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GENERAL DECAY OF SOLUTIONS TO A VISCOELASTIC WAVE EQUATION WITH NONLINEAR LOCALIZED DAMPING

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Abstract. In this paper we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)|u_t|^m u_t + b|u|^\gamma u = 0$$

in a bounded domain. We prove that, for certain class of relaxation functions and certain initial data, the decay rate of the solution energy is similar to that of the relaxation function regardless of the presence or the absence of the frictional damping. This result improves earlier ones in Berrimi and Messaoudi [1] in which only the exponential decay rate is obtained.

1. Introduction

In [1], Berrimi and Messaoudi studied the following nonlinear problem:

$$(1.1)\quad \begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)u_t|u_t|^m + |u|^\gamma u = 0 \quad \text{in } \Omega\times(0,\infty),\\ &u(x,t) = 0, \quad x\in\partial\Omega,\ t\ge 0,\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega, \end{aligned}$$

where Ω is a bounded domain of $\mathbf{R}^n$ $(n\ge 1)$ with a smooth boundary $\partial\Omega$, $\gamma>0$, $m\ge 0$, g is a positive function, and $a\colon\Omega\to\mathbf{R}^+$ is a function, which may vanish on any part of Ω (including Ω itself). Under the condition that

$$g'(t) \le -\xi g(t) \quad \text{for } t\ge 0$$

for some positive constant ξ, the authors obtained an exponential decay result under weaker conditions on both a and g which improved [5].

In fact, in [5], Cavalcanti et al. dealt with the equation

$$(1.2)\quad \begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)u_t + |u|^\gamma u = 0 \quad \text{in } \Omega\times(0,\infty),\\ &u(x,t) = 0, \quad x\in\partial\Omega,\ t\ge 0,\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega, \end{aligned}$$

where $a\colon\Omega\to\mathbf{R}^+$ is a function, which may be null on a part of Ω. Assuming that $a(x)\ge a_0>0$ on $\omega\subset\Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad t\ge 0,$$

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such that $\|g\|_{L^1((0,\infty))}$ is small enough, they obtained an exponential decay result of energy for (1.2). This work extended the result of Zuazua [20], in which he considered (1.2) with $g = 0$ and the linear damping is localized. A related problem, in a bounded domain, of the form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau - \gamma \Delta u_t = 0,$$

for $\rho > 0$, was also studied by Cavalcanti et al. [3]. A global existence result for $\gamma \geq 0$, as well as an exponential decay for $\gamma > 0$, has been established. This last result has been extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [18, 17] and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term.

Recently, Messaoudi [15] studied (1.1) without the nonlinear localized damping term $a(x)|u_t|^m u_t$ and the nonlinear term $|u|^\gamma u$. He proved that the solution energy decays at the same rate of decay of the relaxation function, which is not necessarily decaying in a *polynomial or exponential* fashion. Motivated by the ideas of Messaoudi [15], Han and Wang [8] investigated the general decay of solution energy for the nonlinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + |u_t|^m u_t = 0.$$

Since the term $-\Delta u_{tt}$ was included, the nonlinear damping term $|u_t|^m u_t$ can be controlled there. For other related works, we refer the reader to [6]–[14] and [19].

In the present paper we are also concerned with problem (1.1). Our intention is to show that, for certain class of relaxation functions and certain initial data, the decay rate of the solution energy is similar to that of g (see (G2) below) regardless of the presence or the absence of the frictional damping. Therefore, our result allows a larger class of relaxation functions and improves earlier results in [1] in which only the exponential rate was considered. Our ideas come from [15] while we should overcome the difficulty brought by the control of the nonlinear localized damping term $a(x)u_t|u_t|^m$.

The paper is organized as follows. In the next section, we introduce some notations and prepare some materials. Section 3 contains the statement and proof of our result concerning the general decay of the solution.

2. Preliminaries

In this section, we present some materials needed in the proof of our main result. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. We will also use the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq 2n/(n-2)$ if $n \geq 3$ or $q \geq 2$ if $n = 1, 2$ and $L^r(\Omega) \hookrightarrow L^q(\Omega)$ for $q < r$. We will use the same embedding constant denoted by C_p; i.e.,

$$\|v\|_q \leq C_p \|\nabla v\|_2, \quad \|v\|_q \leq C_p \|v\|_r.$$

For the relaxation function $g(t)$ we assume

(G1) $g\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a bounded $\mathscr{C}^1$ function such that $g(0) > 0$ and

$$1 - \int_0^\infty g(s)\, ds = l > 0.$$

(G2) There exist a positive differentiable functions $\xi(t)$ satisfying

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0,$$

where

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \le k, \quad \xi(t) > 0, \quad \xi'(t) \le 0 \quad \forall\, t > 0, \quad \int_0^{+\infty} \xi(t)\, dt = +\infty.$$

Remark 2.1. (G1) is necessary to guarantee the hyperbolicity of the equation (1.1).

Remark 2.2. Since $\xi(t)$ is nonincreasing then $\xi(t) \le \xi(0) = M$.

We introduce the same "modified" energy functional as in [1]

$$\mathscr{E}(t) := \frac{1}{2}\Big(1 - \int_0^t g(s)\, ds\Big)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}, \tag{2.1}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2\, d\tau, \quad p \ge 1. \tag{2.2}$$

Lemma 2.3. (See Remark 2.3 of [1]) *The modified energy functional satisfies, along the solution of* (1.1),

$$\begin{aligned} \mathscr{E}'(t) &\le -\Big(\int_\Omega a(x)|u_t|^{m+2}\, dx - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|^2\Big) \\ &\le -\int_\Omega a(x)|u_t|^{m+2}\, dx + \frac{1}{2}(g' \circ \nabla u)(t) \le 0. \end{aligned} \tag{2.3}$$

Remark 2.4. This means that the "modified" energy is uniformly bounded (by $\mathscr{E}(0)$) and is decreasing in t.

We also need the following lemma.

Lemma 2.5. (See Lemma 3.1 of [1]) *Let $m \le 2/(n-2)$ for $n \ge 3$. Then there exists a constant C depending on C_p, $\|a\|_\infty$, $\mathscr{E}(0)$, and m only, such that the solution of* (1.1) *satisfies*

$$\int_\Omega a(x)|u|^{m+2}\, dx \le C\big(\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}\big). \tag{2.4}$$

3. General decay of the solution

In this section we state and prove that the rate of decay of energy is similar to that of the relaxation function. We suppose that $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$. Then, the existence of a unique global solution is guaranteed by Proposition 2.1 of [1].

We define the functional

$$F(t) := \mathscr{E}(t) + \varepsilon_1\Psi(t) + \varepsilon_2\chi(t) \tag{3.1}$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\Psi(t) := \xi(t)\int_\Omega u u_t\, dx,$$

$$\chi(t) := -\xi(t)\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau\, dx.$$

Remark 3.1. This functional for $\xi(t) \equiv 1$ was first introduced in [1, 2], and for $\xi(t) \not\equiv 1$ it was first introduced in [15].

Our main result reads as follow.

Theorem 3.2. *Assume that g satisfies (G1) and (G2), such that*

$$0 \le \max\{m, \gamma\} \le \frac{2}{n-2}, \quad n \ge 3.$$

Then, for each $t_0 > 0$, there exist strictly positive constants K and κ such that the solution of (1.1) *satisfies*

$$E(t) \le K e^{-\kappa \int_{t_0}^{t} \xi(s) ds}, \quad t \ge t_0. \tag{3.2}$$

To prove the above result, we establish a series of lemmas by combining the arguments of [1] and [15].

Lemma 3.3. *For ε_1 and ε_2 small, we have*

$$\alpha_1 F(t) \le \mathscr{E}(t) \le \alpha_2 F(t) \tag{3.3}$$

holds for two positive constants α_1 and α_2.

Proof. By using Young's inequality, Sobolev embedding theorem and (2.1), we easily deduce that

$$\begin{aligned}
F(t) \le \mathscr{E}(t) &+ \frac{\varepsilon_1}{2}\xi(t)\int_\Omega |u_t|^2\,dx + \frac{\varepsilon_1}{2}\xi(t)\int_\Omega |u|^2\,dx \\
&+ \frac{\varepsilon_2}{2}\xi(t)\int_\Omega |u_t|^2\,dx + \frac{\varepsilon_2}{2}\xi(t)\int_\Omega \Big(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\Big)^2 dx \\
&\le \mathscr{E}(t) + \frac{\varepsilon_1}{2}M\int_\Omega |u_t|^2\,dx + \frac{\varepsilon_1}{2}C_pM\int_\Omega |\nabla u|^2\,dx \\
&+ \frac{\varepsilon_2}{2}M\int_\Omega |u_t|^2\,dx + \frac{\varepsilon_2}{2}C_p(1-l)M(g\circ\nabla u)(t) \le \frac{1}{\alpha_1}\mathscr{E}(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
F(t) \ge \mathscr{E}(t) &- \frac{\varepsilon_1}{2}M\int_\Omega |u_t|^2\,dx - \frac{\varepsilon_1}{2}C_pM\int_\Omega |\nabla u|^2\,dx \\
&- \frac{\varepsilon_2}{2}M\int_\Omega |u_t|^2\,dx - \frac{\varepsilon_2}{2}C_p(1-l)M(g\circ\nabla u)(t) \\
&\ge \frac{1}{2}l\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\
&- \frac{\varepsilon_1+\varepsilon_2}{2}M\int_\Omega |u_t|^2\,dx - \frac{\varepsilon_1}{2}C_pM\int_\Omega |\nabla u|^2\,dx - \frac{\varepsilon_2}{2}C_p(1-l)M(g\circ\nabla u)(t) \\
&\ge \left(\frac{l}{2} - \frac{\varepsilon_1}{2}C_pM\right)\|\nabla u(t)\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon_1+\varepsilon_2}{2}M\right)\|u_t\|_2^2 \\
&+ \left(\frac{1}{2} - \frac{\varepsilon_2}{2}C_p(1-l)M\right)(g\circ\nabla u)(t) + \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \ge \frac{1}{\alpha_1}\mathscr{E}(t)
\end{aligned}$$

for ε_1 and ε_2 small enough. □

Lemma 3.4. *Under the assumptions of Theorem 3.2, the functional*

$$\Psi(t) := \xi(t) \int_\Omega u u_t \, dx$$

satisfies, along solutions of (1.1),

$$\begin{aligned}\Psi'(t) \le & \left[1 + \frac{2kC_p^2}{l}\right] \xi(t) \int_\Omega u_t^2 \, dx - \frac{l}{8}\xi(t) \int_\Omega |\nabla u|^2 \, dx \\ & - \frac{4-l}{4}\xi(t) \int_\Omega |u|^{\gamma+2} \, dx + \frac{1-l}{2l}\xi(t)(g \circ \nabla u)(t) \\ & + c(\delta)\xi(t) \int_\Omega a(x)|u_t|^{m+2} \, dx.\end{aligned} \tag{3.4}$$

Proof. Using equation (1.1), we easily see that

$$\begin{aligned}\Psi'(t) &= \xi(t) \int_\Omega (u u_{tt} + u_t^2) \, dx + \xi'(t) \int_\Omega u u_t \, dx \\ &= \xi(t) \int_\Omega u_t^2 \, dx - \xi(t) \int_\Omega |\nabla u|^2 \, dx \\ &\quad + \xi(t) \int_\Omega \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau) \, d\tau \, dx - \xi(t) \int_\Omega |u|^{\gamma+2} \, dx \\ &\quad - \xi(t) \int_\Omega a(x)|u_t|^m u_t u \, dx + \xi'(t) \int_\Omega u u_t \, dx.\end{aligned} \tag{3.5}$$

For the third term of the right-hand side of (3.5), we have

$$\begin{aligned}&\int_\Omega \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau) \, d\tau \, dx \\ &\le \frac{1}{2} \int_\Omega |\nabla u(t)|^2 \, dx + \frac{1}{2} \int_\Omega \Big(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) \, d\tau \Big)^2 \, dx.\end{aligned}$$

We then use Cauchy–Schwarz and Young's inequality, and the fact that $\int_0^t g(\tau)\, d\tau \le \int_0^\infty g(\tau)\, d\tau = 1 - l$ to obtain, for any $\eta > 0$ (see also [1])

$$\begin{aligned}&\int_\Omega \Big(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) \, d\tau \Big)^2 dx \\ &\le \int_\Omega \Big(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| \, d\tau \Big)^2 dx + \int_\Omega \Big(\int_0^t g(t-\tau)|\nabla u(t)| \, d\tau \Big)^2 dx \\ &\quad + 2 \int_\Omega \Big(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| \, d\tau \Big) \Big(\int_0^t g(t-\tau)|\nabla u(t)| \, d\tau \Big) dx \\ &\le \left(1 + \frac{1}{\eta}\right) \int_\Omega \int_0^t g(t-\tau) \, d\tau \int_0^t g(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2 \, d\tau \, dx \\ &\quad + (1+\eta) \int_\Omega |\nabla u(t)|^2 \Big(\int_0^t g(t-\tau) \, d\tau \Big)^2 dx\end{aligned}$$

$$\leq (1+\eta)(1-l)^2 \int_\Omega |\nabla u(t)|^2\,dx + \left(1+\frac{1}{\eta}\right)(1-l)\int_\Omega\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\,d\tau\,dx.$$

For the fifth term of the right-hand side of (3.5), we use Young's inequality and Lemma 2.5 to get

$$\begin{aligned}\int_\Omega a(x)|u_t|^m u_t u\,dx &\leq \delta\int_\Omega a(x)|u|^{m+2}\,dx + c(\delta)\int_\Omega a(x)|u_t|^{m+2}\,dx \\ &\leq c(\delta)\int_\Omega a(x)|u_t|^{m+2}\,dx + \delta C\{\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}\}.\end{aligned} \tag{3.6}$$

By combining (3.5)–(3.6) and using

$$\int_\Omega u u_t\,dx \leq \alpha C_p^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|u_t\|_2^2,$$

we have

$$\begin{aligned}\Psi'(t) \leq{}& \left[1+\frac{1}{4\alpha}\left|\frac{\xi'(t)}{\xi(t)}\right|\right]\xi(t)\int_\Omega u_t^2\,dx - \xi(t)\int_\Omega |u|^{\gamma+2}\,dx \\ &- \frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2\left|\frac{\xi'(t)}{\xi(t)}\right|\alpha C_p^2\right]\xi(t)\int_\Omega |\nabla u(t)|^2\,dx \\ &+ \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)\xi(t)(g\circ\nabla u)(t) + c(\delta)\xi(t)\int_\Omega a(x)|u_t|^{m+2}\,dx \\ &+ \delta C\xi(t)\{\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}\}.\end{aligned}$$

Since $\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k$, we get

$$\begin{aligned}\Psi'(t) \leq{}& \left[1+\frac{1}{4\alpha}k\right]\xi(t)\int_\Omega u_t^2\,dx - \xi(t)\int_\Omega |u|^{\gamma+2}\,dx \\ &- \frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2k\alpha C_p^2\right]\xi(t)\int_\Omega |\nabla u(t)|^2\,dx \\ &+ \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)\xi(t)(g\circ\nabla u)(t) + c(\delta)\xi(t)\int_\Omega a(x)|u_t|^{m+2}\,dx \\ &+ \delta C\xi(t)\{\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}\}.\end{aligned}$$

By choosing $\eta = l/(1-l)$, $\delta = l/4C$ and $\alpha = l/8kC_p^2$, (3.4) is established. □

Lemma 3.5. *Under the assumptions of Theorem 3.2, the functional*

$$\chi(t) := -\xi(t)\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx$$

satisfies, along solutions of (1.1),

$$
\begin{aligned}
\chi'(t) \le \delta &\left\{1 + 2(1-l)^2 + C_p\left(\frac{2\mathscr{E}(0)}{l}\right)^{\gamma}\right\}\xi(t)\|\nabla u\|_2^2 \\
&+\left[\left(\frac{1}{2\delta} + 2\delta + \frac{C_p(k+1)}{4\delta}\right)(1-l) + \frac{C(\delta)}{m+2}\|a\|_\infty\left(\frac{2\mathscr{E}(0)}{l}\right)^{\frac{m}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&+\frac{g(0)}{4\delta}C_p\xi(t)(-(g'\circ\nabla u)(t)) + \left[\delta(k+1) - \int_0^t g(s)\,ds\right]\xi(t)\int_\Omega u_t^2\,dx \\
&+\delta(1-l)\frac{m+1}{m+2}\xi(t)\int_\Omega a(x)|u_t|^{m+2}\,dx.
\end{aligned} \tag{3.7}
$$

Proof. Direct computations, using (1.1), yield

$$
\begin{aligned}
\chi'(t) &= -\xi(t)\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&\quad -\xi(t)\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&\quad -\xi(t)\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2\,dx - \xi'(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&= \xi(t)\int_\Omega \nabla u(t)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx \\
&\quad -\xi(t)\int_\Omega\left(\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx \\
&\quad +\xi(t)\int_\Omega a(x)u_t|u_t|^m\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&\quad +\xi(t)\int_\Omega |u|^\gamma u\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&\quad -\xi(t)\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx - \xi(t)\left(\int_0^t g(s)\,ds\right)\int_\Omega u_t^2\,dx \\
&\quad -\xi'(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx.
\end{aligned} \tag{3.8}
$$

Similarly to (3.5), we estimates the right-hand side terms of the above inequality (see also [1]). So for $\delta > 0$, we have: For the first term,

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx \\
&\le \delta\int_\Omega |\nabla u|^2\,dx + \frac{1-l}{4\delta}(g\circ\nabla u)(t).
\end{aligned} \tag{3.9}
$$

For the second term,

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx \\
&\le \delta\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)\,ds\right|^2 dx + \frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right|^2 dx
\end{aligned}
$$

$$\begin{aligned}
&\le \delta \int_\Omega \left(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|\, ds \right)^2 dx \\
&\quad + 2\delta(1-l)^2 \int_\Omega |\nabla u|^2\, dx + \frac{1}{4\delta}(1-l)(g \circ \nabla u)(t) \\
&\le \left(2\delta + \frac{1}{4\delta} \right)(1-l)(g \circ \nabla u)(t) + 2\delta(1-l)^2 \int_\Omega |\nabla u|^2\, dx.
\end{aligned}$$

For the third term, we use Hölder's inequality, Young's inequality and Lemma 2.3 to get

$$\begin{aligned}
&\int_\Omega a(x)u_t|u_t|^m \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \\
&\le \int_0^t g(t-\tau) \left(\int_\Omega a(x)|u_t|^{m+2}\, dx \right)^{\frac{m+1}{m+2}} \left(\int_\Omega a(x)|u(t)-u(\tau)|^{m+2}\, dx \right)^{\frac{1}{m+2}} d\tau \\
&\le \delta \frac{m+1}{m+2} \int_0^t g(t-\tau)\, d\tau \int_\Omega a(x)|u_t|^{m+2}\, dx \\
&\quad + \frac{C(\delta)}{m+2} \int_0^t g(t-\tau) \int_\Omega a(x)|u(t)-u(\tau)|^{m+2}\, dx\, d\tau \\
&\le \delta(1-l)\frac{m+1}{m+2} \int_\Omega a(x)|u_t|^{m+2}\, dx \\
&\quad + \frac{C(\delta)}{m+2}\|a\|_\infty \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^{m+2}\, d\tau \\
&\le \delta(1-l)\frac{m+1}{m+2} \int_\Omega a(x)|u_t|^{m+2}\, dx + \frac{C(\delta)}{m+2}\|a\|_\infty \left(\frac{2\mathscr{E}(0)}{l} \right)^{\frac{m}{2}} (g \circ \nabla u)(t).
\end{aligned}$$

For the fourth term,

$$\begin{aligned}
&\int_\Omega |u|^\gamma u \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \\
&\le \delta \int_\Omega |u|^{2(\gamma+1)}\, dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau \Big)^2 dx.
\end{aligned} \tag{3.10}$$

We use (2.1) and (2.3) to obtain

$$\int_\Omega |u|^{2(\gamma+1)}\, dx \le C_p \|\nabla u\|_2^{2(\gamma+1)} \le C_p \left(\frac{2\mathscr{E}(0)}{l} \right)^\gamma \|\nabla u\|_2^2. \tag{3.11}$$

By inserting (3.11) in (3.10), we get

$$\begin{aligned}
&\int_\Omega |u|^\gamma u \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau\, dx \\
&\le \delta C_p \left(\frac{2\mathscr{E}(0)}{l} \right)^\gamma \|\nabla u\|_2^2 + \frac{C_p(1-l)}{4\delta}(g \circ \nabla u)(t).
\end{aligned}$$

For the fifth term,

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \\
&\le \delta \int_\Omega |u_t|^2\,dx + \frac{g(0)}{4\delta} C_p \int_\Omega \int_0^t -g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx.
\end{aligned} \tag{3.12}
$$

For the last term,

$$
\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx \le \delta \int_\Omega u_t^2\,dx + \frac{C_p(1-l)}{4\delta}(g\circ\nabla u)(t). \tag{3.13}
$$

By combining (3.8)–(3.13), we get

$$
\begin{aligned}
\chi'(t) \le{} & \delta\left\{1+2(1-l)^2 + C_p\left(\frac{\mathscr{E}(0)}{l}\right)^{2\gamma}\right\}\xi(t)\|\nabla u\|_2^2 \\
&+\left[\left(\frac{1}{2\delta}+2\delta+\frac{C_p(k+1)}{4\delta}\right)(1-l)+\frac{C(\delta)}{m+2}\|a\|_\infty\left(\frac{2\mathscr{E}(0)}{l}\right)^{\frac{m}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&+\frac{g(0)}{4\delta}C_p\xi(t)(-(g'\circ\nabla u)(t)) + \left[\delta\left(\left|\frac{\xi'(t)}{\xi(t)}\right|+1\right)-\int_0^t g(s)\,ds\right]\xi(t)\int_\Omega u_t^2\,dx \\
&+\delta(1-l)\frac{m+1}{m+2}\xi(t)\int_\Omega a(x)|u_t|^{m+2}\,dx.
\end{aligned} \tag{3.14}
$$

Since $\left|\frac{\xi'(t)}{\xi(t)}\right| \le k$, the assertion of the lemma is established. □

Proof of Theorem 3.2. Since g is positive, continuous, and $g(0)>0$ then for any $t_0>0$ we have

$$
\int_0^t g(s)\,ds \ge \int_0^{t_0} g(s)\,ds = g_0 > 0 \quad \forall t \ge t_0. \tag{3.15}
$$

Using (3.1), (3.4), (3.7), and (3.15), we obtain

$$
\begin{aligned}
F'(t) \le{} & -\left[1-\varepsilon_1 c(\delta) - \varepsilon_2\delta(1-l)\frac{m+1}{m+2}\right]\xi(t)\int_\Omega a(x)|u_t|^{m+2}\,dx \\
&-\left[\varepsilon_2\{g_0-\delta(k+1)\}-\varepsilon_1\left(1+\frac{2kC_p^2}{l}\right)\right]\xi(t)\int_\Omega u_t^2\,dx \\
&-\left[\frac{\varepsilon_1 l}{8}-\varepsilon_2\delta\left(1+2(1-l)^2+C_p\left(\frac{2\mathscr{E}(0)}{l}\right)^{\gamma}\right)\right]\xi(t)\|\nabla u\|_2^2 \\
&-\varepsilon_1\frac{4-l}{4}\xi(t)\int_\Omega |u|^{\gamma+2}\,dx + \left[\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_pM\right](g'\circ\nabla u)(t) \\
&+\left\{\frac{\varepsilon_1(1-l)}{2l}+\varepsilon_2\left[\left(\frac{1}{2\delta}+2\delta+\frac{C_p(k+1)}{4\delta}\right)(1-l)\right.\right. \\
&\left.\left.+\frac{C(\delta)}{m+2}\|a\|_\infty\left(\frac{2\mathscr{E}(0)}{l}\right)^{\frac{m}{2}}\right]\right\}\xi(t)(g\circ\nabla u)(t).
\end{aligned}
$$

At this point we choose δ so small that

$$\frac{g_0-\delta(k+1)}{1+\frac{2kC_p^2}{l}}>\frac{1}{2}g_0,$$
$$\frac{8}{l}\delta\left\{1+2(1-l)^2+C_p\left(\frac{2\mathscr{E}(0)}{l}\right)^\gamma\right\}<\frac{1}{4}g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2<\varepsilon_1<\frac{1}{2}g_0\varepsilon_2 \tag{3.16}$$

will make

$$k_1:=\varepsilon_2\{g_0-\delta(k+1+\|a\|_\infty)\}-\varepsilon_1\left(1+\frac{2kC_p^2}{l}\right)>0$$
$$k_2:=\frac{\varepsilon_1 l}{8}-\varepsilon_2\delta\left\{1+2(1-l)^2+C_p\left(\frac{2\mathscr{E}(0)}{l}\right)^\gamma\right\}>0.$$

We then pick ε_1 and ε_2 so small that (3.3) and (3.16) remain valid and

$$k_3:=1-\varepsilon_1 c(\delta)-\varepsilon_2\delta(1-l)\frac{m+1}{m+2}>0,$$
$$k_4:=\left[\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_pM\right]-\left\{\frac{\varepsilon_1(1-l)}{2l}+\varepsilon_2\left[\left(\frac{1}{2\delta}+2\delta+\frac{C_p(k+1)}{4\delta}\right)(1-l)\right.\right.$$
$$\left.\left.+\frac{C(\delta)}{m+2}\|a\|_\infty\left(\frac{2\mathscr{E}(0)}{l}\right)^{\frac{m}{2}}\right]\right\}>0.$$

Hence

$$\left[\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_pM\right](g'\circ\nabla u)(t)+\left\{\frac{\varepsilon_1(1-l)}{2l}+\varepsilon_2\left[\left(\frac{1}{2\delta}+2\delta+\frac{C_p(k+1)}{4\delta}\right)(1-l)\right.\right.$$
$$\left.\left.+\frac{C(\delta)}{m+2}\|a\|_\infty\left(\frac{2\mathscr{E}(0)}{l}\right)^{\frac{m}{2}}\right]\right\}\xi(t)(g\circ\nabla u)(t)\le-k_4\xi(t)(g\circ\nabla u)(t),$$

since $\xi(t)$ is nonincreasing. Therefore, we arrive at

$$F'(t)\le-\beta\xi(t)\mathscr{E}(t)\quad\forall t\ge t_0.$$

This inequality and (3.3) yield

$$F'(t)\le-\beta\alpha_1\xi(t)F(t)\quad\forall t\ge t_0.$$

A simple integration leads to

$$F(t)\le F(t_0)e^{-\beta\alpha_1\int_{t_0}^t\xi(s)\,ds}\quad\forall t\ge t_0.$$

This inequality and (3.3) yields

$$\mathscr{E}(t)\le F(t_0)e^{-\beta\alpha_1\int_{t_0}^t\xi(s)\,ds}=Ke^{-\kappa\int_{t_0}^t\xi(s)\,ds}\quad\forall t\ge t_0, \tag{3.17}$$

which completes the proof. □

Similar as in [15], we have the following Remarks.

Remark 3.6. This result generalizes and improves the results of [1]. In particular, it allows some relaxation functions which satisfy $g'\le-ag^p, 1\le p<2$ instead of $p=1$.

Remark 3.7. Note that the exponential decay estimate, given in [1] is only a particular case of (3.17). More precisely, we can obtain exponential decay for $\xi(t) \equiv a$ and polynomial decay for $\xi(t) \equiv a(1+t)^{-1}$, where $a > 0$ is a constant.

Remark 3.8. Estimate (3.17) is also true for $t \in [0, t_0]$ by virtue of the continuity and boundedness of $E(t)$ and $\xi(t)$.

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