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ON GEOMETRIC QUOTIENTS

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Abstract. The following results are established:

i) Let $f\colon M \longrightarrow \mathbf{H}$ be a C^1 map of a compact connected C^1 manifold (without boundary) into a Hilbert space. Then the map f is a C^1 fibre bundle projection onto $f(M)$ if and only if $f^{-1}\colon f(M) \longrightarrow \mathscr{H}(M)$ is Lipschitz. Here, $\mathscr{H}(M)$ denotes the metric space of nonempty closed subsets of M with the Hausdorff metric.

ii) Let M and N be compact connected C^1 manifolds (without boundary) and let $f\colon M \longrightarrow N$ be a C^1 map. Then f is a Lipschitz fibre bundle projection if and only if it is a C^1 fibre bundle projection.

iii) Let $G \times M \longrightarrow M$ be a C^1 action of a compact Lie group on a compact connected C^1 manifold (without boundary) and let $f\colon M \longrightarrow \mathbf{H}$ be an invariant C^1 map. Then the map f induces a bi-Lipschitz embedding of M/G (with respect to the quotient metric) into $\mathbf{H}$ if and only if f induces a C^1 embedding of M/G (with respect to the C^1 quotient structure) into $\mathbf{H}$. Moreover, in contrast to the result of Schwarz in the C^∞ case, such an embedding f exists exactly when the action has a single orbit type.

1. Introduction and main results

In this paper, we develop some general results on smooth and Lipschitz quotients. As applications of this material, we establish two theorems. For a metric space X, we let $(\mathscr{H}(X), d_H)$ denote the metric space of nonempty compact subsets of X with the Hausdorff metric d_H. Our first theorem is the following:

Theorem 1. *Let M be a compact connected C^1 manifold (without boundary) carrying a metric induced by a C^1 embedding into a Euclidean space. Let $f\colon M \longrightarrow \mathbf{H}$ be a C^1 map into a Hilbert space. Then the following three conditions are equivalent:*

(1) $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M), d_H)$, $x \mapsto f^{-1}(x)$, *is Lipschitz.*
(2) $f\colon M \longrightarrow f(M)$ *is a Lipschitz fibre bundle.*
(3) $f(M)$ *is a C^1 submanifold (without boundary) of $\mathbf{H}$ and $f\colon M \longrightarrow f(M)$ is a C^1 fibre bundle.*

Here (3) $\Rightarrow$ (2) is trivial and (2) $\Rightarrow$ (1) is easy. We prove (1) $\Rightarrow$ (3) in Proposition 3.1 below. Then we have:

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Corollary 2. *Let M and N be compact connected C^1 manifolds (without boundary) and let $f\colon M \longrightarrow N$ be a C^1 map. Then f is a Lipschitz fibre bundle projection if and only if it is a C^1 fibre bundle projection.*

To state the second application, let $\mathrm{Lip}(X)$ denote the set of real valued Lipschitz functions on a metric space X, and let $q\colon X \longrightarrow Y$ be a topological quotient map onto a Hausdorff space. Then we set $\mathrm{Lip}_{\mathrm{quot}}(Y) = \{\varphi \mid \varphi \circ q \in \mathrm{Lip}(X)\}$, which may be identified with a closed linear subspace of $\mathrm{Lip}(X)$; see Section 2. If $\mathbf{H}$ is a Hilbert space and $f\colon X \longrightarrow \mathbf{H}$ is a Lipschitz map inducing a topological embedding $\hat{f}\colon Y \hookrightarrow \mathbf{H}$, we set $\mathrm{Lip}_{\mathrm{emb}(f)}(Y) = \hat{f}^*\mathrm{Lip}(\mathbf{H})$. Analogously, for $X = M$ a compact C^1 manifold (with or without boundary), we set $C^1_{\mathrm{quot}}(Y\colon \mathbf{H}) = \{\varphi\colon Y \longrightarrow \mathbf{H} \mid \varphi \circ q \in C^1(M\colon \mathbf{H})\}$. This definition generates an extension of the category of compact C^1 manifolds and C^1 maps; see Section 3. Also, for $f\colon X \longrightarrow \mathbf{H}$ a C^1 map inducing a topological embedding $\hat{f}\colon Y \hookrightarrow \mathbf{H}$, we define $C^1_{\mathrm{emb}(f)}(Y\colon \mathbf{H}) = \hat{f}^*C^1(\mathbf{H}\colon \mathbf{H})$.

Theorem 3. *Let M be a compact connected C^1 manifold (without boundary) carrying a metric induced by a C^1 embedding into a Euclidean space and let $\mathbf{H}$ be a Hilbert space. Let $G \times M \longrightarrow M$ be a C^1 action of a compact Lie group G on M and let $q\colon M \longrightarrow M/G$ be the quotient map onto the orbit space. Then the following four conditions are equivalent:*

1. *There exists a C^1 map $f\colon M \longrightarrow \mathbf{H}$ inducing a topological embedding $\hat{f}\colon M/G \hookrightarrow \mathbf{H}$ so that* $\mathrm{Lip}_{\mathrm{emb}(f)}(M/G) = \mathrm{Lip}_{\mathrm{quot}}(M/G)$.
2. *There exists a C^1 map $f\colon M \longrightarrow \mathbf{H}$ inducing a topological embedding $\hat{f}\colon M/G \hookrightarrow \mathbf{H}$ so that* $C^1_{\mathrm{emb}(f)}(M/G\colon \mathbf{H}) = C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})$.
3. *The quotient C^1 object $\big(M/G, C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})\big)$ is a C^1 manifold (without boundary).*
4. *The action has only one orbit type G/K, where K is a closed subgroup of G.*

In the setting of Theorem 3, there always exists a C^1 map $f\colon M \longrightarrow \mathbf{R}^k \subset \mathbf{H}$, for some k, inducing a topological embedding $\hat{f}\colon M/G \longrightarrow \mathbf{R}^k$; see Section 4. Because the orbit decomposition M/G is *metrically parallel* (see Section 2 for definition), we see by [11, Lemma 2.26] (a result due to Katětov) that (1) is equivalent to the condition that $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M), d_H)$ is Lipschitz. Then (1) $\Rightarrow$ (3) follows from Theorem 1. Also, (3) $\Rightarrow$ (2) is trivial. We prove (2) $\Rightarrow$ (4) in Proposition 4.2 below.

Finally, to establish (4) $\Rightarrow$ (1) in Theorem 3, assume (4). By [22, Theorem B], we may assume that M is a C^∞ manifold and that the action $G \times M \to M$ is C^∞. Then (4) implies by [9, Theorem 4.18] (or by [3, Theorem II.5.8 and p. 308]) that M/G is a C^∞ manifold (without boundary) and that the projection $q\colon M \to M/G$ is a C^∞ fibre bundle with fibre G/K and structure group $N(K)/K$ acting by right translation on G/K. Now choose a C^∞ embedding $\hat{f}\colon M/G \to \mathbf{R}^n \subset \mathbf{H}$ for some n. Then $f = \hat{f} \circ q\colon M \to \mathbf{H}$ is a C^∞ map inducing $\hat{f}$, $f(M)$ is a C^∞ submanifold of $\mathbf{H}$, and $f\colon M \to f(M)$ is a C^∞ fibre bundle. Thus, (3) $\Rightarrow$ (1) in Theorem 1 implies that $f^{-1}\colon f(M) \to (\mathscr{H}, d_H)$ is Lipschitz yielding (1) of Theorem 3.

Thus, when any one of the above four conditions holds, it follows that $f(M)$ is a C^1 submanifold (without boundary) of $\mathbf{H}$, $\hat{f}\colon \big(M/G, C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})\big) \longrightarrow f(M)$ is a C^1 diffeomorphism, and $f\colon M \longrightarrow f(M)$ is a C^1 fibre bundle with fibre G/K and structure group $N(K)/K$ acting by right translation on G/K. The hypothesis in (2) that the map $f\colon M \longrightarrow \mathbf{H}$ is C^1 cannot be dropped, as may be seen from an example (provided by a reader): Let the C^1 action $\mathbf{Z}_2 \times S^1 \longrightarrow S^1$ be defined by the involution $(x, y) \mapsto (x, -y)$ and let $f\colon S^1 \longrightarrow \mathbf{R}^2$ be defined by setting $f(x, y) = (x, |y|)$. Then the induced map $\hat{f}\colon S^1/\mathbf{Z}_2 \longrightarrow \mathbf{R}^2$ is a bi-Lipschitz embedding so that $\mathrm{Lip}_{\mathrm{emb}(f)}(S^1/\mathbf{Z}_2) = \mathrm{Lip}_{\mathrm{quot}}(S^1/\mathbf{Z}_2)$ even though the action has more than one orbit types. However, the map f is not a C^1 map. Still, this example points to even more unfriendly behavior; see the discussion in Section 5.

As a corollary of Theorem 3, we see that the Whitney Embedding Theorem fails for our extended C^1 category, even when the target space is a Hilbert space.

Corollary 4. *Let $G \times M \longrightarrow M$ be a C^1 action of a compact Lie group G on a compact connected C^1 manifold M (without boundary) and let $\mathbf{H}$ be a Hilbert space. Then the quotient C^1 object $\big(M/G, C^1_{quot}(M/G\colon \mathbf{H})\big)$ admits a C^1 embedding into $\mathbf{H}$ if and only if it is a C^1 manifold (without boundary).*

To provide a convenient categorical context for our arguments, we define a modest extension of the C^1 category in two steps. One obtains a modest extension of the C^∞ category in a similar way. Our work here focuses on compact C^1 objects and C^1 actions on manifolds. In part, it shows that the second extension is nontrivial in the C^1 case. In contrast, a deep theorem of Schwarz [23] implies that the second extension is trivial in the C^∞ case; see Section 5. A key ingredient in our proofs is Glaeser's remarkable generalization of the Inverse Function Theorem to *nonmanifold* closed subsets of a Euclidean space [5, Chapitre II, Théorème 1] and a further generalization to certain subsets of a Hilbert space [13]. It is crucial to note that the tangent spaces of such objects are not required to be constant dimensional. These objects have tangent vector *quasibundles* rather than tangent vector bundles; see Section 3, [18], and [13] (for [13], see also the Appendix here).

Acknowledgements. The notion of a quotient metric is an extension of some of the ideas of Jouni Luukkainen in [11]. We would like to thank him for his careful reading of an earlier version of this paper and for sharing his insights with us.

We would also like to thank the referee for many improvements (as well as numerous corrections) among which are the following:

(1) In Theorem 1, the hypothesis that f^{-1} is Lipschitz is an elegant reformulation of our original compound hypothesis that f induces a metrically parallel decomposition and that $\hat{f}$ is bi-Lipschitz from the quotient metric on M/f; see Corollary 2.3.

(2) In the proof of Proposition 3.1, Claim 3.1.2 and Claim 3.1.3 are due to the referee. These constitute a crucial step in extending the result from Euclidean space to Hilbert space. Furthermore, the argument presented to show that $df(y)T_yM \subset T^0_xM$ for $x = f(y)$ is due to the referee. This simplifies (and corrects) our original argument for this claim.

(3) In the proof of Proposition 4.2, the constructions of maps P, J, and F_7 from P_0, J_0, and F_6 respectively, are due to the referee. These correct an error in our earlier version of that proof.

2. Quotient Lipschitz structures

The purpose of this section is to develop the notion of a quotient Lipschitz structure. First, a map $f\colon A \longrightarrow B$ of pseudometric spaces is Lipschitz if there is a constant $\kappa \geq 0$ such that $d_B(f(a), f(a')) \leq \kappa\, d_A(a, a')$ for all $a, a' \in A$. In this case, we set

$$\mathrm{Lip}\,(f) = \inf \left\{ \kappa \;\middle|\; d_B(f(a), f(a')) \leq \kappa\, d_A(a, a') \text{ for all } a, a' \in A \right\}.$$

We say that two maps $f, g\colon A \longrightarrow B$ are *equivalent*, and write $f \sim g$, if $d_B(f(a), g(a)) = 0$ for all $a \in A$. A map $f\colon A \longrightarrow B$ is a *bi-Lipschitz isomorphism* if f is Lipschitz and there exists a Lipschitz map $g\colon B \longrightarrow A$ so that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$. In this case, there is a constant $L \geq 1$ such that

$$\frac{1}{L}\, d_A(a, a') \leq d_B(f(a), f(a')) \leq L\, d_A(a, a')$$

for all $a, a' \in A$ and we say that f is L-bi-Lipschitz. It is clear that bi-Lipschitz isomorphism of pseudometric spaces is an equivalence relation. Furthermore, the (set theoretically) non-invertible isometry $\pi\colon A \longrightarrow A_1$ from a pseudometric space onto its canonical quotient metric space is indeed a bi-Lipschitz isomorphism.

As usual, if A is a pseudometric space, we let $\mathrm{Lip}(A)$ denote the set of all real valued Lipschitz functions on A. It is well known [14] that if B is a subset of a metric space A, then $\mathrm{Lip}(B) = \mathrm{Lip}(A)|_B$.

Now let (X, d) be a compact metric space, let Y be a Hausdorff space, and let $q\colon X \longrightarrow Y$ be a (topological) quotient map. We wish to define a metric ρ on Y so that the map q is a quotient map in the Lipschitz category, that is: q is Lipschitz and if Z is another metric space and $f\colon X \longrightarrow Z$ is a Lipschitz map which factors through q, then the factor map $\hat{f}$ is Lipschitz.

(2.0.1)
$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] & \\ Y \arrow[r, "\hat{f}"] & Z \end{tikzcd}$$

It follows that the defining property (2.0.1) holds in a more general form where Z is only assumed to be a pseudometric space (to see this, add to (2.0.1) the isometry $\pi\colon Z \longrightarrow Z_1$ onto the canonical quotient metric space of Z).

Unfortunately, such a metric does not always exist, but such a pseudometric does always exist and is often a metric (Luukkainen [12]); it is always continuous. Any two such pseudometrics ρ_1 and ρ_2 are bi-Lipschitz equivalent via the identity map (apply (2.0.1) for Y equipped with ρ_i, $Z = (Y, \rho_{3-i})$, $f = q$, and $\hat{f} = \mathrm{id}$ when $i = 1, 2$). Conversely, every pseudometric on the set Y which is bi-Lipschitz equivalent to ρ shares the same Lipschitz quotient property with ρ.

Theorem 2.1. *Let X be a compact metric space, let Y be a Hausdorff space, and let $q\colon X \longrightarrow Y$ be a (topological) quotient map. Then there exists a unique bi-Lipschitz class of pseudometrics on Y so that q becomes a quotient map in the Lipschitz category.*

A maximal bi-Lipschitz class of metrics is what Sullivan [24] calls a *metric gauge*.

In view of [11] and [15], perhaps the most natural way to prove Theorem 2.1 is the following: Let $\mathrm{Lip}\,(X)$ denote the set of real valued Lipschitz functions on X and set $\mathrm{Lip}\,(Y) = \{f\colon Y \longrightarrow \mathbf{R} \mid f\circ q \in \mathrm{Lip}\,(X)\}$, and then we seek ρ on Y so that $\mathrm{Lip}\,(Y)$ is precisely the set of Lipschitz functions with respect to ρ (as it should be).

We recall that $\mathrm{Lip}\,(X)$ is a Banach space with respect to the norm $\|\cdot\|_X$ defined by setting $\|f\|_X = \|f\|_\infty + \mathrm{Lip}\,(f)$, where $\|f\|_\infty$ denotes the L^∞ or sup norm of f. Let $\mathrm{Lip}\,(X)^*$ be the dual Banach space, with norm $\|\cdot\|_X^*$. We note that for each $x\in X$, the evaluation map $\mathrm{ev}(x)\colon \mathrm{Lip}\,(X) \longrightarrow \mathbf{R}$ defined by setting $\mathrm{ev}(x)f = f(x)$ is a member of $\mathrm{Lip}(X)^*$ with $\|\mathrm{ev}(x)\|_X^* = 1$. The key observation here is the fact that the map $\mathrm{ev}\colon X \longrightarrow \mathrm{Lip}\,(X)^*$ given by $\mathrm{ev}\colon x\mapsto \mathrm{ev}(x)$ is a bi-Lipschitz embedding [16], [15]:

$$\frac{1}{1+\mathrm{diam}\,(X)}\, d\,(x,y) \le \|\mathrm{ev}(x)-\mathrm{ev}(y)\|_X^* \le d\,(x,y).$$

We note that $q^*\mathrm{Lip}\,(Y)$ is a Banach subspace of $\mathrm{Lip}\,(X)$; we give $\mathrm{Lip}\,(Y)$ the Banach space structure which makes q^* an isometric Banach isomorphism $\mathrm{Lip}\,(Y) \longrightarrow q^*\mathrm{Lip}\,(Y)$. Then $(q^*)^*\colon \mathrm{Lip}\,(X)^* \longrightarrow \mathrm{Lip}\,(Y)^*$ is a linear Banach quotient map. Let $\|\cdot\|_Y^*$ denote the Banach norm on $\mathrm{Lip}\,(Y)^*$. We may define a map $\mathrm{ev}_Y\colon Y \longrightarrow \mathrm{Lip}\,(Y)^*$ by requiring that $\mathrm{ev}_Y\circ q = (q^*)^*\circ\mathrm{ev}$. Then for $y\in Y$, the map $\mathrm{ev}_Y(y)$ is the evaluation at y. Finally, the pseudometric ρ on Y is defined by setting

$$\rho\,(y_1,y_2) = \|\mathrm{ev}_Y(y_1)-\mathrm{ev}_Y(y_2)\|_Y^*.$$

One checks directly that the pseudometric ρ does satisfy the Lipschitz quotient property and that $\mathrm{Lip}(q)\le 1$. Moreover, the norms on $\mathrm{Lip}(Y)$ induced by q^* and ρ are equivalent: $\|f\circ q\|_X \le \|f\|_Y \le 3\|f\circ q\|_X$ for $f\in\mathrm{Lip}(Y)$.

For a more direct construction of the quotient metric we might try the Hausdorff distance between the inverse images of points in Y. But unfortunately, the map q need not even be continuous with respect to that metric, as the quotient of a sphere by a proper great circle segment shows. However, the map q is actually "Lipschitz" with respect to the *gap function* γ for q, where $\gamma(y,y') = \inf\{d(x,x') \mid q(x)=y,\ q(x')=y'\}$, but the gap function is not necessarily a metric. Referring again to [11], we see that there exists a (unique) largest pseudometric $\rho\le\gamma$. It is then easily verified that this pseudometric has the desired universal quotient property (2.0.1) with Z any pseudometric space. We call this canonical pseudometric ρ *the quotient pseudometric* and it is given by the formula

$$\rho(y,y') = \inf\,\left\{\sum_{i=0}^{n-1}\gamma(y_i,y_{i+1}) \;\middle|\; y_0=y,\ y_n=y',\ \text{and } n\in\mathbf{N}\right\}.$$

We note that with this explicit pseudometric, we have $\mathrm{Lip}(q)\le 1$ and $\mathrm{Lip}(\hat f) = \mathrm{Lip}(f)$ in diagram (2.0.1) above (again with Z a pseudometric space). Furthermore, since $\rho\le\mathrm{Lip}(q)\gamma$, we have $\mathrm{Lip}(q)=1$ if $\mathrm{Lip}(q)>0$ or, equivalently, if $\mathrm{diam}_\rho(Y)>0$.

As Luukkainen [12] points out, consideration of the space filling Peano map shows that the pseudometric ρ is not always a metric. In contrast, if $f\colon X\longrightarrow V$ is Lipschitz onto a metric space, $Y = X/f = \{f^{-1}(v)\mid v\in V\}$ with the quotient topology, and $q(x) = f^{-1}(f(x))$, then the quotient pseudometric ρ is a true metric and the induced homeomorphism $Y\longrightarrow V$ is Lipschitz. More generally, the quotient pseudometric ρ is a metric on Y if and only if $\mathrm{Lip}\,(Y)$ separates points of Y.

In the case of a surjective map $f\colon X\longrightarrow Y$ of a metric space onto a set, the canonical quotient pseudometric has also been defined in [6, Section 1.16_+] (with the name *quotient metric*) as the supremum of the pseudometrics on Y for which f is

1-Lipschitz. It is also observed that in the case of an isometric group action on a metric space with closed orbits, the quotient pseudometric on the orbit space is a metric.

For certain decompositions, the quotient pseudometric class of ρ coincides with that given by the Hausdorff metric d_H; of course, ρ is then a metric. These are the ones for which

$$k\, d_H(A,B) \le \gamma(A,B) \le d_H(A,B)$$

for some constant $0 < k \le 1$, where A and B are elements of the decomposition and $\gamma(A,B) = \inf\{d(x,y) \mid x \in A \text{ and } y \in B\}$. (We say that two points $x \in A$ and $y \in B$ form a *gap pair* if $\gamma(A,B) = d(x,y)$.) We call such a decomposition a *metrically parallel* decomposition; our choice of this term is motivated by the fact that the orbit decomposition is metrically parallel (with $k = 1$) when a compact group acts continuously on a compact metric space by isometries. More generally, when a compact group G acts continuously on a compact metric space X by L-bi-Lipschitz maps with a uniform L, then the orbit decomposition is metrically parallel with $k = 1/L$. In particular, when $L = 1$, then the gap is equal to the Hausdorff metric. We also note that either factor in a compact metric product is a metrically parallel quotient of the product.

The following is a straightforward characterization of a metrically parallel decomposition.

Proposition 2.2. *Let $q\colon X \longrightarrow Y$ be as above and let d_H denote the metric induced on Y by the Hausdorff metric on $\mathscr{H}(X)$. Then the following statements are equivalent:*

(1) *The map q induces a metrically parallel decomposition on X.*
(2) *The map $q\colon (X,\, d) \longrightarrow (Y,\, d_H)$ is Lipschitz.*
(3) *d_H defines a quotient metric on Y.*

Then we have the following corollary.

Corollary 2.3. *Let $f\colon (X,\, d_X) \longrightarrow (Y,\, d_Y)$ be a surjective Lipschitz map of compact metric spaces. Then the following three statements are equivalent:*

(1) *The map f induces a metrically parallel decomposition and $\hat{f}$ is bi-Lipschitz.*
(2) *The map $f^{-1}\colon (Y,\, d_Y) \longrightarrow (\mathscr{H}(X),\, d_H)$ is Lipschitz.*
(3) *There is $L > 0$ such that $f\big(B(x,\varepsilon)\big) \supset B\big(f(x), \varepsilon/L\big)$ for all $x \in X$ and $\varepsilon > 0$. Here, $B(x,\varepsilon)$ denotes the closed ball of radius ε centered at x.*

The map $f\colon [-1,\, 1] \longrightarrow [-1/2,\, 1/2]$ given by

$$f(x) = \begin{cases} x + 1/2 & \text{for } -1 \le x \le -1/2, \\ 0 & \text{for } -1/2 \le x \le 1/2, \\ x - 1/2 & \text{for } 1/2 \le x \le 1 \end{cases}$$

is an example of a Lipschitz map for which $\hat{f}$ is bi-Lipschitz but the induced decomposition is not metrically parallel. The map $f\colon [0,\, 1] \times [0,\, 1] \longrightarrow [0,\, 1]$ given by $f(x,y) = x^2$ is an example of a Lipschitz map for which the induced decomposition is metrically parallel but $\hat{f}$ is not bi-Lipschitz.

In [6, Section $1.25\frac{1}{2}{}_+$] a map $f\colon X \longrightarrow Y$ of metric spaces is defined to be *co-Lipschitz* if it satisfies condition (2) of Corollary 2.3. In [1, Definition 3.1] condition (3) of Corollary 2.3 is used as the definition of a co-Lipschitz map. In this case, the co-Lipschitz constant co-Lip(f) of f is defined as the infimum of the constants L satisfying (3). It now follows from (2) $\Leftrightarrow$ (3) that in fact co-Lip(f) = Lip(f^{-1}). Moreover, in [1, Definition 3.2] a map between metric spaces is called a *Lipschitz quotient* if it is both Lipschitz and co-Lipschitz. A metric space Y is then called a Lipschitz quotient of a metric space X if there is a surjective Lipschitz quotient map of X onto Y. See also [2, Definition 11.10].

For completeness, we say that a decomposition is *topologically parallel* if the quotient map $q\colon (X, d) \longrightarrow (Y, d_H)$ is only continuous. This condition, of course, is equivalent to requiring the Hausdorff metric d_H to define the quotient topology on Y. Then q is open. Thus, we have:

Corollary 2.4. *Topologically parallel (and hence metrically parallel) decompositions are continuous.*

Finally, by a *Lipschitz fibre bundle* $f\colon X \longrightarrow Y$ we mean a surjective Lipschitz map of metric spaces such that, for any $y \in Y$, there exists a neighborhood U of y and a bi-Lipschitz isomorphism $\varphi\colon U \times f^{-1}(y) \longrightarrow f^{-1}(U)$ making the diagram

$$\begin{tikzcd} U \times f^{-1}(y) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{proj}_U"'] & & f^{-1}(U) \arrow[dl, "f|_{f^{-1}(U)}"] \\ & U & \end{tikzcd}$$

commute. We note that if $f\colon X \longrightarrow Y$ is a Lipschitz fibre bundle with X compact, then f induces a metrically parallel decomposition of X with $f^{-1}\colon Y \longrightarrow (\mathscr{H}(X), d_H)$ Lipschitz.

3. The extended C^1 category and proof of Theorem 1

We now define a modest extension of the C^1 category in two steps. First, we may regard the usual C^1 category as having for objects the finite dimensional closed C^1 submanifolds (with or without boundary) of a Hilbert space $\mathbf{H}$. We extend this category by allowing more objects, the closed subsets of $\mathbf{H}$. More specifically, we say that a map $F\colon \mathbf{H} \longrightarrow \mathbf{H}$ is C^1 if and only if the Gâteaux derivative $dF(x)\colon \mathbf{H} \longrightarrow \mathbf{H}$, defined by $dF(x)y = d/dt|_{t=0}F(x + ty)$, exists and is a bounded linear map for each $x \in \mathbf{H}$, and the map $x \mapsto dF(x)$ is continuous with respect to the operator norm on its range. This last condition implies that $dF(x)$ is then the Fréchet derivative. For X and Y closed subsets of $\mathbf{H}$, we say that a map $f\colon X \longrightarrow Y$ is C^1 if and only if it is the restriction of a C^1 map $F\colon \mathbf{H} \longrightarrow \mathbf{H}$. Because the closed convex hull of a compact set is compact, $\|dF(x)\|$ is bounded for x in the convex hull of a compact set X. Then the Mean Value Theorem implies that, when X is compact, a C^1 map $f\colon X \longrightarrow Y$ is Lipschitz. Since $\mathbf{H}$ admits C^1 partitions of unity, it follows that if X and Y are finite dimensional closed C^1 submanifolds (with or without boundary) of $\mathbf{H}$, then a C^1 map $f\colon X \longrightarrow Y$ in the classical sense can indeed be extended to a C^1 map $F\colon \mathbf{H} \longrightarrow \mathbf{H}$. Obviously, when X is closed, a C^1 map $f\colon X \longrightarrow Y$ is continuous. These are the maps of the once extended category $\mathscr{E}^1(1)$ and we write $C^1(X\colon Y) = \{f \mid f\colon X \longrightarrow Y \text{ is } C^1\}$. A C^1 diffeomorphism $f\colon X \longrightarrow Y$ is simply an isomorphism in this category. If a C^1 diffeomorphism $f\colon X \longrightarrow Y$ exists, we write $X \cong Y$. As usual, a C^1 embedding is a C^1 map f which is a C^1 diffeomorphism onto its image. As pointed out by the referee, it is known that C^1 differentiability (in the usual Fréchet sense) for a map to $\mathbf{H}$ cannot be characterized by requiring the compositions of this map with the C^1 functions $\mathbf{H} \longrightarrow \mathbf{R}$ to be C^1. In fact, it is easy to construct a continuous map $f\colon \mathbf{R} \longrightarrow \mathbf{H}$ which is C^1 outside 0 and with the property that $\lim_{t\to 0}\big(f(t) - f(0)\big)/t$ does not exist, but for which $\varphi \circ f$ is C^1 for each C^1 function $\varphi\colon \mathbf{H} \longrightarrow \mathbf{R}$. This difficulty is the reason we must take $C^1(X\colon \mathbf{H})$, instead of $C^1(X\colon \mathbf{R})$, as the structure module for X; see (3.0.1). See also [20] for some related issues.

Given two objects X and Y, a product $X \times Y$ is readily defined (though is not canonical; it is unique up to a diffeomorphism $\mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{H}$) because the product $\mathbf{H} \times \mathbf{H}$ is the underlying space for the Hilbert space $\mathbf{H} \oplus \mathbf{H}$ ($\cong \mathbf{H}$ when $\mathbf{H}$ is infinite dimensional). Now that we know what we mean by C^1 maps $\mathbf{H} \oplus \mathbf{H} \longrightarrow \mathbf{H}$ and $\mathbf{H} \longrightarrow \mathbf{H} \oplus \mathbf{H}$, we obtain definitions of C^1 maps $X \times Y \longrightarrow Z$ and $Z \longrightarrow X \times Y$ by restriction. Then we have

(1) the projections $\mathrm{proj}_X\colon X \times Y \longrightarrow X$ and $\mathrm{proj}_Y\colon X \times Y \longrightarrow Y$ are C^1,
(2) $(X \times Y,\ \mathrm{proj}_X,\ \mathrm{proj}_Y)$ has the product universal property in our category, and
(3) all injections $X \longrightarrow X \times Y$ by $x \mapsto x \times y_0$, for $y_0 \in Y$ fixed, are C^1 embeddings.

With these preliminaries, in our category the definitions of a group object $(G,\ \cdot\ ,\ (\ )^{-1})$, abbreviated G, and a group action $G \times X \longrightarrow X$ are clear. (Of course, any Lie group is such a group object.)

The definition of the quotient topological space is the usual one, but we lack an interpretation of this quotient as a C^1 object. To provide this interpretation, we extend our extended C^1 category $\mathscr{E}^1(1)$ once more, to $\mathscr{E}^1(2)$, in order to allow these quotients to appear as C^1 objects. Specifically, we note that thus far a map $f\colon X \longrightarrow Y$ is C^1 if and only if

$$f^* C^1(Y\colon \mathbf{H}) = \big\{ \varphi \circ f \;\big|\; \varphi \in C^1(Y\colon \mathbf{H}) \big\} \subset C^1(X\colon \mathbf{H}). \tag{3.0.1}$$

Hence, we may redefine $\mathscr{E}^1(1)$ as the category with objects the pairs $(X,\ C^1(X\colon \mathbf{H}))$ and maps $f\colon\ (X,\ C^1(X\colon \mathbf{H})) \longrightarrow (Y,\ C^1(Y\colon \mathbf{H}))$ the maps $f\colon X \longrightarrow Y$ such that $f^* C^1(Y\colon \mathbf{H}) \subset C^1(X\colon \mathbf{H})$. Again products and group objects exist, although now the direct definition of $C^1(X \times Y\colon \mathbf{H})$ from $C^1(X\colon \mathbf{H})$ and $C^1(Y\colon \mathbf{H})$ is messy. For a group object G in $\mathscr{E}^1(1)$ acting C^1 on an object X in $\mathscr{E}^1(1)$, we let X/G be the topological quotient and $q\colon X \longrightarrow X/G$ the topological quotient map. Then, the quotient object $(X/G,\ C^1(X/G\colon \mathbf{H}))$ is defined by setting

$$C^1(X/G\colon \mathbf{H}) = \big\{ f\colon X/G \longrightarrow \mathbf{H} \;\big|\; q^* f \in C^1(X\colon \mathbf{H}) \big\}\,.$$

Then the elements of $C^1(X/G\colon \mathbf{H})$ are continuous. Our second extension $\mathscr{E}^1(2)$ has for objects those of $\mathscr{E}^1(1)$ together with the quotient objects just defined. A map $f\colon\ (A,\ C^1(A\colon \mathbf{H})) \longrightarrow (B,\ C^1(B\colon \mathbf{H}))$ in this category is defined to be a *continuous* map $f\colon A \longrightarrow B$ such that $f^* C^1(B\colon \mathbf{H}) \subset C^1(A\colon \mathbf{H})$. It is clear that for two objects A and B in $\mathscr{E}^1(1)$, this definition results in the same C^1 maps. The resulting category $\mathscr{E}^1(2)$ is our modest extension of the C^1 category. In it the quotient map $q\colon X \longrightarrow X/G$ is a *categorical* quotient map. That is, if $f\colon X \longrightarrow Y$ is a C^1 map and the object G in $\mathscr{E}^1(1)$ is a group object so that the diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\cdot"] \arrow[d, "\mathrm{proj}_X"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes, then there is a unique C^1 map $\hat{f}$ so that the diagram

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] & \\ X/G \arrow[r, "\hat{f}"] & Y \end{tikzcd}$$

also commutes.

For the proof of Proposition 3.1 below we need the notion of the Glaeser tangent space of a nonmanifold subset of a Hilbert space, as defined in [5] and [13]. Let X be a subset of a Hilbert space $\mathbf{H}$ (which could be finite dimensional) and let $x \in X$. We define

$$C_x^0 X = \Big\{ \xi \in \mathbf{H} \;\Big|\; \xi = \lim_{n\to\infty} \frac{x_n - y_n}{\|x_n - y_n\|}, \text{ where } \{x_n\}_{n\geq 1} \text{ and } \{y_n\}_{n\geq 1} \text{ are} \\ \text{sequences in } X \text{ converging to } x \in X \text{ with } x_n \neq y_n \Big\}$$

and let $T_x^0 X$ be the closed linear span of $C_x^0 X$. For $\alpha > 0$ an ordinal, we set inductively

$$C_x^\alpha X = \Big\{ \xi \in \mathbf{H} \;\Big|\; \xi = \lim_{n\to\infty} \xi_n, \text{ where } \xi_n \in T_{x_n}^{\beta_n} X \text{ with } \beta_n < \alpha \text{ and} \\ \{x_n\}_{n\geq 1} \text{ is a sequence in } X \text{ converging to } x \Big\}$$

and let $T_x^\alpha X$ be the closed linear span of $C_x^\alpha X$. Then the Glaeser tangent space of X at x is defined by setting $T_x X = \bigcup_\alpha T_x^\alpha X$; this is a closed linear subspace of $\mathbf{H}$ as $T_x^\alpha X \subset T_x^\beta X$ for $\alpha < \beta$ and because $\alpha \mapsto T_x^\alpha X$ is eventually constant. Clearly, if X is a C^1 submanifold of $\mathbf{H}$, then $T_x^0 X = T_x X$ is the usual tangent space of X at x.

We will also need two other related concepts: spherically compact sets and quasibundles. For the first, we recall from [16], [17], [13], [19] that a subset $X \subset \mathbf{H}$ is said to be *spherically compact* if the set

$$U(X) = \left\{ \frac{x-y}{\|x-y\|} \;\middle|\; x, y \in X \text{ and } x \neq y \right\}$$

has compact closure in the norm topology of $\mathbf{H}$. For the second, let $\mathbf{G}(\mathbf{H})$ denote the set of all finite dimensional linear subspaces of $\mathbf{H}$. We may define a topology on $\mathbf{G}(\mathbf{H})$ characterized by the condition that $\lim_{n\to\infty} A_n = A$ if and only if for each subsequence $\{A_{n_i}\}_{i\geq 1}$ of $\{A_n\}_{n\geq 1}$, each sequence $\{a_i\}_{i\geq 1} \subset \mathbf{H}$ with $a_i \in A_{n_i}$ and $\|a_i\| = 1$ for each i, has a subsequence converging to an element of A. This topology on $\mathbf{G}(\mathbf{H})$ is first countable but not Hausdorff. If X is a closed subset of $\mathbf{H}$, a *right quasibundle* over X is simply a continuous map $X \longrightarrow \mathbf{G}(\mathbf{H})$. For further details see [18] and [13].

In [13, Theorem 3.1] it is shown that if $X \subset \mathbf{H}$ is compact and spherically compact with the map $TX\colon x \mapsto T_x X$ a right quasibundle and if $x \in X$ with $n = \dim T_x X$, then $T_x X = T_x^{n-1} X$. Since the question is local, the same result holds for X *any* closed subset of a finite dimensional Euclidean space; this is an improvement of [5, Chapitre II, Proposition VII] where it is shown that $T_x X = T_x^{2n} X$ for $X \subset \mathbf{R}^n$. Also, $|TX| = \bigcup_{x\in X} \{x\} \times T_x X$ is closed in $X \times \mathbf{H}$ by [13, Lemma 4.1]. In [5, Chapitre II, Proposition IV], in the case of a closed subset of a Euclidean space, this is included in the definition (upper semicontinuity). By [18, Lemma 2.5], TX is then a right quasibundle if $X \subset \mathbf{R}^n$ is a closed subset. But for $\dim \mathbf{H} = \infty$, there exists a compact subset $X \subset \mathbf{H}$ so that $\dim T_xX < \infty$ for all $x \in X$ while TX is *not* a right quasibundle [13, Example 8.7]. Furthermore, it is shown in [13, Theorem 2.1] that for a compact subset $X \subset \mathbf{H}$ the map TX is a right quasibundle if and only if $\dim T_xX < \infty$, for all $x \in X$, and there exists a right quasibundle $\Omega\colon x \mapsto \Omega_x$ with $C_x^0X \subset \Omega_x$ for all $x \in X$. In this case, we see that $T_xX \subset \Omega_x$ for all $x \in X$. This property of TX is taken as the definition of TX in [5, Chapitre II, Proposition IV] for a closed subset $X \subset \mathbf{R}^n$. We note that this definition of TX applies only to closed subsets of $\mathbf{H}$ in our extended C^1 category and that we do not define TX for X a quotient object.

For X and Y closed subsets of $\mathbf{H}$ and $f\colon X \longrightarrow Y$ a C^1 map, let $F, \Phi\colon \mathbf{H} \longrightarrow \mathbf{H}$ be any two C^1 extensions of f. We note that for $x \in X$ we have $dF(x)|_{T_xX} = d\Phi(x)|_{T_xX}$ and that the image lies in $T_{f(x)}Y$. Thus, $df(x)\colon T_xX \longrightarrow T_{f(x)}Y$ is well defined as $dF(x)|_{T_xX}$. Of course, not having defined TX for quotient objects, we do not define df for C^1 maps into or out of these.

We recall from the Introduction that for a compact metric space X, we let $(\mathscr{H}(X),\, d_H)$ denote the metric space of nonempty closed subsets of X with the Hausdorff metric d_H.

Proposition 3.1. *Let $f\colon M \longrightarrow \mathbf{H}$ be a C^1 map of a compact connected C^1 manifold M (without boundary) into a Hilbert space $\mathbf{H}$. If $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M),\, d_H)$ is Lipschitz, then $f(M)$ is a C^1 submanifold (without boundary) of $\mathbf{H}$ and $f\colon M \longrightarrow f(M)$ is a C^1 fibre bundle.*

We note that the hypothesis that f be C^1 does not imply that the decomposition of M is metrically parallel. However, with the crucial hypothesis that $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M),\, d_H)$ is Lipschitz, Corollary 2.3 implies that we have a metrically parallel decomposition of M and that $\hat{f}\colon M/f \longrightarrow f(M)$ is a bi-Lipschitz equivalence. Finally, if $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M),\, d_H)$ is only assumed to be continuous, then $f(M)$ need not be a submanifold of $\mathbf{H}$; see remark VII in Section 5.

For the proof of Proposition 3.1, we will need the following lemma, which may be regarded as an easy version of the proposition.

Lemma 3.2. *Let M be a compact connected C^1 manifold (without boundary) and let $f\colon M \longrightarrow \mathbf{R}^n$ be a C^1 map. Then the differential $df(y)\colon T_yM \longrightarrow T_{f(y)}f(M)$ is surjective for all $y \in M$ if and only if $f\colon M \longrightarrow f(M)$ is a C^1 fibre bundle. In this case, $f(M)$ is a C^1 submanifold (without boundary) of $\mathbf{R}^n$.*

Proof. One direction is trivial. For the other, we may assume by the Whitney Embedding Theorem [8, Theorem 1.3.5 and Theorem 2.2.9] that M is a C^1 submanifold of $\mathbf{R}^m$ for some m. Let $X = f(M) \subset \mathbf{R}^n$. Since we do not yet know that the function $x \mapsto \dim T_xX$ is constant, we must use Glaeser's generalization of the Inverse Function Theorem [5, Chapitre II, Théorème 1] or [13, Theorem 5.1] to overcome this difficulty in order to conclude that X is indeed a C^1 submanifold of $\mathbf{R}^n$: To this end, let $y \in M$ and $f(y) = x_0$. Without loss of generality, we may assume $0 = x_0 \in X \subset \mathbf{R}^n$. Glaeser's Inverse Function Theorem [5, Chapitre II, Théorème 1] implies that near $0 \in X$, the closed set X is the graph of a C^1 function $g\colon K \longrightarrow T_0X^{\perp}$ with $g(0) = 0$ and $dg(0) = 0$, where $K \subset \pi(X)$ is a closed set with $0 \in K$ and $\pi\colon \mathbf{R}^n \longrightarrow T_0X$ is the orthogonal projection. Thus, by restricting to the closure Z of an open neighborhood X_0 of 0 in X, we may assume that $\pi(Z) \subset K$ and that $(\pi|_Z)^{-1}$ is a C^1 diffeomorphism. Furthermore, by identifying $(\mathscr{O}, y)$ with $(\mathbf{R}^p, 0)$ for some neighborhood $\mathscr{O}$ of y in M, we may assume that $f\colon \mathbf{R}^p \longrightarrow X_0$. Then $d(\pi \circ f)(y)T_yM = d\pi(0)T_0X = T_0(T_0X)$ so that by the usual Implicit Function Theorem some open neighborhood U of y in M is C^1 submersed by $\pi \circ f$ onto some open neighborhood V of 0 in the manifold T_0X. Finally, $\pi|_{(\pi|_{X_0})^{-1}(V)}$ is a chart near 0 in X and so our original X is a C^1 submanifold of $\mathbf{R}^n$. Then the function $x \mapsto \dim T_xX$ is constant because X is connected.

To summarize, our argument thus far shows that the set $X \subset \mathbf{R}^n$ is a compact connected C^1 manifold and that *every* value of $f\colon M \longrightarrow X$ is a regular value. Then, because M is also compact and connected, a standard argument (see [10, p. 55] or [4]) shows that f is a C^1 fibre bundle projection. $\square$

Proof of Proposition 3.1. There is a real number $\kappa \geq 0$ with $d_H\left(f^{-1}(u), f^{-1}(v)\right) \leq \kappa\|u - v\|$ for all $u, v \in f(M) \subset \mathbf{H}$. The following easy fact will smooth out the proof:

Claim 3.1.1. *Let $\kappa \geq 0$ be as above. Then for any $u' \in f^{-1}(u)$ there exists $v' \in f^{-1}(v)$ so that $\|u' - v'\| \leq \kappa\|u - v\|$.*

To see this fact, we may again assume without loss of generality that M is a C^1 submanifold of $\mathbf{R}^m$ for some m. We choose v' so that

$$\|u' - v'\| = \inf\left\{\|u' - v''\| \;\middle|\; v'' \in f^{-1}(v)\right\} \leq d_H\left(f^{-1}(u), f^{-1}(v)\right).$$

Now let $X = f(M) \subset \mathbf{H}$, let $x = f(y) \in X$, and let $\xi \in C_x^0X$. We may choose sequences $\{y_j\}_{j\geq 1}$ and $\{z_j\}_{j\geq 1}$ in M such that $f(y_j) \neq f(z_j)$, for each j, with both sequences $\{f(y_j)\}_{j\geq 1}$ and $\{f(z_j)\}_{j\geq 1}$ converging to x, and such that

$$\xi = \lim_{j\to\infty} \frac{f(y_j) - f(z_j)}{\|f(y_j) - f(z_j)\|}.$$

By Claim 3.1.1, for each $j \geq 1$, we may assume that y_j has been chosen so that $\|y - y_j\| \leq \kappa\|f(y) - f(y_j)\|$. Similarly, for each $j \geq 1$, we may assume that z_j has been chosen so that $\|y_j - z_j\| \leq \kappa\|f(y_j) - f(z_j)\|$. Consequently, we have $\lim_{j\to\infty} y_j = y = \lim_{j\to\infty} z_j$. Finally, by taking subsequences of $\{y_j\}_{j\geq 1}$ and $\{z_j\}_{j\geq 1}$, we may assume that the sequence $\{(y_j - z_j)/\|y_j - z_j\|\}_{j\geq 1}$ converges to some $\eta \in T_yM \subset \mathbf{R}^m$. Then, because f is C^1, we have

$$\lim_{j\to\infty} \frac{f(y_j) - f(z_j)}{\|y_j - z_j\|} = df(y)\,\eta = c\,\xi$$

for some $c \geq 1/\kappa > 0$. Hence, the map $df(y)\colon T_yM \longrightarrow T_xX$ contains C_x^0X in its image, which is a vector space, so it actually contains the span T_x^0X of C_x^0X.

Next, using C^1 paths in M to define tangent vectors on M, we show that $df(y)T_yM \subset T_x^0X$ for $x = f(y)$. For this, we note that if $\xi \in df(y)T_yM$ and $\|\xi\| = 1$, then there is some $0 \neq \eta \in T_yM$ such that $df(y)\eta = \xi$. Let $t \mapsto \alpha(t)$ be a C^1 path in M with $\alpha(0) = y$ and $d\alpha/dt|_{t=0} = \eta$. Choose a sequence $t_1 > t_2 > \cdots > 0$ converging to 0 and, for each i, let $v_i = \alpha(t_i) - \alpha(0)$. Then the sequence $\{v_i/t_i\}_{i\geq 1}$ converges to η. Writing $f\big(\alpha(t_i)\big) - f\big(\alpha(0)\big) = df(y)v_i + \|v_i\|\varepsilon_i$ with $\varepsilon_i \to 0$, we see that

$$\lim_{i\to\infty} \frac{f\big(\alpha(t_i)\big) - f\big(\alpha(0)\big)}{t_i} = df(y)\eta + \|\eta\|0 = \xi$$

and hence

$$\lim_{i\to\infty}\frac{f\big(\alpha(t_i)\big)-f\big(\alpha(0)\big)}{\|f\big(\alpha(t_i)\big)-f\big(\alpha(0)\big)\|}=\lim_{i\to\infty}\frac{\big[f\big(\alpha(t_i)\big)-f\big(\alpha(0)\big)\big]/t_i}{\|\big[f\big(\alpha(t_i)\big)-f\big(\alpha(0)\big)\big]/t_i\|}=\frac{\xi}{\|\xi\|}=\xi.$$

Therefore, $\xi\in C^0_xX$.

It now follows that $df(y)T_yM=T^0_xX$ and that $C^0_xX=\{\xi\in T^0_xX \mid \|\xi\|=1\}$. We note that the latter equality is *not* valid for all closed subsets of $\mathbf{H}$. The following fact will be needed in the sequel:

Claim 3.1.2. *Let $\kappa\geq 0$ be as above. Then for any $y\in M$ and any $\xi\in df(y)T_yM$ there exists $\eta_0\in T_yM$ with $df(y)\eta_0=\xi$ and $\|\eta_0\|\leq\kappa\|\xi\|$. In other words, the norm of the natural linear isomorphism $T^0_xX=df(y)T_yM\longrightarrow T_yM/\ker df(y)$ is at most κ.*

To see this, we may assume that $\|\xi\|=1$ so that $\xi\in C^0_xX$ for $x=f(y)$. Then, as established above, there is a unit vector $\eta\in T_yM$ with $df(y)\eta=c\xi$ for some $c\geq 1/\kappa$. Now $\eta_0=\eta/c$ satisfies the requirements of the claim.

We next check (by induction) that the image of $df(y)$ contains T^a_xX for each ordinal a. The case $a=0$ is known (above). Let $a>0$ and suppose that $T^b_xX\subset df(y)T_yM$ for $b<a$. Let $\xi\in C^a_xX$. Then there exist a sequence $\{x_j\}_{j\geq1}\subset X$ converging to $x\in X$ and, for each $j\geq1$, a vector $\xi_j\in T^{b_j}_{x_j}X$, for some ordinal $b_j<a$, so that $\xi_j\to\xi$. Again, as in the case $a=0$, we find a sequence $\{y_j\}_{j\geq1}\subset M$ converging to y and so that $f(y_j)=x_j$ for each $j\geq1$. Since $T^{b_j}_{x_j}X\subset df(y_j)T_{y_j}M=T^0_{x_j}X$, by Claim 3.1.2 above, for each $j\geq1$ there exists a vector $\eta_j\in T_{y_j}M$ with $df(y_j)\eta_j=\xi_j$ and $\|\eta_j\|\leq\kappa\|\xi_j\|$. Then $\sup_{j\geq1}\|\eta_j\|<\infty$ and we may assume, by taking subsequences, that $\eta_j\to\eta$ for some $\eta\in T_yM$. Consequently, $df(y)\eta=\xi$ and so $C^a_xX\subset df(y)T_yM$ implying that $T^a_xX\subset df(y)T_yM$. Hence, we have that $df(y)T_yM=T_xX$ is finite dimensional for each $y\in M$.

Now there are two cases to consider:

Case 1: $\dim(\mathbf{H})<\infty$. This case is Lemma 3.2.

Case 2: The General Case. It suffices to show that there is a C^1 embedding $\iota\colon X=f(M)\longrightarrow\mathbf{R}^n$ of $f(M)$ into some Euclidean space. It is shown in [13, Theorem 5.3] that such an embedding exists if and only if the compact set X is spherically compact with TX a quasibundle; see the Appendix.

Claim 3.1.3. *The set X is spherically compact.*

To see this, we let $\{x_j\}_{j\geq1}$ and $\{x'_j\}_{j\geq1}$ be sequences in X with $x_j\neq x'_j$ and both converging to some $x\in X$. We must show that the sequence $\{\xi_j=(x_j-x'_j)/\|x_j-x'_j\|\}_{j\geq1}\subset\mathbf{H}$ of unit vectors has a norm convergent subsequence. As above, choose $\kappa>0$ and $y,y_j,y'_j\in M$ so that $f(y)=x$, $f(y_j)=x_j$, $f(y'_j)=x'_j$ with $\|y-y_j\|\leq\kappa\|x-x_j\|$ and $\|y_j-y'_j\|\leq\kappa\|x_j-x'_j\|$. Then $\lim_{j\to\infty}y_j=y=\lim_{j\to\infty}y'_j$. We may assume that $(y_j-y'_j)/\|y_j-y'_j\|$ converges (in norm) to some $\eta\in T_yM$. Then $(x_j-x'_j)/\|y_j-y'_j\|\to df(y)\eta$ with $\|df(y)\eta\|\geq1/\kappa$. Hence, $\xi_j\to df(y)\eta/\|df(y)\eta\|$.

Claim 3.1.4. *TX is a quasibundle.*

For this, we note that by [13, Lemma 2.3], TX is a quasibundle, provided that $\sigma(TX)=\{(x,\xi)\mid x\in X,\xi\in T_xX,\|\xi\|=1\}\subset X\times\mathbf{H}$ is compact. To this end, if $\{(x_j,\xi_j)\}_{j\geq1}$ is a sequence in $\sigma(TX)$, then there exist $\kappa>0$, $y_j\in M$, and $\eta_j\in T_{y_j}M$ such that $f(y_j)=x_j$, $df(y_j)\eta_j=\xi_j$, and $\|\eta_j\|\leq\kappa$. Hence, we may assume that

$y_j \to y \in M$ and $\eta_j \to \eta \in T_yM$. Consequently, we have $x_j \to x = f(y) \in X$ and $\xi_j \to df(y)\eta \in T_xX$. $\square$

As pointed out by the referee, one can prove Proposition 3.1 without using the embedding result [13, Theorem 5.3]. Specifically, Lemma 3.2 has a generalization in which $\mathbf{R}^n$ is replaced by a Hilbert space $\mathbf{H}$ and the compact subset $X = f(M) \subset \mathbf{H}$ is assumed to be spherically compact with TX a quasibundle. In the proof of this generalization we use [13, Theorem 4.2] (see the Appendix) instead of [5, Chapitre II, Théorème 1] in the case where $X = X_k$ for some k. This is possible since the proof of Lemma 3.2 shows that, by [13, Lemma 4.3], we may first conclude that X is a topological (even a Lipschitz) k-submanifold of $\mathbf{H}$ for some k so that $\dim T_xX = k$ for each $x \in X$.

Proof of Corollary 2. If the map $f\colon M \longrightarrow f(M)$ is a Lipschitz fibre bundle projection, then $\hat{f}$ is bi-Lipschitz with respect to the Hausdorff metric and the decomposition is metrically parallel. Hence, Corollary 2 follows. $\square$

4. Proof of Theorem 3

We begin with the following lemma, whose proof is straightforward and is left to the reader.

Lemma 4.1. *Let $\theta\colon \mathbf{R}^k \longrightarrow \mathbf{R}$ and $\omega\colon \mathbf{R}^k \longrightarrow \mathbf{H}$ be C^1 functions with $\theta(0) = 0$, $d\theta(0) = 0$, $\omega(0) = 0$, and $d\omega(0) = 0$. Then there exists an increasing C^1 function $h\colon ([0,\infty), 0, (0,\infty)) \longrightarrow ([0,\infty), 0, (0,\infty))$ with $h'(0) = 0$, $h'(t) > 0$ for $t > 0$, $|\theta(x)|/h(\|x\|) \le 1$, and $\|\omega(x)\|/h(\|x\|) \le 1$ for $x \neq 0$.*

Let $G \times M \longrightarrow M$ be a C^1 action of a compact Lie group G on a compact C^1 manifold M. By [22, Theorem B], we may assume that M is a C^∞ manifold and that the action $G \times M \longrightarrow M$ is C^∞. By [22, Theorem 2.1], there is an equivariant C^∞ embedding ι of M into some orthogonal representation space $\mathbf{R}^n$ of G; see also [25]. Letting $\mathbf{R}^k$ be a trivial representation space of G, we say that a polynomial map $H\colon \mathbf{R}^n \longrightarrow \mathbf{R}^k$ is a *Hilbert invariant polynomial map* if H is invariant and if, for any invariant polynomial function $p\colon \mathbf{R}^n \longrightarrow \mathbf{R}$, there exists a polynomial function $\tilde{p}\colon \mathbf{R}^k \longrightarrow \mathbf{R}$ so that $p = \tilde{p} \circ H$. That a Hilbert invariant polynomial map always exists is a classical result of Hilbert; see [3, p. 326] and [7, Chapter X, Theorem 5.6]. Let $H\colon \mathbf{R}^n \longrightarrow \mathbf{R}^k$ be a Hilbert invariant polynomial map and then factor $H \circ \iota$ through M/G to define a topological embedding $\widehat{H \circ \iota}\colon M/G \longrightarrow \mathbf{R}^k$.

For $f\colon M \longrightarrow \mathbf{H}$ as in the hypotheses for Theorem 3, let $F = (\iota \times f) \circ \mathit{\Delta}_M\colon M \longrightarrow \mathbf{R}^n \times \mathbf{H}$, where $\mathit{\Delta}_M$ denotes the diagonal map for M. Finally, let $F_1 = (H \times \mathrm{id}) \circ F$.

Proposition 4.2. *With the hypotheses as in Theorem 3, if there exists a C^1 map $f\colon M \longrightarrow \mathbf{H}$ inducing a topological embedding $\hat{f}\colon M/G \longrightarrow \mathbf{H}$ such that $C^1_{\mathrm{emb}(f)}(M/G\colon \mathbf{H}) = C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})$, then the action has only one orbit type.*

In what follows, we will have occasion to deal with a left action $G \times M \longrightarrow M$ and rarely a right action $M \times G \longrightarrow M$. In the first case, we will denote the orbit space by the conventional M/G (rather than $G\backslash M$) and in the second case, by $M/_rG$. We note that Lipschitz matters occur neither in the statement nor in the proof of Proposition 4.2.

Proof of Proposition 4.2. The outline of the proof is as follows: We first carry out seven reduction steps. Then we complete the proof by arriving at a contradiction when the action has more than one orbit type.

Let $F = (\iota \times f) \circ \varDelta_M \colon M \longrightarrow \mathbf{R}^n \times \mathbf{H}$ be as above and let $F_1 = P_0 \circ F$, where $P_0 = H \times \mathrm{id}$, and H is a Hilbert invariant polynomial map. Then the diagram

(4.2.1)
$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "q"'] \arrow[dr, "F_1"] & \mathbf{K} = \mathbf{R}^n \times \mathbf{H} \arrow[d, "P_0"] \\ M/G \arrow[r, "\hat{F}_1"] & \mathbf{L} = \mathbf{R}^k \times \mathbf{H} \end{tikzcd}$$

commutes, with $\mathbf{K}$ a representation Hilbert space of G, $\mathbf{L}$ a trivial representation Hilbert space of G, and $\hat{F}_1 \colon M/G \longrightarrow \mathbf{L}$ the induced topological embedding.

Step 1. Let $\mathrm{pr}_2 \colon \mathbf{R}^k \times \mathbf{H} \longrightarrow \mathbf{H}$ be the projection on the second factor. Then we have the following inclusions:

$$\begin{aligned} C^1_{\mathrm{quot}}(M/G \colon \mathbf{H}) &= C^1_{\mathrm{emb}(f)}(M/G \colon \mathbf{H}) = \hat{f}^* C^1(\mathbf{H} \colon \mathbf{H}) \\ &= \hat{F}_1^* \mathrm{pr}_2^* C^1(\mathbf{H} \colon \mathbf{H}) \subset C^1_{\mathrm{emb}(F_1)}(M/G \colon \mathbf{H}) \subset C^1_{\mathrm{quot}}(M/G \colon \mathbf{H}). \end{aligned}$$

Hence,

$$C^1_{\mathrm{emb}(F_1)}(M/G \colon \mathbf{H}) = C^1_{\mathrm{quot}}(M/G \colon \mathbf{H}).$$

Step 2. By [22] and [8, p. 51], we may assume that M is a C^∞ manifold and that the action $G \times M \longrightarrow M$ is C^∞. Let Γ be an invariant Riemannian metric on M and let $\exp \colon TM \longrightarrow M$ be the associated exponential map. Then by [10, pp. 42–46], there exists some $\varepsilon > 0$ so that $\exp \colon \nu_\varepsilon(Gx \colon M) \longrightarrow N_\varepsilon(Gx) = \{y \in M \mid \mathrm{dist}_\Gamma(y, Gx) \le \varepsilon\}$ is an equivariant C^∞ (and hence C^1) diffeomorphism. Here, as usual, $\nu(Gx \colon M)$ denotes the normal bundle of Gx in M and $\nu_\varepsilon(Gx \colon M)$ denotes the normal closed ε-disk bundle of Gx in M. Also, we will use $\nu(Gx \colon M)_x$ and $\nu_\varepsilon(Gx \colon M)_x$ to denote the respective fibres over the point $x \in M$. Furthermore, we let the isotropy group G_x of $x \in M$ act on the *right* of $\nu(Gx \colon M)_x$ by setting $\xi h = dh^{-1}(x)\xi$. Because this action is orthogonal, it restricts to one on $\nu_\varepsilon(Gx \colon M)_x$.

For any $x \in M$, the restriction $F_2 = F_1|_{N_\varepsilon(Gx)} \colon N_\varepsilon(Gx) \longrightarrow \mathbf{L}$ factors through a topological embedding $\hat{F}_2 \colon N_\varepsilon(Gx)/G \longrightarrow \mathbf{L}$ and then Step 1 together with well-known C^1 extension and G-averaging theorems imply that

$$C^1_{\mathrm{emb}(F_2)}\big(N_\varepsilon(Gx)/G \colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(N_\varepsilon(Gx)/G \colon \mathbf{H}\big).$$

Step 3. For any $x \in M$, the composition $F_3 = F_2 \circ \exp \colon \nu_\varepsilon(Gx \colon M) \longrightarrow \mathbf{L}$ factors through a topological embedding $\hat{F}_3 \colon \nu_\varepsilon(Gx \colon M)/G \longrightarrow \mathbf{L}$ and then Step 2 implies that

$$C^1_{\mathrm{emb}(F_3)}\big(\nu_\varepsilon(Gx \colon M)/G \colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(\nu_\varepsilon(Gx \colon M)/G \colon \mathbf{H}\big).$$

Step 4. Given $x \in M$, for brevity we write $D = \nu_\varepsilon(Gx \colon M)_x$. We note that the map $((g,\xi),h) \mapsto (gh, dh^{-1}(x)\xi)$ is a C^∞ free *right* action $(G \times D) \times G_x \longrightarrow G \times D$ of the isotropy subgroup G_x of x on $G \times D$. Then $(G \times D)/_r G_x$ is a C^∞ manifold with boundary, and the C^∞ map $G \times D \longrightarrow \nu_\varepsilon(Gx \colon M)$, given by $(g,\xi) \mapsto dg(x)\xi$, factors through a usual C^∞ (and hence C^1) diffeomorphism $\varPhi_1 \colon (G \times D)/_r G_x \longrightarrow \nu_\varepsilon(Gx \colon M)$. We define a C^∞ *left* action $G \times ((G \times D)/_r G_x) \longrightarrow (G \times D)/_r G_x$ by setting $g \cdot [g', \xi] = [gg', \xi]$. Here, $[g,\xi] = (g,\xi) \cdot G_x$ denotes the orbit of (g,ξ) under G_x. This action is well-defined and we note that the diffeomorphism $\varPhi_1$ is G-equivariant.

For any $x \in M$, the composition $F_4 = F_3 \circ \Phi_1\colon (G \times D)/_r G_x \longrightarrow \mathbf{L}$ factors through a topological embedding $\hat{F}_4\colon ((G \times D)/_r G_x)/G \longrightarrow \mathbf{L}$ and then Step 3 implies that

$$C^1_{\mathrm{emb}(F_4)}\big(((G \times D)/_r G_x)/G\colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(((G \times D)/_r G_x)/G\colon \mathbf{H}\big).$$

Step 5. Using the G-equivariant diffeomorphism $\exp \circ \Phi_1\colon (G \times D)/_r G_x \longrightarrow N_\varepsilon(Gx)$, we see that whenever the action of G_x on D is trivial, then for any $y \in N_\varepsilon(Gx)$, the isotropy subgroup G_y is conjugate to all of G_x. Hence, because M is connected, if the right action of G_x on $\nu_\varepsilon(Gx\colon M)_x$ is trivial for every $x \in M$, then the action $G \times M \longrightarrow M$ has only one orbit type G/G_x.

From now on we assume that the action $G \times M \longrightarrow M$ has more than one orbit types. Then there exists some $x \in M$ for which the action of G_x on $D = \nu_\varepsilon(Gx\colon M)_x$ is nontrivial. Fixing such $x \in M$, we write

$$A = \{\xi \in \nu(Gx\colon M)_x \mid \xi h = \xi \text{ for all } h \in G_x\}$$

and let $B \neq 0$ denote the orthogonal complement of A in $\nu(Gx\colon M)_x$. Then

$$B = \left\{ \sum_{i=1}^{m} (\xi_i h_i - \xi_i) \;\Big|\; m \in \mathbf{N},\ h_i \in G_x,\ \text{and } \xi_i \in \nu(Gx\colon M)_x \right\}.$$

Here we may replace each ξ_i with its orthogonal projection in B. Let $B_\varepsilon = \big\{\xi \in B \;\big|\; \|\xi\|_\Gamma \leq \varepsilon\big\}$. Then the equivariant inclusion $B_\varepsilon \subset D$ induces a usual C^∞ (and hence C^1) G-equivariant embedding $\Phi_2\colon (G \times B_\varepsilon)/_r G_x \longrightarrow (G \times D)/_r G_x$.

The composition $F_5 = F_4 \circ \Phi_2\colon (G \times B_\varepsilon)/_r G_x \longrightarrow \mathbf{L}$ factors through a topological embedding $\hat{F}_5\colon ((G \times B_\varepsilon)/_r G_x)/G \longrightarrow \mathbf{L}$ and then Step 4 implies that

$$C^1_{\mathrm{emb}(F_5)}\big(((G \times B_\varepsilon)/_r G_x)/G\colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(((G \times B_\varepsilon)/_r G_x)/G\colon \mathbf{H}\big).$$

Step 6. The inclusion $B_\varepsilon \hookrightarrow G \times B_\varepsilon$, given by $\xi \mapsto (e_G, \xi)$, induces a usual C^∞ (and hence C^1) left G_x-equivariant embedding $\Phi_3\colon B_\varepsilon \longrightarrow (G \times B_\varepsilon)/_r G_x$. Hence, the composition $F_6 = F_5 \circ \Phi_3\colon B_\varepsilon \longrightarrow \mathbf{L}$ factors through a topological embedding $\hat{F}_6\colon B_\varepsilon/G_x \longrightarrow \mathbf{L}$. Then the diagram

$$\begin{tikzcd}
B_\varepsilon \arrow[rr, "\Phi_3"] \arrow[d, "q_x"'] \arrow[drr, "j"] & & (G \times B_\varepsilon)/_r G_x \arrow[d, "q"] \arrow[drr, "F_5"] & & \\
B_\varepsilon/G_x \arrow[rr, "\hat{j}"] & & ((G \times B_\varepsilon)/_r G_x)/G \arrow[rr, "\hat{F}_5"] & & \mathbf{L}
\end{tikzcd}$$

commutes, with q_x and q quotient maps, j defined as the obvious composition, and with $\hat{j}$ as the induced map, which is a homeomorphism. Clearly, we have $\hat{F}_6 = \hat{F}_5 \circ \hat{j}$ and

$$\hat{j}^*\colon C^1_{\mathrm{quot}}\big(((G \times B_\varepsilon)/_r G_x)/G\colon \mathbf{H}\big) \longrightarrow C^1_{\mathrm{quot}}(B_\varepsilon/G_x\colon \mathbf{H}).$$

To check that this map is surjective, it suffices to show that

$$\Phi_3^*\colon C^1\big((G \times B_\varepsilon)/_r G_x\colon \mathbf{H}\big)^G \longrightarrow C^1(B_\varepsilon\colon \mathbf{H})^{G_x}$$

is surjective. (Here the superscripts G and G_x refer to G-invariant and G_x-invariant functions, respectively.) To this end, let $\alpha \in C^1(B_\varepsilon\colon \mathbf{H})^{G_x}$ and define $\tilde{\alpha}[g, \xi] = \alpha(\xi)$.

Then $\tilde{\alpha}$ is a well-defined member of $C^1\big((G \times B_\varepsilon)/_r G_x \colon \mathbf{H}\big)^G$ and $\Phi_3^* \tilde{\alpha} = \alpha$. Using Step 5, we calculate

$$\begin{aligned} C^1_{\mathrm{emb}(F_6)}\big(B_\varepsilon/G_x \colon \mathbf{H}\big) &= \hat{j}^* C^1_{\mathrm{emb}(F_5)}\big(((G \times B_\varepsilon)/_r G_x)/G \colon \mathbf{H}\big) \\ &= \hat{j}^* C^1_{\mathrm{quot}}\big(((G \times B_\varepsilon)/_r G_x)/G \colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\left(B_\varepsilon/G_x \colon \mathbf{H}\right). \end{aligned}$$

Thus,

$$C^1_{\mathrm{emb}(F_6)}\big(B_\varepsilon/G_x \colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(B_\varepsilon/G_x \colon \mathbf{H}\big).$$

That is, $\hat{F}_6$ is a C^1 embedding of B_ε/G_x equipped with the quotient C^1 structure.

Step 7. Let $J_0 = F \circ \exp \circ \Phi_1 \circ \Phi_2 \circ \Phi_3 \colon B_\varepsilon \longrightarrow \mathbf{K}$. Then J_0 is a G_x-equivariant C^1 embedding and the diagram

$$\begin{tikzcd} B_\varepsilon \arrow[r, "J_0"] \arrow[d, "q_x"'] \arrow[dr, "F_6"] & \mathbf{K} \arrow[d, "P_0"] \\ B_\varepsilon/G_x \arrow[r, "\hat{F}_6"] & \mathbf{L} \end{tikzcd}$$

commutes, with $P_0 \colon \mathbf{K} \longrightarrow \mathbf{L}$ as in (4.2.1). We note that the action of G_x on the positive dimensional disk B_ε is linear (in fact orthogonal) with 0 as the only fixed point. We abbreviate $dh(x)\xi$ as $h\xi$ for $h \in G_x$ and $\xi \in B_\varepsilon$. Clearly, we have $h\xi = \xi h^{-1}$. Of course, the map J_0 need not be linear, and as $J_0(0) = F(x) = \big(\iota(x), f(x)\big)$, we even have $J_0(0) \neq 0$ whenever $G_x \neq G$. Let $\tau \colon \mathbf{K} \longrightarrow \mathbf{K}$ be the G_x-equivariant C^1 diffeomorphism $y \mapsto y - J_0(0)$. Then $J = \tau \circ J_0 \colon B_\varepsilon \longrightarrow \mathbf{K}$ is a G_x-equivariant C^1 embedding with $J(0) = 0$. Let $\mu \colon \mathbf{L} \longrightarrow \mathbf{L}$ be the C^1 diffeomorphism $y \mapsto y - P_0(J_0(0))$. Then $P = \mu \circ P_0 \circ \tau^{-1} \colon \mathbf{K} \longrightarrow \mathbf{L}$ is a G_x-invariant polynomial map $(y, z) \mapsto \big(H(y + \iota(x)) - H(\iota(x)), z\big)$ with $P(0) = 0$ and $F_7 = \mu \circ F_6$ is a G_x-invariant map inducing a topological embedding $\hat{F}_7 = \mu \circ \hat{F}_6$. Then the diagram

(4.2.2)
$$\begin{tikzcd} B_\varepsilon \arrow[r, "J"] \arrow[d, "q_x"'] \arrow[dr, "F_7"] & \mathbf{K} \arrow[d, "P"] \\ B_\varepsilon/G_x \arrow[r, "\hat{F}_7"] & \mathbf{L} \end{tikzcd}$$

is commutative, and Step 6 implies that

$$C^1_{\mathrm{emb}(F_7)}\big(B_\varepsilon/G_x \colon \mathbf{H}\big) = C^1_{\mathrm{quot}}\big(B_\varepsilon/G_x \colon \mathbf{H}\big).$$

We are now in a position to complete the proof of the Proposition. By Step 5, since the map $dJ(0) \colon B \longrightarrow T_0 J(B_\varepsilon)$ is a G_x-equivariant linear isomorphism, any $\eta \in T_0 J(B_\varepsilon)$ may be written as $\eta = \sum_{i=1}^m (h_i \eta_i - \eta_i)$ with $\eta_i \in T_0 J(B_\varepsilon)$ and $h_i \in G_x$. Also, since $P \colon \mathbf{K} \longrightarrow \mathbf{L}$ is equivariant, so is $dP(0)$ and we have $dP(0)\eta = 0$ for $\eta \in T_0 J(B_\varepsilon)$. That is, $dP(0)|_{T_0 J(B_\varepsilon)} = 0$.

Next, we use the Implicit Function Theorem to see that, for $\varepsilon > 0$ sufficiently small, $J(B_\varepsilon)$ is the graph of a C^1 function $\omega \colon \pi J(B_\varepsilon) \longrightarrow T_0 J(B_\varepsilon)^\perp$ with $\omega(0) = 0$ and $d\omega(0) = 0$, where $\pi \colon \mathbf{K} \longrightarrow T_0 J(B_\varepsilon)$ is the orthogonal projection. For $z \in T_0 J(B_\varepsilon)$, let $\theta(z) = \|z\|^{3/2}$. Then θ is C^1 with $\theta(0) = 0$ and $d\theta(0) = 0$. It follows from Lemma 4.1 that there is an increasing C^1 function $h \colon [0, \infty) \longrightarrow [0, \infty)$ with $h(0) = h'(0) = 0$ and $h'(t) > 0$ for $t > 0$ so that

(4.2.3)
$$\|\omega(\pi y)\| \leq h(\|\pi y\|), \quad \text{and} \quad \lim_{z \to 0} \|z\|^2 / h(\|z\|) = 0.$$

Define an increasing C^1 function $u\colon \big([0,\infty),0,(0,\infty)\big) \longrightarrow \big([0,\infty),0,(0,\infty)\big)$ by setting

$$u(r) = \int_0^r \sqrt{h'(t)}\,dt$$

and note that $u(0) = u'(0) = 0$ with $\lim_{r\to 0} u(r)/h(r) = \infty$. The function $\psi\colon B_\varepsilon \longrightarrow \mathbf{R} \subset \mathbf{H}$ defined by setting $\psi(\xi) = u(\|\pi J(\xi)\|)$ is C^1 and G_x-invariant, and so defines $\hat\psi \in C^1_{\mathrm{quot}}(B_\varepsilon/G_x\colon \mathbf{R})$. Because $\hat F_7$ is a C^1 embedding of the quotient space $\big(B_\varepsilon/G_x, C^1_{\mathrm{quot}}(B_\varepsilon/G_x\colon \mathbf{H})\big)$, there exists a function $\varphi \in C^1(\mathbf{L}\colon \mathbf{H})$ with $\varphi\circ\hat F_7 = \hat\psi$. We may assume that $\varphi(\mathbf{L}) \subset \mathbf{R}$. Using diagram (4.2.2), we conclude that $u(\|\pi J(\xi)\|) = \varphi\big(P(J(\xi))\big)$. We have

$$\lim_{\xi\to 0} u(\|\pi J(\xi)\|)/h(\|\pi J(\xi)\|) = \infty, \tag{4.2.4}$$

and we will obtain a contradiction by showing that the function

$$u(\|\pi J(\xi)\|)/h(\|\pi J(\xi)\|) = \varphi\big(P(J(\xi))\big)/h(\|\pi J(\xi)\|)$$

is bounded on B_ε.

To this end, we write

$$P(J(\xi)) = P(\pi J(\xi)) + dP(\pi J(\xi))\omega(\pi J(\xi)) + \Pi\big(\pi J(\xi), \omega(\pi J(\xi))\big),$$

where Π is a finite sum of polynomial maps, each multilinear of order ≥ 2 in its second variable. By (4.2.3), we see that

$$\lim_{\xi\to 0} \Pi\big(\pi J(\xi), \omega(\pi J(\xi))\big)/h(\|\pi J(\xi)\|) = 0$$

and that

$$\left\| dP(\pi J(\xi)) \frac{\omega(\pi J(\xi))}{h(\|\pi J(\xi)\|)} \right\| \leq |||dP(\pi J(\xi))|||,$$

where the operator norm on the right hand side is bounded near $\xi = 0$. Furthermore, because $P(0) = 0$ and $dP(0)|_{T_0 J(B_\varepsilon)} = 0$ and because P is a polynomial map, we must have that $\|P(\pi J(\xi))\|/\|\pi J(\xi)\|^2$ is bounded on B_ε. Consequently, we have $\lim_{\xi\to 0} \|P(\pi J(\xi))\|/h(\|\pi J(\xi)\|) = 0$ by (4.2.3). Thus, we have shown that $P(J(\xi))/h(\|\pi J(\xi)\|)$ is bounded on B_ε.

By the Mean Value Theorem, and since $\varphi(0) = 0$, we may write

$$\varphi\big(P(J(\xi))\big) = d\varphi\big(\zeta(P(J(\xi)))\big)\cdot P(J(\xi)), \tag{4.2.5}$$

where $\zeta(P(J(\xi)))$ is on the line segment $[0, P(J(\xi))]$ and $\|d\varphi\big(\zeta(P(J(\xi)))\big)\|$ is bounded on B_ε. Then, because $P(J(\xi))/h(\|\pi J(\xi)\|)$ is bounded on B_ε, we finally see that $\varphi\big(P(J(\xi))\big)/h(\|\pi J(\xi)\|)$ is also bounded on B_ε.

This boundedeness contradicts (4.2.4) to complete the proof. $\square$

We note that the above proof fails to go through in the Lipschitz case at least at equation (4.2.5) where φ must be C^1. Of course, as pointed out in the Introduction, certain quotients with multiple orbit types admit bi-Lipschitz embeddings even into Euclidean space.

Proof of Corollary 4. This follows trivially from Theorem 3. If $\hat f\colon M/G \longrightarrow \mathbf{H}$ is a C^1 embedding, then $C^1_{\mathrm{emb}(f)}(M/G\colon \mathbf{H}) = C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})$. Hence, $\big(M/G, C^1_{\mathrm{quot}}(M/G\colon \mathbf{H})\big)$ is a C^1 manifold (without boundary). $\square$

5. Concluding remarks

We finish this paper with several remarks:

I. The unfriendly behavior pointed to by the example in the Introduction is actually quite hostile, and it follows from a deep theorem of Schwarz [23]. He shows that, for a C^∞ action $G \times M \longrightarrow M$, an equivariant C^∞ embedding $\iota\colon M \longrightarrow \mathbf{R}^n$, a Hilbert invariant polynomial map $H\colon \mathbf{R}^n \longrightarrow \mathbf{R}^k$, and the map $\sigma = H \circ \iota$, we always have $C^\infty_{\text{quot}}(M/G\colon \mathbf{R}) = C^\infty_{\text{emb}(\sigma)}(M/G\colon \mathbf{R})$. It follows then that the embedding $\hat\sigma\colon M/G \longrightarrow \mathbf{R}^k$ from the beginning of Section 4 is in fact a C^∞ embedding. It is easy to see that unless the quotient C^1 object M/G is a C^1 manifold (without boundary), the same map $\hat\sigma$ cannot be a C^1 embedding. In fact, by Corollary 4, *no* C^1 embedding exists, even into $\mathbf{H}$, unless M/G is a C^1 manifold. As it happens, the map $\hat\sigma$ is actually C^1. The difficulty is that the map $\hat\sigma^{-1}\colon \hat\sigma(M/G) \longrightarrow M/G$ is *not* C^1 unless M/G is C^1 manifold.

More starkly, the map $\hat\sigma^{-1}$ is a C^∞ map which fails to be C^1. Our definition of the quotient C^1 structure on M/G is forced by the requirement that it be a categorical quotient. The standard definitions of the C^∞ quotient [23], metric quotient [11] and here, and the topological quotient are precisely parallel. Thus, either one agrees to never mention C^1 quotient structures again or one accepts that there exists a map that is somehow “infinitely differentiable” but not “once differentiable.”

II. Even if one rejects the notion of a quotient C^1 structure, the same problem persists at the metric level: According to Theorem 3, unless M/G is a C^1 manifold (without boundary), the C^∞ embedding $\hat\sigma$ cannot be a bi-Lipschitz embedding with respect to the quotient metric.

More precisely, there is no forgetful functor from the extended C^∞ category to the Lipschitz category which commutes with both quotients and restrictions.

III. In [22], Palais shows that each C^1 action $G\times M \longrightarrow M$ is C^1-equivalent to an essentially unique C^∞ action. However, the fact that in general $C^1_{\text{emb}(f)}(M/G\colon \mathbf{H}) \neq C^1_{\text{quot}}(M/G\colon \mathbf{H})$ implies that nonetheless there is an essential difference between C^1 and C^∞ quotients. In particular, there is no forgetful functor from the extended C^∞ category to the extended C^1 category which commutes with both quotients and restrictions.

IV. Returning to the example $\mathbf{Z}_2 \times S^1 \longrightarrow S^1$ from the Introduction, we see that a choice for an equivariant embedding $\iota\colon S^1 \longrightarrow \mathbf{R}^n$ is given by $n = 2$ with $\iota(x,y) = (x,y)$. The action of $\mathbf{Z}_2$ here on $\mathbf{R}^2$ is by the involution $(x,y) \mapsto (x,-y)$ so that $k = n = 2$ and $H\colon \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ can be chosen as the map $H(x,y) = (x,y^2)$. Then the Schwarz–Hilbert C^∞ embedding [23] is given by $\hat H\colon S^1/\mathbf{Z}_2 \longrightarrow \mathbf{R}^2$ with image $P = \{(x,z)\,|\, z \geq 0 \text{ and } x^2 + z = 1\}$. The map $\hat H^{-1}\colon P \longrightarrow S^1/\mathbf{Z}_2$ is C^∞ (according to Schwarz) but not C^1. More specifically, the invariant C^1 function $\varphi\colon (x,y) \mapsto y^{4/3}$ defines a C^1 function $\hat\varphi\colon S^1/\mathbf{Z}_2 \longrightarrow \mathbf{R}$ with $(\hat H^{-1})^*\hat\varphi$ not C^1. In fact, $(\hat H^{-1})^*\hat\varphi(x,z) = z^{2/3}$.

V. It follows from Theorem 1 of Schwarz [23] that, for a representation $G\times\mathbf{R}^n \longrightarrow \mathbf{R}^n$, the invariant C^∞ functions are the C^∞ functions of invariant polynomials; see also [3, p. 326]. This consequence fails trivially in the C^1 case; this failure is the germ of the failure of the C^1 Whitney Embedding Theorem [8, Theorem 1.3.5 and Theorem 2.2.9] for singular quotients M/G.

VI. For the sake of further concreteness, we offer the following simple examples. Let $\mathbf{Z}_n$ denote the cyclic group of order n and let $\mathbf{Z}_n \times \mathbf{C} \longrightarrow \mathbf{C}$ be the usual action, i.e., the generator rotates $\mathbf{C}$ by an angle $2\pi/n$. By the Hilbert Invariant Polynomial Theorem ([3, p. 326] or [7, Chapter X, Theorem 5.6]), an embedding $\hat{H}_n$ is induced by the map $H_n\colon \mathbf{C} \longrightarrow \mathbf{R}^3$ given by $H_n(z) = (\mathrm{Re}\{z^n\},\ \mathrm{Im}\{z^n\},\ z\bar{z})$. Then we may ask whether the topological planes $\mathbf{C}/\mathbf{Z}_n$ and $\mathbf{C}/\mathbf{Z}_m$ are C^∞ diffeomorphic for $n \neq m$. In fact, for $2 \leq n < m$, near the origin, $\hat{H}_n(\mathbf{C}/\mathbf{Z}_n)$ and $\hat{H}_m(\mathbf{C}/\mathbf{Z}_m)$ are not bi-Lipschitz equivalent with respect to the metrics inherited from $\mathbf{R}^3$. Consequently, they are not C^1 diffeomorphic and hence, as must be well known, not C^∞ diffeomorphic. Thus, by Schwarz [23], $\mathbf{C}/\mathbf{Z}_n$ and $\mathbf{C}/\mathbf{Z}_m$ are not C^∞ diffeomorphic when $2 \leq n < m$.

VII. In Proposition 3.1, if $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M),\ d_H)$ is only continuous, then $f(M)$ need not even be a topological manifold as shown by the example $f\colon S^3 \longrightarrow \mathbf{R}^7$ given by $f(x,y,z,w) = (x^2, y^2, z^2, xy, xz, yz, w)$. Then $f(S^3)$ is homeomorphic to the suspension $\sum \mathbf{RP}^2$ of the real 2-dimensional projective space. To see this homeomorphism quickly, we note that f is a Hilbert invariant polynomial map for the action $(x,y,z,w) \mapsto (-x,-y,-z,w)$ of $\mathbf{Z}_2$ on S^3 which is antipodal on the equatorial S^2 and fixed on the poles. To generate many such examples, we observe that for a C^1 action $G \times M \longrightarrow M$, a C^1 equivariant embedding $\iota\colon M \longrightarrow \mathbf{R}^n$, and a Hilbert invariant polynomial map $H\colon \mathbf{R}^n \longrightarrow \mathbf{R}^k$ as in the beginning of Section 4, the C^1 map $f = H \circ \iota$ has $f^{-1}\colon f(M) \longrightarrow (\mathscr{H}(M),\ d_H)$ continuous because the decomposition is metrically parallel.

If $f\colon M \longrightarrow \mathbf{H}$ is an injective C^1 map, then $f(M)$ is a topological manifold, but $f(M)$ need not be a C^1 manifold (not even a Lipschitz manifold), as shown by the following example due to the referee: Let $f\colon S^1 \longrightarrow \mathbf{R}^2$ be an injective C^1 map given near $(1,0)$ by $f(\cos t, \sin t) = (t^2, 0)$ if $t \geq 0$, and by $f(\cos t, \sin t) = (t^2, t^4)$ if $t \leq 0$. Then $f(S^1)$ has a cusp at $(0,0) = f(1,0)$.

Appendix

On smooth finite dimensional embeddings. Our original version of Proposition 3.1 above had target space $\mathbf{R}^n$ but noted that the main result of [13] could be used to improve the proposition to have target space $\mathbf{H}$, a Hilbert space, if only it could be shown that $f(M)$ is spherically compact. The referee showed (Claim 3.1.3) that $f(M) \subset \mathbf{H}$ is spherically compact. Then the implication (1) $\Rightarrow$ (3) in Theorem 3 becomes a corollary of Theorem 1, doing away with our original tedious proof. However, the referee also noted that in order to carry out this program, a portion of [13] needed to be clarified. The results of this clarification are:

(1) In the statement of [13, Lemma 3.1] it should be assumed that TX is a quasibundle.
(2) The statement and the proof of [13, Theorem 4.2] change slightly, as displayed below, to replace the condition $p \in X$ with $p \in X_k$.
(3) The beginning of the proof of [13, Lemma 4.6] is clarified as below.
(4) Theorem [13, Theorem 5.2] is incorrect as stated. The correct statement, displayed below, includes the additional hypothesis that the point p is an isolated

point of $f^{-1}(f(p))$. Also, the conclusion is changed slightly: the neighborhood U is replaced by a smaller one $V \subset U$, and $f(V)$ is a neighborhood of $f(p)$ in $f(U)$—not in Y—with $f|_V$ a C^1 diffeomorphism.

(5) In the statements of [13, Theorem 7.3] and [13, Theorem 7.4] the subset Y should be assumed to be closed.

Finally, as noted in Section 3 above, a C^1 map on a compact set is Lipschitz. Hence, a C^1 map on a locally compact set is locally Lipschitz and so continuous; all the sets in [13] are at least locally compact.

Because each object of interest $X \subset \mathbf{H}$ is compact, we may assume, without loss of generality, that $\mathbf{H}$ is separable. Also, throughout [13] "projection" means "bounded linear projection". With these clarifications, the results of [13] remain valid. In particular, the main result stands as stated:

Theorem. [13, Theorem 5.3] *Let $\mathbf{H}$ be a Hilbert space and let X be a compact and spherically compact subset of $\mathbf{H}$ with TX a quasibundle. Then there is a C^1 embedding of X into $\mathbf{R}^N$ for some N finite.*

Of course, for X compact, the converse is immediate. It follows from Lemma 2.4 in the Authors' paper [21] (proof due to Lang) that if X is a compact and spherically compact subset of a normed linear space, then there exists a bi-Lipschitz embedding $X \hookrightarrow \mathbf{R}^N$ for some $N < \infty$.

In the gloss that follows, we use the terminology and notation of [13]. In particular, we recall that $X_k = \{p \in X \mid \dim T_pX = k\}$ and $X^k = \bigcup_{i \geq k} X_i$. Also, we recall that for a compact and spherically compact subset $X \subset \mathbf{H}$ with TX a quasibundle and a point $p \in X$ the orthogonal projection $\pi\colon \mathbf{H} \longrightarrow T_pX$ has four useful properties for a sufficiently small open neighborhood (in X) U of p:

(1) $\pi|_U$ is a bi-Lipschitz homeomorphism onto $\pi(U)$. We write $g = (\pi|_U)^{-1}$.
(2) For $q \in U$, the linear map $\pi\colon T_qX \longrightarrow T_{\pi(q)}\pi(U) \subset T_{\pi(q)}(T_pX)$ is a linear isomorphism. We write $\lambda(\pi(q)) = \left(\pi|_{T_qX}\right)^{-1}$.
(3) The function $\lambda\colon (x,\xi) \mapsto \lambda(x)\xi$ is continuous $T\pi(U) \longrightarrow \mathbf{H}$.
(4) We have $U_k = U \cap X_k$ and $\pi(U_k) = \pi(U)_k$.

Theorem. [13, Theorem 4.2] *Let $\mathbf{H}$ be a Hilbert space and let X be a compact and spherically compact subset of $\mathbf{H}$ with TX a quasibundle. For $k \geq 0$, let $p \in X_k$ and let U be a sufficiently small neighborhood (in X) of p. Then the map $(\pi|_{U_k})^{-1}$ has a C^1 extension $g_p\colon T_pX \longrightarrow \mathbf{H}$ such that:*

(i) *$dg_p(\pi(q))\xi = \lambda(\pi(q))\xi$ for $q \in U \cap X_k$ and $\xi \in T_{\pi(q)}\pi(U) \subset T_{\pi(q)}(T_pX)$.*
(ii) *$g_p(x) = x + G_p(x)$ for any $x \in T_pX$, where $G_p\colon T_pX \longrightarrow (T_pX)^\perp$ is a C^1 map.*

Proof. Let V be an open neighborhood (in X) of p, sufficiently small so that $V \cap X^{k+1} = \emptyset$, $\pi|_V\colon V \longrightarrow \pi(V)$ is a bi-Lipschitz homeomorphism, and $\pi|_{T_qX}\colon T_qX \longrightarrow T_{\pi(q)}\pi(V)$ is a linear isomorphism for each $q \in V$, as above. Let U be a sufficiently small open neighborhood of p in V with $\overline{U}$ compact and contained in V. We note that $\overline{U}$ is spherically compact and that, by [13, Theorem 2.1], $T\overline{U}$ is a quasibundle. Then the four conditions above (before the statement of the theorem) hold with $\left(\overline{U}, \overline{U}\right)$ in place of (X, U). Thus we may assume that we have:

(1) $X^k = X_k$; hence, X_k is compact.
(2) $\pi|_X\colon X \longrightarrow \pi(X)$ is bi-Lipschitz with $g = (\pi|_X)^{-1}$.

(3) $\pi|_{T_qX}\colon T_qX \longrightarrow T_{\pi(q)}\pi(X)$ is a linear isomorphism for all $q \in X$ with $\lambda(x) = \left(\pi|_{T_{g(x)}X}\right)^{-1}$.

(4) $\lambda\colon (x,\xi) \mapsto \lambda(x)\xi$ defines a continuous map $T\pi(X) \longrightarrow \mathbf{H}$.

For $x \in \pi(X) \subset T_pX$, we let $\varphi(x)\colon T_pX \longrightarrow T_x\pi(X)$ be the orthogonal projection. Then we see as in [13] that the map $\pi(X)_k \times T_pX \longrightarrow \mathbf{H}$ defined by $(x,\xi) \mapsto \lambda(x)\left(\varphi(x)\xi\right)$ is continuous. We write

$$\lambda(x)\left(\varphi(x)\xi\right) = \varphi(x)\xi + \Lambda(x)\xi$$

with $x \in \pi(X)_k \subset T_pX$ and $\Lambda(x)\xi \in T_pX^\perp$. (For compatibility with [13], we note that $\Lambda(x) = \Lambda(x)\varphi(x)$ here.) Then we have:

(5) $\Lambda\colon \pi(X)_k \times T_pX \longrightarrow T_pX^\perp$ is continuous.

The remainder of the proof of Theorem [13, Theorem 4.2] now goes through as in [13] with every occurrence of U replaced with X. The crucial point is that the two sequences $\{x_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ now converge to $x \in \pi(X)_k \subset \pi(X)$. □

Corollary A.1. *Let* $\mathbf{H}$ *be a Hilbert space and let* X *be a compact and spherically compact subset of* $\mathbf{H}$ *with* TX *a quasibundle. For* $k \geq 0$, *let* $p \in X_k$. *Then there exist an open neighborhood* Q *of* p *in* $\mathbf{H}$ *and a* k*-dimensional* C^1 *submanifold* N *of* $\mathbf{H}$, *closed in* $\mathbf{H}$, *so that*

(i) $\overline{Q} \cap X^{k+1} = \emptyset$ *and* $\overline{Q} \cap X$ *is compact.*

(ii) $p \in Q \cap X_k \subset N$.

(iii) $T_qX = T_qN$ *for each* $q \in Q \cap X_k$.

Proof. The first assertion follows from the fact that X^{k+1} is closed and $p \notin X^{k+1}$. Let $g_p\colon T_pX \longrightarrow \mathbf{H}$ be the C^1 map of the above theorem. Then $N = g_p(T_pX)$ will do, and (ii) is clear. Finally, letting Q be a sufficiently small neighborhood of p in $\mathbf{H}$, we see that (iii) follows from the inclusion $T_qX \subset T_qN$ because both vector spaces have dimension k. To see that the inclusion holds, let $\eta \in T_qX$ and recall that $\pi|_{T_qX}\colon T_qX \longrightarrow T_{\pi(q)}\pi(Q \cap X) \subset T_pX$ is a linear isomorphism with inverse $\lambda(\pi(q))$. Then

$$\eta = \lambda(\pi(q))\left(\pi|_{T_qX}\right)\eta = dg_p(\pi(q))\left(\pi|_{T_qX}\right)\eta \in T_qN. \qquad \square$$

We need the easy fact that Corollary A.1 is hereditary in the following sense.

Corollary A.2. *Let* X *and* Q *be as in Corollary A.1 and let* W *be an open subset of* Q. *Then we have*

(i) $\overline{W} \cap X^{k+1} = \emptyset$ *and* $\overline{W} \cap X$ *is compact.*

(ii) $W \cap X_k \subset N$.

(iii) $T_qX = T_qN$ *for each* $q \in W \cap X_k$.

Lemma. [13, Lemma 4.6] *Let* $\mathbf{H}$ *be a Hilbert space and let* X *be a compact and spherically compact subset of* $\mathbf{H}$ *with* TX *a quasibundle. Let* $k \geq 0$. *Then there is a* k*-dimensional* C^1 *submanifold* M_k *of* $\mathbf{H}$ *so that* $X_k \subset M_k$ *and* $T_pX = T_pM_k$ *for all* $p \in X_k$.

Proof. Clearly Corollary A.1 is a local version of this lemma. It allows us to choose for each $p \in X_k$ a corresponding open subset Q_p of $\mathbf{H}$ and a closed submanifold N_p of $\mathbf{H}$ satisfying the three conditions of the corollary. Let $\mathscr{Q}'$ be the resulting family of open subsets with union containing X_k. Then $\mathscr{O} = \bigcup \mathscr{Q}'$ is metric and so paracompact. Let

$$\mathscr{B} = \left\{ B \mid B \text{ is an open ball in } \mathbf{H} \text{ with } \overline{B} \subset \text{ some } Q \in \mathscr{Q}' \right\}.$$

Then $\mathscr{B}$ is an open cover of $\mathscr{O}$. Let $\mathscr{Q}$ be an open locally finite (in $\mathscr{O}$) refinement of $\mathscr{B}$, and hence of $\mathscr{Q}'$, still covering $\mathscr{O}$. Therefore, for each $Q \in \mathscr{Q}$, we may choose a point $p(Q) \in X_k$ so that $Q \subset Q_{p(Q)}$. Now $\mathscr{O}$ is Lindelöf since $\mathbf{H}$ satisfies the Second Axiom of Countability, and hence we may assume that $\mathscr{Q}$ is a countable family: $\mathscr{Q} = \{Q_i\}_{i\geq 1}$. For each $i \geq 1$, let $p_i = p(Q_i)$, $N_i = N_{p_i}$, and let $\pi_i \colon \mathbf{H} \longrightarrow T_{p_i}X$ denote the orthogonal projection. Using Corollary A.2, we see that we have a locally finite (in $\mathscr{O}$) countable family $\{Q_i\}_{i\geq 1}$ of open subsets of $\mathbf{H}$ and a countable family $\{N_i\}_{i\geq 1}$ of closed submanifolds of $\mathbf{H}$, with each pair (Q_i, N_i) satisfying the corollary, and with

$$\overline{Q}_j \subset \mathscr{O} = \bigcup_{i\geq 1} Q_i.$$

As usual, we construct a shrinking $\{P_i\}_{i\geq 1}$ of $\{Q_i\}_{i\geq 1}$. That is, $\{P_i\}_{i\geq 1}$ is also an open cover of $\mathscr{O}$ with $\overline{P}_i \subset Q_i$ for all $i \geq 1$; then $\overline{P}_i \cap X_k$ is compact for all $i \geq 1$.

The remainder of the proof of Lemma [13, Lemma 4.6] now goes through as in [13] to obtain new k-dimensional (nonclosed) C^1 submanifolds $N_j^{(i)}$ of $\mathbf{H}$. These are obtained by isotoping and shrinking suitable open subsets of N_j, keeping X_k fixed and respecting π_j. The resulting manifolds $N_j^{(i)}$, for $j = 1, 2, \ldots, i$, intersect pairwise in mutually open subsets in such a way that the union

$$M_k^{(i)} = \bigcup_{j=1}^{i} N_j^{(i)}$$

is a C^1 submanifold of $\mathbf{H}$ and $M_k^{(i)}$ contains $X_k \cap \left(\bigcup_{j=i}^{i} \overline{P}_j \right)$. To begin the construction, let $U_i = Q_i \cap X$. There exists an open neighborhood V_i, with compact closure, of $\pi_i(U_i)$ in $T_{p_i}X$ so that $\pi_i|_{X_k \cap \pi_i^{-1}(V_i)}$ is bi-Lipschitz and so that $d\pi_i(q) \colon T_qX \longrightarrow T_{\pi_i(q)}(T_{p_i}X)$ is an isomorphism for $q \in X_k \cap \pi_i^{-1}(V_i)$. The family $\{N_j^{(i)}\}_{j\geq 1}$ is constructed by induction on i so that $X_k \cap \overline{P}_i \subset N_j^{(i)}$ for $j = 1, 2, \ldots, i$, and $N_j^{(i)} = N_j^{(1)}$ for $j > i$. We begin the induction ($i = 1$) by setting

$$N_j^{(1)} = \pi_j^{-1}(V_j) \cap N_j \quad \text{for } j \geq 1.$$

Then $N_j^{(1)}$ is a C^1 submanifold of $\mathbf{H}$ because it is an open subset of N_j. In addition, because N_j is closed in $\mathbf{H}$, the manifold $N_j^{(1)}$ is closed in $\pi_j^{-1}(V_j)$.

To carry out the induction step (from $i-1$ to i):

(1) We replace the C^1 submanifold $M_k^{(i-1)}$ with an open subset $M_k^{(i-1,0)}$ which still contains the compact set $X_k \cap \left(\bigcup_{j=1}^{i-1} \overline{P}_j \right)$ and so that

$$\pi_i|_{M_k^{(i-1,0)} \cap \pi_i^{-1}(V_i)} \colon M_k^{(i-1,0)} \cap \pi_i^{-1}(V_i) \longrightarrow T_{p_i}X$$

is a C^1 embedding.

(2) We isotope $N_i^{(1)}$ to $N_i^{(i-1,0)}$, keeping its closed subset $X_k \cap N_i^{(1)}$ fixed and respecting π_i, so that $N_i^{(i-1,0)} \cap M_k^{(i-1,0)}$ contains a set W, open in both $N_i^{(i-1,0)}$ and $M_k^{(i-1,0)}$, which in turn contains $X_k \cap \left(\bigcup_{j=1}^{i-1} \overline{P}_j \right) \cap \overline{P}_i$.

(3) There exists an open neighborhood $M_k^{(i-1,1)}$ of $X_k \cap \left(\bigcup_{j=1}^{i-1} \overline{P}_j\right)$ in $M_k^{(i-1,0)}$, and there exists an open neighborhood $N_i^{(i)}$ of $X_k \cap \overline{P}_i$ in $N_i^{(i-1,0)}$, so that
  (a) $M_k^{(i-1,1)} \cap N_i^{(i)}$ is an open subset of both $M_k^{(i-1,1)}$ and $N_i^{(i)}$; and
  (b) $\overline{M_k^{(i-1,1)} \cap N_i^{(i)}} \subset W$.

  It follows that $M_k^{(i-1,1)} \cup N_i^{(i)}$ is a C^1 manifold. In addition, we may assume that
  (c) $M_k^{(i-1,1)} \setminus Q_i = M_k^{(i-1,0)} \setminus Q_i$.

(4) We set $N_j^{(i)} = N_j^{(i-1)} \cap M_k^{(i-1,1)}$ for $j = 1, 2, \ldots, i-1$, and $N_j^{(i)} = N_j^{(1)}$ for $j > i$.

Now the induction step is complete. Because $\mathscr{Q}$ is locally finite in the open set $\mathscr{O}$, for each $p \in X_k$ there is a set O_p which is an open neighborhood of p in every manifold $M_k^{(i)}$ for i large. The union of these neighborhoods is the desired manifold M_k.

For further details, see [13]. □

Theorem. [13, Theorem 5.2] (The Inverse Function Theorem) *Let* $\mathbf{H}$ *be a Hilbert space and let* $X \subset \mathbf{H}$ *and* $Y \subset \mathbf{H}$ *be compact and spherically compact subsets with* TX *and* TY *quasibundles. Let* f *be a* C^1 *map from* X *onto* Y. *If a point* $p \in X$ *is an isolated point of* $f^{-1}(f(p))$ *and* $df(p)$ *is a linear isomorphism, then there is a neighborhood* U *of* p *in* X *such that, for any neighborhood* V *of* p *in* U, *the set* $f(V)$ *is a neighborhood of* $f(p)$ *in* $f(U)$ *and* $f|_V$ *is a* C^1 *diffeomorphism.*

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