the specific form of η as in previous proofs. Nonetheless, the inequality follows essentially as above. □

Proof of Theorem 2. It follows from Theorem 2.3 that there is a metric bi-Lipschitz $\widehat{d}$ equivalent to $\widehat{q}$. This implies that the proof of the quasymmetry of $\mathrm{Id}\colon (X,d_X) \to (X,\widehat{d})$ reduces to the establishment of the 3-point condition for $\widehat{q}$. This was done in Lemma 2.12, 2.13, 2.14, 2.15, 2.16 and 2.17.

Lemma 2.2 implies that the restriction $\mathrm{Id}\colon (B(x,\delta(x)/2),d_X) \to (X\setminus Y,\widehat{d})$ is locally bi-Lipschitz of factors equivalent to $\widehat{\delta}(x)/\delta(x)$. Hence Id is locally quasisimilar.

Finally, the definition of $\widehat{q}$ makes $\mathrm{Id}\colon (Y,d_Y) \to (Y,\widehat{d})$ uniformly bi-Lipschitz. □

2.3. Further properties of the new metrics. Since the map $\mathrm{Id}\colon (X,d_X) \to (X,\widehat{d})$ is quasisymmetric, $(X,\widehat{d})$ has all the properties of (X,d_X) which are invariant under quasisymmetric maps. Furthermore, it is also easy to see the following relation between the Hausdorff dimensions: $\dim(X,\widehat{d}) = \max\{\dim(Y,d_Y), \dim(X\setminus Y,d_X)\}$.

We now turn to the proof of Corollary 3:

Proof of Corollary 3. Let $\varepsilon > 0$; we first choose a metric d_X in the gauge of X such that $\dim(X,d_X) \le \dim_c X + \varepsilon$. If $\dim(Y,d_X) \le \dim_c Y + \varepsilon$ holds as well, then we are done. Otherwise, we choose a metric d_Y in the gauge of Y such that $\dim(Y,d_Y) \le \dim_c Y + \varepsilon$. Applying Theorem 2 to the embedding $(Y,d_Y) \to (X,d_X)$ establishes the corollary. □

Proposition 2.18. *We choose X and Y as in Theorem 2. We assume furthermore that*

(i) *the space (X,d_X) is Q-Ahlfors regular for some $Q>0$;*
(ii) *the subset $f(Y)$ is porous in X, and, constants $\theta \in (0,1)$ and $C>0$ exist such that, for any $r \le \operatorname{diam} Y$ and $\beta \ge 1$, any d_Y-ball $B_Y(y,r) \subset Y$ can be covered by at most $Ce^{Q\beta}\theta^\beta$ balls of radius $e^{-\beta}r$.*

Then $(X,\widehat{d})$ is Q-Ahlfors regular.

Let us note that the second assumption in (ii) is an upper bound for the box dimension of (Y, d_Y), and it holds for porous subsets of Q-regular metric spaces (see § 1.1). Moreover, Theorem 2 and Proposition 1.2 imply that if Y is porous in (X, d_X), then it has to be porous in $(X, \widehat{d})$ also, so this assumption is also necessary for the Ahlfors regularity of $(X, \widehat{d})$ since d_Y and $\widehat{d}$ are bi-Lipschitz.

Proof. Let us denote by μ the Q-Hausdorff measure in $(X, \widehat{d})$. We will write $B_X(x,r)$ for a ball in (X, d_X) and $\widehat{B}(x,r)$ for a ball in $(X, \widehat{d})$.

Since $\mathrm{Id}\colon (X \setminus Y, d_X) \to (X \setminus Y, \widehat{d})$ is a quasisimilarity, it follows that, for any $x \in X \setminus Y$ and any $r \leq 1/2$, the ball $B_X(x, r\delta(x))$ corresponds approximately to a ball of radius $Cr\widehat{\delta}(x)$ in $(X, \widehat{d})$ for some universal constant $C > 0$, so that $\mu(\widehat{B}(x, r\widehat{\delta}(x))) \asymp (r\widehat{\delta}(x))^Q$.

Let us fix a point $y \in Y$ and $r > 0$. Since Y is porous, a constant $c > 0$ exists such that $\widehat{B}(y,r)$ contains a ball $\widehat{B}(x, cr)$ disjoint from Y. Therefore $cr \leq \widehat{\delta}(x)$ so

$$\mu(\widehat{B}(y,r)) \geq \mu(\widehat{B}(x, \widehat{\delta}(x))) \gtrsim \widehat{\delta}(x)^Q \gtrsim r^Q.$$

For the converse inequality, we first note that $\mu(Y) = 0$ by assumption (ii). Thus it is enough to bound $\mu(\widehat{B}(y,r) \setminus Y)$. We cover $\widehat{B}(y,r) \setminus Y$ by balls $B_X(x, \delta(x)/10)$. We extract an at most countable subfamily $B_X(x_j, \delta_j/10)$ of balls pairwise disjoint such that $\widehat{B}(y,r) \setminus Y \subset \cup B_X(x_j, \delta_j/2)$ (Theorem 1.2 in [13]).

Denote by A_n the set of centers (x_j) such that $re^{-(n+1)} < \widehat{\delta}(x_j) \leq re^{-n}$. It follows that if $x_j \in A_n$, then $\mu(B_X(x_j, \delta_j/2)) \asymp \widehat{\delta}(x_j)^Q \asymp r^Q e^{-nQ}$. For each x_j, choose a point $y_j \in Y$ such that $\widehat{\delta}(x_j) = \widehat{d}(x, y_j)$. Since $(X, \widehat{d})$ is doubling and the balls $\{B_X(x_j, \delta_j/10)\}_j$ are disjoint, the nerve of the family of balls $\{\widehat{B}(y_j, \widehat{\delta}(x_j)),\ x_j \in A_n\}$ has uniformly bounded valence V (independent from n). Therefore, we may split this family of balls into $V + 1$ families of pairwise disjoint balls. The assumption (ii) and the fact that $f\colon (Y, d_Y) \to (Y, \widehat{d})$ is bi-Lipschitz now imply that the number of balls involved in A_n is bounded by $e^{Qn}\theta^n$ up to a factor (which depends on the bi-Lipschitz constant, the uniform perfectness of (Y, d_Y) and V). Thus

$$\sum_{A_n} \mu(B_X(x_j, \delta_j/2)) \lesssim e^{Qn}\theta^n \left(e^{-n}r\right)^Q \lesssim \theta^n r^Q.$$

Therefore,

$$\mu(\widehat{B}(y,r)) \leq \sum_{n\geq 0} \sum_{A_n} \mu(B_X(x_j, \delta_j/2)) \lesssim \sum_{n \geq 0} \theta^n r^Q \lesssim r^Q.$$

Let us consider a point $x \in X \setminus Y$, and let $y \in Y$ be such that $\widehat{\delta}(x) = \widehat{d}(x,y)$. If $r \in [\widehat{\delta}(x)/2, 2\widehat{\delta}(x)]$, then

$$\mu(\widehat{B}(x,r)) \geq \mu(\widehat{B}(x, \widehat{\delta}(x)/2)) \gtrsim \widehat{\delta}(x)^Q \gtrsim r^Q.$$

On the other hand,

$$\mu(\widehat{B}(x,r)) \leq \mu(\widehat{B}(y,2r)) \lesssim r^Q.$$

If $r \geq 2\widehat{\delta}(x)$, then $\widehat{B}(y, r - \widehat{\delta}(x)) \subset \widehat{B}(x,r) \subset \widehat{B}(y, r + \widehat{\delta}(x))$ with $r - \widehat{\delta}(x) \geq r/2$ and $r + \widehat{\delta}(x) \leq (3/2)r$, so $\mu(\widehat{B}(x,r)) \asymp r^Q$. □

Corollary 2.19. *Let us assume that X and Y satisfy the assumptions of Proposition 2.18. If (X, d_X) is Q-Loewner, then $(X, \widehat{d})$ is Q-Loewner too.*

Proof. From Proposition 2.18, it is known that (X, d_X) and $(X, \widehat{d})$ are both Q-Ahlfors regular. A theorem of Tyson implies under these assumptions that the Loewner condition is preserved under quasisymmetric equivalence (cf. [21]). □

3. Angles at seams

Definition. Let X_1 and X_2 be two closed subsets of a metric space X such that $X_1 \cap X_2 \neq \emptyset$. The *seam* is by definition the closed set $Y = X_1 \cap X_2$. Following Agard and Gehring [1], the *angle* $\angle(X_1, X_2)$ between X_1 and X_2 is by definition the supremum over all $c > 0$ such that, for any $(x_1, x_2) \in X_1 \times X_2$,

$$|x_1 - x_2| \geq c \cdot \inf_{y \in Y}\{|x_1 - y| + |y - x_2|\}.$$

Theorem 3.1. *Let $X = X_1 \cup X_2$ and $X' = X_1' \cup X_2'$ be metric spaces with positive angles. Let us assume that $Y = X_1 \cap X_2$ and $Y' = X_1' \cap X_2'$ are λ-uniformly perfect subspaces such that $\operatorname{diam} Y \geq \mu \operatorname{diam} X_1$ for some $\mu \in (0, 1)$.*

If $f \colon X \to X'$ is a homeomorphism such that $f|_{X_j}$ is η-quasisymmetric and $f(X_j) = X_j'$, then f is globally $\widehat{\eta}$-quasisymmetric quantitatively.

Remark. A condition such as $\operatorname{diam} Y \geq \mu \operatorname{diam} X_1$ is necessary for the theorem to be true. A counter-example will be given after the proof of Theorem 1.

We will reduce its proof to the following result due to Aseev, Kuzin and Tetenov [3, Thm. 3.1].

Theorem 3.2. *A map $f \colon X_1 \cup X_2 \to X'$ is quasisymmetric under the following conditions:*

(a1) *f is η-quasisymmetric on each set X_i;*
(a2) *$f(X_1) \cap f(X_2) = f(X_1 \cap X_2)$;*
(a3) *$\angle(X_1, X_2) > 0$ and $\angle(f(X_1), f(X_2)) > 0$;*
(a4) *the map f is η-quasisymmetric at y for any $y \in X_1 \cap X_2$ i.e., for all $y \in X_1 \cap X_2$, for any $x_1, x_2 \in X_1 \cup X_2$,*

$$\frac{|f(x_1) - f(y)|}{|f(x_2) - f(y)|} \leq \eta\left(\frac{|x_1 - y|}{|x_2 - y|}\right).$$

Proof of Theorem 3.1. In order to apply Theorem 3.2, it just remains to verify (a4) for $x_1 \in X_1$, $x_2 \in X_2$ and $y \in Y$.

Since $\operatorname{diam} Y \geq \mu \operatorname{diam} X_1$, there is some $\widehat{y} \in Y$ such that $|x_1 - y| \geq |y - \widehat{y}| \geq \lambda\mu |x_1 - y|$. Therefore

$$|f(x_1) - f(y)| \eta(1) \geq |f(\widehat{y}) - f(y)| \geq \frac{1}{\eta(1/(\lambda\mu))} |f(x_1) - f(y)|$$

so that

$$\frac{|f(x_2) - f(y)|}{|f(x_1) - f(y)|} \lesssim \frac{|f(x_2) - f(y)|}{|f(\widehat{y}) - f(y)|} \leq \eta\left(\frac{|x_2 - y|}{|\widehat{y} - y|}\right) \leq \eta\left(\frac{|x_2 - y|}{\lambda\mu |x_1 - y|}\right)$$

and

$$\frac{|f(x_1) - f(y)|}{|f(x_2) - f(y)|} \lesssim \frac{|f(\widehat{y}) - f(y)|}{|f(x_2) - f(y)|} \leq \eta\left(\frac{|\widehat{y} - y|}{|x_2 - y|}\right) \leq \eta\left(\frac{|x_1 - y|}{|x_2 - y|}\right)$$

Thus (a4) holds. □

4. Sewing metric spaces

In this section, we prove Theorem 1. Then we look at how properties are preserved under gluing and provide a proof of Theorem 4.

Proof of Theorem 1. Let d_j be the metric on X_j, $j = 1, 2$. We may normalise them so that $\operatorname{diam} Y_1 = \operatorname{diam} Y_2$. We apply Theorem 2 to (X_1, d_1), (Y_2, d_2) and to the quasisymmetric embedding $f^{-1}\colon Y_2 \to X_1$. We obtain a metric $\widehat{d_1}$ on X_1 which is power quasisymmetric to d_1 on X_1 and its restriction to Y_1 is bi-Lipschitz to d_2 on Y_2 via the map f. Moreover, the identity is also locally quasisimilar on $X_1 \setminus Y_1$ with respect to d_1 and $\widehat{d_1}$.

Let us first define a function $\widehat{q}\colon \widehat{X} \times \widehat{X} \to \mathbf{R}_+$.

- If $x_1, x_1' \in X_1$, set $\widehat{q}(x_1, x_1') = \widehat{d_1}(x_1, x_1')$.
- If $x_2, x_2' \in X_2$, set $\widehat{q}(x_2, x_2') = d_2(x_2, x_2')$.
- If $x_1 \in X_1$ and $x_2 \in X_2$, set $\widehat{q}(x_1, x_2) = \widehat{q}(x_2, x_1) = \inf_{y \in Y_1}\{\widehat{d_1}(x_1, y) + d_2(x_2, f(y))\}$.

Define finally
$$\widehat{d}(x, y) = \inf \sum \widehat{q}(x_j, x_{j+1})$$
where the infimum is taken over all finite chains $x_0, \ldots, x_n$ with $x_0 = x$ and $x_n = y$. Then $\widehat{d}$ is a metric, equal to $\widehat{d_1}$ on X_1 by (5) and bi-Lipschitz to d_2 on X_2 (since both metrics $\widehat{d_1}$ and d_2 are just bi-Lipschitz on the seam, it is not clear why the restriction of $\widehat{d}$ to X_2 should coincide with d_2). Hence $\widehat{d}$ is locally quasisimilar on $(X_1 \setminus Y_1) \cup (X_2 \setminus Y_2)$.

It also follows from the definition of $\widehat{d}$ that the seam of this sewing has a positive angle.

It remains to prove that the gauge of $\widehat{X}$ is well defined. Let us assume that we are given two metrics $\widehat{d}$ and $\widehat{d}'$ on $\widehat{X}$ such that the angle at the seam $Y = Y_1 = Y_2$ is positive in both cases, and the embeddings $\mathrm{Id}\colon X_j \to \widehat{X}$ are all quasisymmetric. Then the identity map $\mathrm{Id}\colon (\widehat{X}, \widehat{d}) \to (\widehat{X}, \widehat{d}')$ is a quasisymmetric map restricted to each X_j, $j = 1, 2$. The assumptions of Theorem 3.1 are clearly satisfied, so the map $\mathrm{Id}\colon (\widehat{X}, \widehat{d}) \to (\widehat{X}, \widehat{d}')$ is globally quasisymmetric: both metrics $\widehat{d}$ and $\widehat{d}'$ determine the same gauge. This implies in particular that if instead of d_1 and d_2, we had started with other metrics in their respective gauge, then the metric obtained would also define the same gauge on $\widehat{X}$. □

Remark. If we assume that Y is bounded but (X_1, d_1) and (X_2, d_2) are both unbounded, then the gauge of the gluing is not well-defined. Let $\widehat{d}$ be the metric on $\widehat{X}$ obtained with d_1 and d_2, and let $\widehat{d}'$ be the metric obtained with d_1 and $d_2^{1/2}$. Let us fix $y \in Y$. For $x_1 \in X_1$ far enough from Y, it follows that $\widehat{d}(x_1, y) \asymp \widehat{d}(x_1, y) \asymp d_1(x_1, y)$. Similarly, if $x_2 \in X_2$ is far enough from Y, then $\widehat{d}(x_2, y) \asymp d_2(x_2, y)$ and $\widehat{d}'(x_2, y) \asymp d_2(x_2, y)^{1/2}$. Therefore, if we choose x_1 and x_2 such that $d_1(x_1, y) \asymp d_2(x_2, y)$ then $\widehat{d}(x_1, y) \asymp \widehat{d}(x_2, y)$ but $\widehat{d}'(x_2, y) \asymp \widehat{d}'(x_1, y)^{1/2}$ so the identity map is not quasisymmetric.

We now turn to properties of $\widehat{X}$ inherited from X_1 and X_2. We note that since the embedding $(X_1 \setminus Y_1, d_1) \to (\widehat{X}, \widehat{d})$ is a local quasisimilarity and the embedding

$(X_2, d_2) \to (\widehat{X}, \widehat{d})$ is bi-Lipschitz, the Hausdorff dimensions are related by $\dim \widehat{X} = \max\{\dim(X_1 \setminus Y_1), \dim X_2\} \le \max\{\dim X_1, \dim X_2\}$. Relabeling the spaces so that $\dim Y_2 \ge \dim Y_1$ would yield the equality.

Since the gauge of the gluing in Theorem 4 does not depend on the initial metrics of the conformal gauges of X_1 and X_2, it follows from Corollary 3 that

$$\dim_c \widehat{X} = \max\{\dim_c X_1, \dim_c X_2\}\,.$$

Proposition 4.1. *We consider metric spaces as in Theorem 1. If Y_1 and Y_2 are porous, and X_1 and X_2 are both Q-Ahlfors regular, then $\widehat{X}$ is Q-Ahlfors regular as well.*

Proof. Since Y_2 is porous in an Q-regular space, the assumptions of Proposition 2.18 are satisfied (cf. §1.1), so $(X_1, \widehat{d_1})$ is Q-Ahlfors regular. Therefore $\widehat{X}$ is also Ahlfors regular since the metrics are bi-Lipschitz equivalent. □

We restate Theorem 4:

Corollary 4.2. *Let us assume that X_1 and X_2 are Q-Loewner and Q-regular proper metric spaces, each containing a uniformly perfect and porous closed subset Y_1 and Y_2 which are quasisymmetrically equivalent. We assume that X_1 is bounded if Y_1 is. Then $\widehat{X}$ is also Q-Ahlfors regular and Q-Loewner.*

Proof. From Proposition 4.1, it is known that $\widehat{X}$ is Q-Ahlfors regular. Furthermore, $(X_1, \widehat{d_1})$ is also Loewner. Thus, $(X_1, \widehat{d})$ and $(X_2, \widehat{d})$ are both Loewner and Ahlfors regular.

Since the gluing makes a positive angle, the space $\widehat{X}$ is quasiconvex. Therefore, it is enough to prove that $\widehat{X}$ admits a $(1, Q)$-Poincaré inequality. We already know that $(X_1, \widehat{d})$ and $(X_2, \widehat{d})$ both carry $(1, Q)$-Poincaré inequalities.

Since $(Y, \widehat{d})$ is uniformly perfect in a Q-Ahlfors regular space, it follows from Theorem 1.1 that there is some $s \in (0, Q)$ such that $\mathscr{H}_s(Y \cap B(y, r)) \gtrsim r^s$. Thus, the assumptions of Theorem 6.15 in [14] are satisfied and $\widehat{X}$ admits a $(1, Q)$-Poincaré inequality: $\widehat{X}$ is also a Loewner space. □

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