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THE TEICHMÜLLER PROBLEM FOR MEAN DISTORTION

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Abstract. The classical Teichmüller problem asks one to identify the deformation of a disk which holds the boundary fixed, moves the origin to a given point and which minimises the maximal conformal distortion. The minimiser exists and is quasiconformal—Teichmüller identified the extremal. Here we study the same problem, but instead of the maximal conformal distortion we consider the mean conformal distortion. In this setting many of the usual tools of quasiconformal mappings are lost. In surprising contrast to this classical case, we show that there cannot be a minimiser. However we give asymptotically sharp bounds for the minimal mean distortion and conjectured extrema. These exhibit quite different behaviour to that observed for the maximal conformal distortion and lend themselves to possibly modeling other phenomena in material science, for instance tearing. The key tools for the proofs of the main results are based on our recent joint work with Astala, Iwaniec and Onninen.

1. Introduction

In the last few years researchers have considered analogues of extremal problems in the theory of quasiconformal mappings, both in the plane and in space, with a view to minimising more general functionals than the maximal distortion for more general sorts of mappings. For instance in [5] the boundary value problem for self mappings of the disk with finite distortion was investigated. Fascinating connections between minimisers of mean distortion and harmonic functions were discovered and subsequently connections to the Nitsche conjecture (1962) were found [4, 6] when investigating mappings between annuli which minimise mean distortion. This novel phenomena (non-existence of expected minima outside a range of moduli) may persist in other situations, depending on curvature and topology. This circle of ideas has applications in theoretical materials science and critical phase phenomena, as distortion functionals are natural measures of change in a system and address fundamental questions relating microstructure and length scales.

In this note we study another classical problem—Teichmüller's problem, see [16, 1]. Thus we consider a domain Ω and points $z_1, z_2 \in \Omega$ and seek the homeomorphic mapping of finite distortion $f\colon \Omega \to \Omega$ with $f|\partial\Omega = \text{Identity}$ and $f(z_1) = z_2$ which minimises some integral average of distortion. The classical problem has found applications in the theory of homogeneity of domains as introduced in [10] and more recently in the homogeneity constants of surfaces, see [7]. Surprisingly we see a completely different phenomena to that observed classically.

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2. The problem

Let $\Omega \subset \mathbf{C}$ be a planar domain. A homeomorphism $f\colon \Omega \to \Omega$ of Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega)$ has *finite distortion* if the Jacobian determinant $J(z,f)$ is non-negative and there is a distortion function $\mathbf{K}(z,f)$, finite almost everywhere, such that

$$\|Df(z)\|^2 \le \mathbf{K}(z,f)\, J(z,f).$$

These mappings are generalisations of quasiconformal homeomorphism and have found considerable recent application in geometric function theory [3, 11, 12].

The function $\mathbf{K}$ has far better convexity properties than the more usual linear distortion K (defined by the inequality $|Df(z)|^2 \le K(z,f)\, J(z,f)$) and is more suited to minimisation problems. In terms of the complex derivatives f_z and $f_{\overline{z}}$ we have

$$\|Df\|^2 = |f_z|^2 + |f_{\overline{z}}|^2, \quad |Df|^2 = |f_z| + |f_{\overline{z}}|$$

and so

$$\mathbf{K}(z,f) = \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2 - |f_{\overline{z}}|^2}, \quad K(z,f) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|},$$

then $K = \mathbf{K} + \sqrt{\mathbf{K}^2 - 1}$ is increasing so that $\|K\|_\infty$ and $\|\mathbf{K}\|_\infty$ have the same minimisers.

We consider the following problem of minimising mean distortion:

Problem. For $0 \le r < 1$, let

$$\mathscr{M}_T(r) = \inf \left\{ \frac{1}{\pi} \iint_{\mathbf{D}} \mathbf{K}(z,f)\, dz \right\} \tag{1}$$

where the infimum is taken over all mapping $f\colon \mathbf{D} \to \mathbf{D}$ of finite distortion such that f has a homeomorphic extension to $\overline{\mathbf{D}}$ and

- $f|\partial\mathbf{D} \to \partial\mathbf{D}$ is the identity mapping,
- $f(0) = r$.

We have postponed a remark concerning the hypothesised existence of boundary values to Section 8.

2.1. The classical result. Before giving our main result we state the classical result for the maximal distortion $\|\mathbf{K}\|_\infty$. There are two reasons for this; from it we obtain trivial bounds on the mean distortion which will show that there must be different minimisers; and also we get to introduce some special functions we'll have to analyse later.

Let $\mathscr{K}$ be the complete elliptic integral of the first kind,

$$\mathscr{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \quad \mu(r) = \frac{\pi}{2}\,\frac{\mathscr{K}(\sqrt{1-r^2})}{\mathscr{K}(r)}. \tag{2}$$

The function μ gives us the Grötzch and Teichmüller ring moduli,

$$\gamma_2(s) = \frac{4}{\pi}\,\mu\left(\frac{s-1}{s+1}\right), \ s > 1, \quad \text{and} \quad \tau_2(s) = \frac{2}{\pi}\,\mu\left(\frac{\sqrt{1+s}-1}{\sqrt{1+s}+1}\right), \ s > 0,$$

respectively, [1, §8.56]. We want to note the estimate

$$\mu(x) = \log\left(\frac{4}{x} - x - \delta(x)\right) \tag{3}$$

where $x^3/4 < \delta(x) < 2x^3$ given in [13, p. 62].

When formulated in terms of $\mathbf{K}$ Teichmüller's theorem states

Theorem 1. *Let $f\colon \mathbf{D}\to\mathbf{D}$ be a quasiconformal map with $f(0)=r\geq 0$ and which extends to the identity on the unit circle. Then*

$$\|\mathbf{K}(z,f)\|_\infty \geq \frac{1}{2}\left(\coth^2\left(\frac{\mu(r)}{2}\right)+\tanh^2\left(\frac{\mu(r)}{2}\right)\right). \tag{4}$$

This estimate is sharp. If we let $d=\rho_{\mathbf{D}}(0,r)=\log\frac{1+r}{1-r}$ denote the hyperbolic distance between 0 and r, we have the asymptotics

- $\|\mathbf{K}(z,f)\|_\infty \approx \frac{8}{\pi^4}\,d^2$, *as* $d\to\infty$,
- $\|\mathbf{K}(z,f)\|_\infty \approx 1+\frac{d^2}{4}$, *as* $d\to 0$.

The quadratic decay near 0 is a feature of the distortion $\mathbf{K}$ and not that of the linear distortion (which behaves as $1+d/2$ for d near 0).

Our main result here is the following. In the body of the paper explicit formulae are given for all $r\in[0,1]$, but these are quite complicated—involving dilogarithm functions and the like.

Theorem 2. *The minimal mean distortion function $\mathscr{M}_T(r)$ defined at (1) for the Teichmüller problem has the following asymptotics*

- *as* $r\to 1$
$$\frac{2}{\pi^2}\log\frac{1+r}{1-r}+\frac{17\log 2}{4\pi^2}\leq \mathscr{M}_T(r)\leq \frac{2}{\pi^2}\log\frac{1+r}{1-r}+\frac{4}{3}-\frac{7+8\log 2}{2\pi^2}$$
up to an $O(1-r)$ *term,*
- *as* $r\to 0$
$$\mathscr{M}_T(r)\leq 1+\left(\frac{20-8\log(2)}{\pi^2}-\frac{7}{6}\right)r^2+O(r^4).$$

The minimum value $\mathscr{M}_T(r)$, for $r>0$, is never attained for a homeomorphism of finite distortion.

In terms of the hyperbolic distance $d=\rho_{\mathbf{D}}(0,r)$ we have

- $\mathscr{M}_T(r)\approx\frac{2}{\pi^2}\,d$, as $d\to\infty$,
- $\mathscr{M}_T(r)\leq 1+\left(\frac{5-2\log(2)}{\pi^2}-\frac{7}{24}\right)d^2$, as $d\to 0$.

The estimate as $d\to\infty$ is best possible but it is not clear what happens as $r\to 0$. Notice the constant here $\frac{5-2\log(2)}{\pi^2}-\frac{7}{24}=0.07447\ldots$ is less than $\frac{1}{4}$, the constant for the maximal distortion.

3. Non-existence of minima

The connection between minimisers of mean distortion and harmonic maps is the focus of our earlier work [5, 4]. We recall the following theorem of [5].

Theorem 3. *Let $g_o\colon \mathbf{S}\to\mathbf{S}$ be a homeomorphism. Consider the minimisation problem*

$$\inf_g\left\{\frac{1}{\pi}\iint_{\mathbf{D}}\mathbf{K}_g(z)\,|dz|^2\right\}$$

where the infimum is taken over all homeomorphisms $g\colon\mathbf{D}\to\mathbf{D}$ of finite distortion for which $g|\mathbf{S}=g_o$. Then there is a minimizer if and only if g_0 has finite energy, *this minimiser is unique and attains the minimum value* $\mathscr{E}(g_o)$, *and* $g^{-1}\colon \mathbf{D}\to\mathbf{D}$ *is harmonic.*

Here, the energy of the boundary values are defined by

$$\mathscr{E}(g_o) = -\frac{1}{2\pi^2}\iint_{\mathbf{S}\times\mathbf{S}} \log|g_o(\zeta)-g_o(\eta)|\,d\zeta\,d\overline{\eta}$$

In regard to our problem, this has the following consequence.

Theorem 4. *If* $r>0$, *the value* $\mathscr{M}_T(r)$ *is never attained by a mapping of finite distortion.*

Proof. Suppose that $f\colon \mathbf{D}\to\mathbf{D}$ is a homeomorpism of finite distortion and does minimise $\iint_{\mathbf{D}} \mathbf{K}(z,f)\,|dz|^2$ and has $f(0)=r>0$ and $f|\mathbf{S} = \text{Identity}$. Let $z\in\mathbf{D}\setminus\{0\}$, $\delta<\min\{|z|,1-|z|\}$ and the disk $D=\mathbf{D}(z,\delta)\subset\mathbf{D}\setminus\{0\}$. The image $f(D)$ is not a round disk but it is a Jordan domain conformally equivalent, say by φ, to D. Let $g\colon D\to D$ be the homeomorphic minimiser of mean distortion with harmonic inverse which has boundary values $\varphi\circ f\colon \mathbf{S}(z,\delta)\to\mathbf{S}(z,\delta)$. The map g exists since $\varphi\circ f\colon D\to D$ is a competitor with finite mean distortion—so the minimum is finite. Consider the new map

$$\hat{f} = \begin{cases} f(z), & z\in\mathbf{D}\setminus D,\\ \varphi^{-1}\circ g, & z\in D.\end{cases} \tag{5}$$

Of course $\mathbf{K}(z,\hat{f}) = \mathbf{K}(z,f)$ on $\mathbf{D}\setminus D$ while

$$\iint_D \mathbf{K}(z,f)\,|dz|^2 \geq \iint_D \mathbf{K}(z,\hat{f})\,|dz|^2 = \iint_D \mathbf{K}(z,g)\,|dz|^2 \tag{6}$$

by virtue of the fact that g is a minimiser. The map $\hat{f}\colon\mathbf{D}\to\mathbf{D}$ is clearly a homeomorphism as the boundary values match up by construction. However there is a slight technical problem in showing that $\hat{f}$ is a mapping of finite distortion, but this is easily dealt with in exactly the same was as in [11, p. 124]. Next notice that $\hat{f}^{-1} = g^{-1}\circ\varphi^{-1}$ is harmonic since g^{-1} is. Recall the minimiser g is unique as well. Therefore, unless $f=\hat{f}$ on D we have found a mapping with strictly smaller mean distortion. This is not the case as f is chosen to be a minimiser. Therefore this process has not changed f and so f^{-1} is harmonic near z. As $z\in\mathbf{D}\setminus\{0\}$ is arbitrary we find that $f^{-1}\colon \mathbf{D}\setminus\{r\}\to\mathbf{D}\setminus\{0\}$ is harmonic. We can't carry this process out at $z=0$ since this would change the problem—we could no longer guarantee $f(0)=r$. However the following Lemma 1 of [8, p. 403] shows that $f^{-1}\colon\mathbf{D}\to\mathbf{D}$ is harmonic. But the boundary values of f^{-1} are the identity and so by uniqueness for the Poisson problem $f^{-1} = \text{Identity}$ which is a contradiction. Thus there is no minimiser f. □

Lemma 1. *If* h *is harmonic in the punctured disc* $\mathbf{D}\setminus\{0\}$ *and satisfies*

$$\liminf_{r\downarrow 0} \frac{\int_0^{2\pi}|h(re^{i\theta})|\,d\theta}{-\log r} = 0,$$

then h *has a harmonic extension to* $\mathbf{D}$.

Lemma 1 shows any minimising sequence must develop quite a strong singularity near the origin. Theorem 4 begs the question as to what the minimum value might be. One could even expect that the minimum value is always 1, given by a sequence degenerating to the identity whose (large) distortion is supported on a set of small measure. We show this is not the case by giving an asymptotically sharp lower bound. However first we need to have at hand some conformal mappings.

4. Conformal mapping

Let $\mathbf{H}$ denote the upper half plane and $\mathbf{D}_+$ the upper half-disk,

$$\mathbf{H} = \{z \in \mathbf{C} : \Im(z) > 0\}, \quad \mathbf{D}_+ = \{z \in \mathbf{D} : \Im(z) > 0\}.$$

From Nehari [15, §8 p. 209] we see that the conformal mapping $\varphi\colon \mathbf{H} \to \mathbf{D}_+$ has the form

$$\varphi(z) = \frac{-i\sqrt{z+1} + \sqrt{z-1}}{-i\sqrt{z+1} - \sqrt{z-1}} \tag{7}$$

and we settle on the formula

$$\varphi(z) = \frac{1}{z}\left(1 + i\sqrt{z^2-1}\right) = 1 - 2\left(1 + i\sqrt{\frac{z+1}{z-1}}\right)^{-1} \tag{8}$$

though some care must be taken with branches here and the formula at (7) is more computationally reliable. The inverse of this map is the conformal map $\mathbf{D}_+ \to \mathbf{H}$ is given by

$$\frac{2z}{1+z^2}. \tag{9}$$

Next, the map we will need the conformal map $\Phi\colon \mathbf{D}_+ \to \mathbf{D}$,

$$\Phi(w) = \frac{\left(\frac{1+w}{1-w}\right)^2 - i}{\left(\frac{1+w}{1-w}\right)^2 + i}, \tag{10}$$

and note that $\Phi(\pm 1) = \pm 1$, $\Phi(0) = -i$. Note that $(1+e^{it})/(1-e^{it}) = i\cot(t/2)$ and so on the boundary of $\mathbf{D}_+$ we have the map described by

$$\Phi(e^{it}) = -i + \frac{2}{\cos(t) - i}, \quad 0 \le t \le \pi,$$
$$\Phi(t) = -i + \frac{4t}{1 - 2it + t^2}, \quad -1 \le t \le 1.$$

5. Lower bounds for the mean distortion

Consider the two continua $E = [-1, 0]$ and $F = \{e^{i\theta} : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$. Let Γ be the family of all curves in the disk joining the two continua E and F in $\mathbf{D}$,

$$\Gamma = \Gamma(E, F; \mathbf{D}) \tag{11}$$

This is half the planar Mori ring [1, p. 172]. Since $\operatorname{dist}(E, F) \ge 1$, the function $\rho(z) \equiv 1$ is an admissible Borel density for the curve family Γ. Let $f\colon \mathbf{D} \to \mathbf{D}$ be a mapping of finite distortion with $f(0) = r$ and $f|\partial\mathbf{D} = \text{Identity}$. Then $f(F) = F$ while $f(E)$ is a continua joining -1 and r. The curve family $f\Gamma$ consists of all curves joining these two continua. The linear fractional transformation $(z-r)/(1-rz)$ conformally maps the curve family E to a curve joining -1 to 0 and F to the arc $\{e^{i\theta} :$

$|\theta| \le \arctan\left(-\frac{1-r^2}{2r}\right)\}$ whose endpoints are $\frac{i-r}{1-ir}$ and $-\frac{i+r}{1+ir}$. The symmetrisation principle (or the extremal property of the Mori ring [1, Theorem 8.54]) tells us that

$$\operatorname{Mod}(f\Gamma) \ge \operatorname{Mod}(\Gamma') \tag{12}$$

where Γ' consists of all curves joining the continuum $[-1,0]$ to F. We want to compute $\operatorname{Mod}(\Gamma')$. First note that $\sqrt{\cdot}$ maps $\mathbf{D}\setminus[-1,0]$ conformally onto $-i\mathbf{D}_+$ and F to the arc $\left\{e^{i\theta} : |\theta| \le \frac{1}{2}\arctan\left(-\frac{1-r^2}{2r}\right)\right\}$. It follows from the formula at (9) that the map

$$\frac{2iz}{1-z^2} \tag{13}$$

takes this half disk to the upper half-plane. The interval $[-i,i]$ is mapped to $[-1,1]$ while the circular arc, connecting the point $\sqrt{\frac{i-r}{1-ir}}$ and its complex conjugate is mapped to the complement (in $\mathbf{R}$) of the interval $[-\alpha,\alpha]$, where

$$\alpha = \sqrt{2}\,\frac{\sqrt{1+r^2}}{1+r}. \tag{14}$$

Then the map $z \mapsto (\alpha-1)(z-1)/(2z+2\alpha)$ is conformal, preserves $\mathbf{R}$, maps the interval $[-1,1]$ to $[-1,0]$ and the complement of $[-\alpha,\alpha]$ to the ray

$$[(\alpha-1)^2/4\alpha,\infty] = \left[\frac{(1+r-\sqrt{2}\sqrt{1+r^2})^2}{4\sqrt{2}(1+r)\sqrt{1+r^2}},\infty\right].$$

By conformality, the modulus $\operatorname{Mod}(\Gamma')$ is equal to the modulus of all curves joining these two intervals in the upper half-plane. By subadditivity this is exactly half the modulus of all curves joining these continua in $\mathbf{C}$ and this latter modulus is precisely the modulus of the Teichmüller ring, $\tau_2(t)$ at the point

$$t_r = \frac{(1+r-\sqrt{2}\sqrt{1+r^2})^2}{4\sqrt{2}(1+r)\sqrt{1+r^2}}. \tag{15}$$

Lemma 2. *Let E' be a continuum in the disk joining -1 to $r>0$ and let F' be the semicircle with endpoint $\pm i$. If Γ' consists of all curves joining E' to F' in $\mathbf{D}$, then*

$$\operatorname{Mod}(\Gamma') \ge \frac{1}{2}\,\tau_2(t_r) = \frac{\pi}{4}\,\mu\left(\frac{\sqrt{1+t_r}-1}{\sqrt{1+t_r}+1}\right) \tag{16}$$

where t_r is defined at (15).

5.1. Q-homeomorphisms. We now need to recall some modulus estimates for mappings of finite distortion. We will take these from [14] where the theory was developed using curve families instead of the analytic approach of [3, 2, 11].

Let Ω be a domain in $\mathbf{C}$. A mapping $f\colon\Omega\to\mathbf{C}$ is called a Q-homeomorphism if for every family of paths Γ in Ω we have

$$\operatorname{Mod}(f\Gamma) \le \iint_\Omega Q(z)\rho^2(z)\,|dz|^2 \tag{17}$$

for all admissible functions ρ for the curve family Γ. Recall that ρ is admissible means $\int_\gamma \rho(s)\,ds \ge 1$ for all curves $\gamma\in\Gamma$. In [14, Theorem 2.19] it is shown that if

$f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $K(z,f) \in L^1_{\mathrm{loc}}(\Omega)$, then (17) holds with $Q(z) = K(z,f)$, the linear distortion.

5.2. Lower bounds. In the situation as described in the previous subsection we note that for the curve family Γ as defined at (11) the function $\rho(z) \equiv 1$ is admissible. Thus we may put together the estimates of (12), (16) and (17) to get

$$\frac{1}{2}\,\tau_2(t_r) \le \mathrm{Mod}(f\Gamma) \le \iint_{\mathbf{D}} K(z,f)\,|dz|^2 \tag{18}$$

We now apply Jensen's inequality with the convex function $t \mapsto \frac{1}{2}(t+t^{-1})$, now requiring the left hand side to be at least 1, to give us a lower bound.

Corollary 1. *Let $f\colon \mathbf{D} \to \mathbf{D}$ be a $W^{1,2}_{\mathrm{loc}}(\mathbf{D})$ homeomorphism of finite distortion which extends continuously to the identity on $\mathbf{S}$ and has $f(0) = r > 0$. Then*

$$\frac{\tau_2(t_r)}{4\pi} + \frac{\pi}{\tau_2(t_r)} \le \frac{1}{\pi}\iint_{\mathbf{D}} \mathbf{K}(z,f)\,|dz|^2. \tag{19}$$

The lower bound here is really only any good if r is not near 0. The difficulty being our choice of $\rho \equiv 1$ as the admissible function is suboptimal. Thus we want to discuss here the asymptotic behaviour as $r \to 1$ (and so $t_r \to 0$). It is the term $\frac{1}{4\pi}\tau_2(t_r)$ that we must study.

In terms of μ, we note the oscillating series expansion of $\frac{\sqrt{1+t_r}-1}{\sqrt{1+t_r}+1}$ and a slight refinement of (3) leads to the estimate

$$\begin{aligned}\frac{\tau_2(t_r)}{4\pi} &= \frac{1}{2\pi^2}\,\mu\left(\frac{\sqrt{1+t_r}-1}{\sqrt{1+t_r}+1}\right) \approx \frac{1}{2\pi^2}\,\mu\left(\frac{\sqrt{1+\frac{1}{256}(1-r)^4}-1}{\sqrt{1+\frac{1}{256}(1-r)^4}+1}\right)\\ &\approx \frac{2}{\pi^2}\log\frac{1+r}{1-r} + \frac{17\log 2}{4\pi^2} = \frac{2}{\pi^2}\rho_{\mathbf{D}}(0,r) + \frac{17\log 2}{4\pi^2}\end{aligned}$$

where we have also used the expansion $t_r = \frac{1}{256}(1-r)^4 + \frac{1}{128}(1-r)^5 + \cdots$. Thus,

$$\frac{2}{\pi^2}\rho_{\mathbf{D}}(0,r) + \frac{17\log 2}{4\pi^2} + O(1-r) \le \frac{1}{\pi}\iint_{\mathbf{D}} \mathbf{K}(z,f)\,|dz|^2 \tag{20}$$

We compare this linear bound as $d = \rho_{\mathbf{D}}(0,r) \to \infty$ as $r \to 1$ with the implicit quadratic bound (since the L^1 mean is smaller than the L^∞ bound) of Theorem 1 of $\frac{8}{\pi^4}d^2$. We will see this linear bound is best by example.

6. Minimisers

In this section we give a formula for the mean distortion of a natural family of candidate examples which have asymptotically correct behaviour—that is it matches the lower bounds established in the previous section. Although at this point we know there is no minimiser, it is reasonable to believe that the symmetry of the problem across the real axis is reflected in minimising sequences. We also note that on a convex region in $\mathbf{D}$ we decrease the mean distortion by replacing the map with an inverse harmonic function.

For these reasons we consider a mappings $g\colon \mathbf{D}_+ \to \mathbf{D}_+$ which have harmonic inverses, are the identity on the semi-circle and are some as yet undefined self homeomorphism h of the interval $[-1,1]$. We get candidates for minimising sequences by by by reflecting g across the real line. We will seek to minimise the mean distortion by good choices of h. Post composing g with a conformal map $\mathbf{D}_+ \to \mathbf{D}$ means we can use our explicit formulas for the mean distortion.

Indeed, if there is a minimising sequence $\{f_i\}$ for the problem (1) which maps the interval $[-1,1]$ to itself, then replacing f_i by the inverses of the harmonic extension of f_i^{-1} on the upper and lower disks and using the compactness properties, the harmonic version of the Hurwitz theorem and other results found in [9] one can see that the inverses of modified minimising sequence will converge to a mapping of the disk to itself which is a homeomorphism away from a subinterval of $[-1,1]$ (the limit will be monotone). Thus we expect that a minimising sequence "converges" to a map f which is a homeomorphism (with harmonic inverse) away from the origin and $f(\mathbf{D}\setminus\{0\}) = \mathbf{D}\setminus[a,b]$. Presumably $[a,b]=[0,r]$.

6.1. The minimisation problem. Suppose we have a homeomorphism $f\colon \partial\mathbf{D}_+ \to \partial\mathbf{D}$ of the form

$$f(z) = \begin{cases} \Phi(z), & |z|=1, \\ \Phi(h(z)), & -1\le z\le 1, \end{cases} \tag{21}$$

where $\Phi\colon \mathbf{D}_+ \to \mathbf{D}$ is defined at (10) and $h\colon [-1,1]\to[-1,1]$ is an increasing homeomorphism, $h(\pm 1)=\pm 1$. We want to compute the conformal energy of the map f as a function of h and choose h to minimise this conformal energy. Γ will denote the obvious parameterisation of the boundary of $\mathbf{D}_+$ into the upper circle $C_+ = \{e^{it} : 0\le t\le \pi\}$ and the interval $[-1,1]$.

Then

$$\begin{aligned}
\mathscr{E}_{\mathbf{D}_+}(f) &= -\frac{1}{\pi^2}\iint_{\Gamma\times\Gamma} \log|f(\zeta)-f(\eta)|\, d\zeta\, d\overline{\eta} \\
&= -\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log|\Phi(e^{it})-\Phi(e^{is})|\, e^{it}\, dt\; e^{-is}\, ds \\
&\quad -\frac{1}{\pi^2}\int_0^\pi\int_{-1}^1 \log|\Phi(e^{is})-\Phi(h(t))|\, dt\, (-ie^{-is})\, ds \\
&\quad -\frac{1}{\pi^2}\int_{-1}^1\int_0^\pi \log|\Phi(e^{it})-\Phi(h(s))|\, ie^{it}\, dt\, ds \\
&\quad -\frac{1}{\pi^2}\int_{-1}^1\int_{-1}^1 \log|\Phi(h(t))-\Phi(h(s))|\, dt\, ds \\
&= -\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log\left|\frac{2}{\cos(t)-i}-\frac{2}{\cos(s)-i}\right| e^{i(t-s)}\, dt\, ds \\
&\quad +\frac{2}{\pi^2}\int_{-1}^1\int_0^\pi \log|\Phi(e^{it})-\Phi(h(s))|\, \sin(t)\, dt\, ds \\
&\quad -\frac{1}{\pi^2}\int_{-1}^1\int_{-1}^1 \log|\Phi(h(t))-\Phi(h(s))|\, dt\, ds.
\end{aligned}$$

We compute the constant term (that is independent of h) to be

$$-\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log\left|\frac{2}{\cos(t)-i}-\frac{2}{\cos(s)-i}\right| e^{i(t-s)} dt\, ds = \frac{4\pi+\pi^2-4-8\log(2)}{2\pi^2}.$$

We substitute the values of Φ (see (10)) to obtain the two integrals

$$\mathscr{I}_1 = \frac{2}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log\left|\frac{2}{\cos(t)-i} - \frac{4h(s)}{1-2ih(s)+h^2(s)}\right| \sin(t)\, dt\, ds,$$
$$\mathscr{I}_2 = -\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \log\left|\frac{4h(t)}{1-2ih(t)+h^2(t)} - \frac{4h(s)}{1-2ih(s)+h^2(s)}\right| dt\, ds.$$

Notice that if $h(s) = s$, we have

$$\mathscr{I}_1 = \frac{1}{\pi^2}\left(4 - 4\sqrt{2}\pi + \pi^2 + 8\log(2)\right), \tag{22}$$
$$\mathscr{I}_2 = \frac{-1}{2\pi^2}\left(4 + \pi(4 - 8\sqrt{2} + \pi) + 8\log(2)\right). \tag{23}$$

Summing these three terms gives us

Corollary 2. *With $h(s) = s$ we have $\mathscr{E}_{\mathbf{D}_+}(f) = 1$.*

Now we want to consider the integral

$$\begin{aligned}
\mathscr{I}_1 &= \frac{1}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log\left|\frac{2+2h^2(s)-4h(s)\cos(t)}{(\cos(t)-i)(1-2ih(s)+h^2(s))}\right|^2 \sin(t)\, dt\, ds \\
&= \frac{1}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log\left|2+2h^2(s)-4h(s)\cos(t)\right|^2 \sin(t)\, dt\, ds \\
&\quad - \frac{1}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log[1+\cos^2(t)]\, \sin(t)\, dt\, ds \\
&\quad - \frac{1}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log(1+6h^2(s)+h^4(s))\, \sin(t)\, dt\, ds \\
&= \frac{1}{\pi^2}\int_{-1}^{1}\int_{0}^{\pi} \log(1+h^2(s)-2h(s)\cos(t))^2\, \sin(t)\, dt\, ds \\
&\quad - \frac{1}{\pi^2}\int_{-1}^{1} \log(1+6h^2(s)+h^4(s))\, ds \int_{0}^{\pi} \sin(t)\, dt \\
&\quad + \frac{8-\pi+6\log(2)}{\pi^2}.
\end{aligned}$$

We can integrate out the t variable in the first integral here so that we are left with

$$\begin{aligned}
\mathscr{I}_1 &= -\frac{2}{\pi^2}\int_{-1}^{1} \log(1+6h^2(s)+h^4(s))\, ds + \frac{4\log(2)-2\pi}{\pi^2} \\
&\quad + \frac{2}{\pi^2}\int_{-1}^{1} \left(h(s) + \frac{1}{h(s)}\right) \log\frac{1+h(s)}{1-h(s)} + 2\log(1-h^2(s))\, ds.
\end{aligned} \tag{24}$$

Next, the other integral is

$$
\begin{aligned}
\mathscr{I}_2 &= -\frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log\left|\frac{4h(t)(1+h^2(s))-4h(s)(1+h^2(t))}{(1-2ih(t)+h^2(t))(1-2ih(s)+h^2(s))}\right|^2 dt\,ds\\
&= -\frac{8\log 2}{\pi^2}-\frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log|(h(t)-h(s))(1-h(t)h(s))|^2\,dt\,ds\\
&\quad+\frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log(1+6h^2(t)+h^4(t))(1+6h^2(s)+h^4(s))\,dt\,ds\\
&= -\frac{8\log 2}{\pi^2}-\frac{1}{2\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log|(h(t)-h(s))(1-h(t)h(s))|^2\,dt\,ds\\
&\quad+\frac{2}{\pi^2}\int_{-1}^{1}\log(1+6h^2(t)+h^4(t))\,dt
\end{aligned}
$$

Adding these together gives us a formula for the conformal energy of f,

$$
\begin{aligned}
\mathscr{E}_{\mathbf{D}_+}(f) = &-\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log|h(t)-h(s)|\,dt\,ds\\
&-\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\log|1-h(t)h(s)|\,dt\,ds\\
&+\frac{2}{\pi^2}\int_{-1}^{1}\left(h(s)+\frac{1}{h(s)}\right)\log\frac{1+h(s)}{1-h(s)}+2\log(1-h^2(s))\,ds\;+\;A
\end{aligned}
$$

where A is an absolute constant. From Corollary 2 if we put $h(t)=t$, then $\mathscr{E}_{\mathbf{D}_+}(f)=1$ and therefore with $\mathscr{Q}=[-1,1]\times[-1,1]$ we have

Theorem 5.

$$
\begin{aligned}
\mathscr{E}_{\mathbf{D}_+}(f) = &-\frac{1}{\pi^2}\iint_{\mathscr{Q}}\log\left|\frac{h(t)-h(s)}{t-s}\right|dt\,ds\;-\;\frac{1}{\pi^2}\iint_{\mathscr{Q}}\log\left|\frac{1-h(t)h(s)}{1-st}\right|dt\,ds\\
&+\frac{2}{\pi^2}\int_{-1}^{1}\left(h(s)+\frac{1}{h(s)}\right)\log\frac{1+h(s)}{1-h(s)}+2\log(1-h^2(s))\,ds\;+\;C
\end{aligned}
$$

where

$$
C=\frac{12-16\log(2)}{\pi^2}.
$$

We write the energy in this form as there is obvious simplification of the singularity in the first two integrals for linear functions. It is these terms that determine the asymptotic behaviour for large r. We seek the approriate choice of h to minimise this.

6.2. A choice of h. To obtain upper bounds for the minimal mean distortion we need to find examples of homeomorphisms $h\colon[-1,1]\to[-1,1]$ with $h(0)=r>0$ and compute the value of $\mathscr{E}_{\mathbf{D}_+}(f)$ in Theorem 5. The first natural example to try is

$$
h(t)=\begin{cases}(1+r)t+r, & -1\le t\le 0,\\ (1-r)t+r, & 0\le t\le 1.\end{cases} \tag{25}
$$

With this map we compute

$$
\begin{aligned}
(26)\qquad & -\frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds \\
&= -\frac{1}{\pi^2}\int_{-1}^{0}\int_{-1}^{0} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds - \frac{1}{\pi^2}\int_{0}^{1}\int_{0}^{1} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds \\
&\quad - \frac{2}{\pi^2}\int_{-1}^{0}\int_{0}^{1} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds \\
&= -\frac{1}{\pi^2}\int_{-1}^{0}\int_{-1}^{0} \log 1+r\,dt\,ds - \frac{1}{\pi^2}\int_{0}^{1}\int_{0}^{1} \log 1-r\,dt\,ds \\
&\quad - \frac{2}{\pi^2}\int_{-1}^{0}\int_{0}^{1} \log\left|\frac{(1+r)t-(1-r)s}{t-s}\right| ds\,dt \\
&= -\frac{1}{\pi^2}\log(1-r^2) \\
&\quad + \frac{1}{\pi^2(1-r^2)}\left(-4r^2\log(2) + (1+r)^2\log(1+r) + (1-r)^2\log(1-r)\right).
\end{aligned}
$$

Thus, near $r=1$ we have

$$
(27)\qquad -\frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds \approx \frac{-1}{\pi^2} + \frac{1}{\pi^2}\log\frac{1+r}{1-r} + O(1-r),
$$

and near $r=0$ we see that (26) behaves like as

$$
(28)\qquad -\frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds \approx \frac{4(1-\log(2))}{\pi^2}\,r^2 + O(r^4).
$$

Next

$$
\begin{aligned}
(29)\qquad & -\frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{1-h(t)h(s)}{1-st}\right| dt\,ds \\
(30)\qquad &= -\frac{1}{\pi^2}\int_{0}^{1}\int_{0}^{1} \log\left|\frac{1-(r+s-rs)(r+t-rt)}{1-st}\right| dt\,ds \\
(31)\qquad &\quad - \frac{1}{\pi^2}\int_{-1}^{0}\int_{-1}^{0} \log\left|\frac{1-(r+s+rs)(r+t+rt)}{1-st}\right| dt\,ds \\
(32)\qquad &\quad - \frac{2}{\pi^2}\int_{-1}^{0}\int_{0}^{1} \log\left|\frac{1-(r+t-rt)(r+s+rs)}{1-st}\right| dt\,ds
\end{aligned}
$$

First, (30) is equal to

$$
\begin{aligned}
&\frac{1}{6} + \frac{1}{\pi^2}\log\frac{1+r}{1-r} + \frac{1}{\pi^2(1-r)^2} \\
&\quad \cdot \left(2r(1-r)\log(1+r) + \log(r^2)\log(1+r) - 2Li_2(1-r) + Li_2(1-r^2)\right)
\end{aligned}
$$

and has the expansions

$$
\begin{aligned}
r \text{ near } 1 &\approx \frac{1}{\pi^2}\log\frac{1+r}{1-r} + \frac{1}{6\pi^2}\left(\pi^2 - 3(1+6\log(2))\right) + O(r-1), \\
r \text{ near } 0 &\approx \left(\frac{4}{\pi^2} - \frac{1}{3}\right) r + \left(\frac{13}{2\pi^2} - \frac{1}{2}\right) r^2 - \left(\frac{2}{3} - \frac{62}{9\pi^2}\right) r^3 + O(r^4).
\end{aligned}
$$

Next, (31) is equal to

$$\frac{1}{6} - \frac{1}{\pi^2(1+r)^2}\big((r^2-1)\log(1-r) - 2\log(r)\log(1-r) + (1+r)^2\log(1+r) + 2Li_2(1+r) - Li_2(1-r^2)\big)$$

and has the expansions

$$r \text{ near } 1 \approx \frac{1}{24} - \frac{\log(2)}{\pi^2} + O(r-1),$$
$$r \text{ near } 0 \approx \left(\frac{1}{3} - \frac{4}{\pi^2}\right) r + \left(\frac{13}{2\pi^2} - \frac{1}{2}\right) r^2 + \left(\frac{2}{3} - \frac{62}{9\pi^2}\right) r^3 + O(r^4).$$

Next, (32) is equal to

$$\frac{1}{6} + \frac{4\log(2)}{\pi^2} + \frac{2}{\pi^2(1-r^2)}\bigg(-\frac{\pi^2}{4} - \log(4) + \log(1-r)(r(r-1) - \log(r)) + (r + r^2 - \log(r))\log(1+r) + Li_2(1-r) + Li_2(1+r) - Li_2(1-r^2)\bigg)$$

and has the expansions

$$r \text{ near } 1 \approx \frac{1}{6} + \frac{2\log(2) - 2}{\pi^2} + O(r-1),$$
$$r \text{ near } 0 \approx -\left(\frac{1}{6} + \frac{3 - 4\log(2)}{\pi^2}\right) r^2 + O(r^4).$$

The third integral can be computed as

$$\frac{2}{\pi^2}\int_{-1}^{0}\left(r+s+rs+\frac{1}{r+s+rs}\right)\log\frac{1+(1+r)s+r}{1-(1+r)s-r} + 2\log(1-(r+s+rs)^2)\, ds$$
$$= \frac{1}{2\pi^2(1+r)}\bigg(\pi^2 - 12(1+r) + 16\log(2) + 2(3 + r^2 + 2\log(r))\log\frac{1+r}{1-r} + 8r\log(1-r^2) - 4Li_2(1-r) + 4Li_2(1+r)\bigg),$$

$$\frac{2}{\pi^2}\int_{0}^{1}\left(r+s-rs+\frac{1}{r+s-rs}\right)\log\frac{1+(1-r)s+r}{1-(1-r)s-r} + 2\log(1-(r+s-rs)^2)\, ds$$
$$= \frac{1}{2\pi^2(1-r)}\big(\pi^2 + 4(3r + 4\log(2) - 3) + 2\log(1-r)(3 - 4r + r^2 + 2\log(r)) - 2(1 + 4r + r^2) + 2\log(r))\log(1+r) + 4Li_2(1-r) - 4Li_2(1+r)\big).$$

Together these terms behave as

$$\frac{16\log(2) - 6}{\pi^2} + \frac{1}{2} + O(1-r) \tag{33}$$

near $r = 1$ and

$$1 + \frac{16\log(2) - 12}{\pi^2} + \left(1 + \frac{4(4\log(2) - 5)}{\pi^2}\right) r^2 + O(r^4) \tag{34}$$

near $r = 0$.

Now adding together all the asymptotic expansions near $r = 1$ gives

$$\begin{aligned}\mathscr{E}_{\mathbf{D}_+}(f) &\approx \frac{2}{\pi^2}\log\frac{1+r}{1-r} + \left(\frac{7}{8} + \frac{5-4\log(2)}{2\pi^2}\right) + O(1-r)\\ &= \frac{2}{\pi^2}\log\frac{1+r}{1-r} + 0.98742\ldots + O(1-r)\end{aligned}$$

and near $r = 0$ we have

$$\begin{aligned}\mathscr{E}_{\mathbf{D}_+}(f) &\approx 1 + \left(\frac{8\log(2)}{\pi^2} - \frac{1}{6}\right) r^2 + O(r^4)\\ &= 1 + 0.395177\ldots r^2 + O(r^4).\end{aligned}$$

At this point we observe that the asymptotic behaviours in terms of the hyperbolic distance $d = \rho_{\mathbf{D}}(0, r)$ are

$$\begin{aligned}\frac{2d}{\pi^2} + 0.987842 &\quad \text{as } d \to \infty,\\ 1 + 0.09879\ldots d^2 + O(d^4) &\quad \text{as } d \to 0.\end{aligned}$$

7. Minimising sequences?

The lower semicontinuity of the integral of the distortion function (in two dimensions) shows that no sequence of minimisers can have uniformly bounded distortion (i.e. quasiconformal—if sufficiently regular). Thus in order to describe a minimising sequence we must look to some degenerating behaviour. Some computational experiments motivate the following examples. Given $\delta > 0$ we set

$$h_\delta(t) = \begin{cases} t, & -1 \le t \le -\delta,\\ (\frac{r}{\delta}+1)t + r, & -\delta \le t \le 0,\\ (1-r)t + r, & 0 \le t \le 1.\end{cases} \tag{35}$$

We are interested here in the behviour as $\delta \to 0$ and $r \to 1$ of the function

$$\begin{aligned}\mathscr{E}_{\mathbf{D}_+}(f) = &-\frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{h(t)-h(s)}{t-s}\right| dt\,ds - \frac{1}{\pi^2}\iint_{\mathscr{Q}} \log\left|\frac{1-h(t)h(s)}{1-st}\right| dt\,ds\\ &+ \frac{2}{\pi^2}\int_{-1}^{1}\left(h(s) + \frac{1}{h(s)}\right)\log\frac{1+h(s)}{1-h(s)} + 2\log(1-h^2(s))\,ds \,+\, C\end{aligned}$$

where

$$C = \frac{12-16\log(2)}{\pi^2}.$$

Following the arguments quite similar to those above, and letting $\delta \to 0$ we obtain the bounds stated in Theorem 2 with the constant term $0.805714\ldots$ as $r \to 1$. This is the smallest constant we were able to achieve. However this sequence cannot be a minimising sequence (at least for r small) for the behaviour as $r \to 0$ is of the incorrect order, $1 + O(r^{2-\epsilon})$ and not $1 + O(r^2)$.

8. A note concerning boundary values

It is not true that every homeomorphism $g\colon \mathbf{D} \to \mathbf{D}$ of finite distortion with

$$\iint_{\mathbf{D}} \mathbf{K}(z,g)\,dz < \infty$$

has a homeomorphic (or even continuous) extension to $\overline{\mathbf{D}}$. We give a construction based on the following interesting example. Let

$$A_1 = \{z : 0 < |z| < 1\} \quad \text{and} \quad A_2 = \{z : 1 < |z| < 1+\sqrt{2}\}.$$

As A_1 is the punctured disk and so there is no quasiconformal mapping $A_1 \to A_2$. However,

$$f(z) = (|z| + \sqrt{|z|^2+1})\,\frac{z}{|z|} : A_1 \to A_2$$

is of finite distortion and using the formulas of [11, p. 220], which give the distortion functions of radial stretchings such as f, we find

$$\mathbf{K}(z,f) = \frac{1}{2}\left(\frac{|z|}{\sqrt{|z|^2+1}} + \frac{\sqrt{|z|^2+1}}{|z|}\right),$$

whence

$$\frac{1}{\pi}\iint_{A_1} \mathbf{K}(z,f)\,dz = \int_0^1 \left(\frac{t^2}{\sqrt{t^2+1}} + \sqrt{t^2+1}\right)dt = \sqrt{2}.$$

Note that the inverse of f is the harmonic map $h\colon A_2 \to A_1$,

$$h(z) = z - 1/\overline{z}. \tag{36}$$

Let us consider the mapping

$$g_1 = f|\mathbf{D}(\tfrac{1}{2},\tfrac{1}{2}) \tag{37}$$

which is a mapping of finite distortion. Notice that for $\theta \in (-\pi/2, \pi/2)$ we have

$$\lim_{r\to 0} g_1(re^{i\theta}) = \lim_{r\to 0} f(re^{i\theta}) = e^{i\theta}. \tag{38}$$

The image domain is a sort of lune. Now there is a conformal mapping $\varphi\colon g_1(\mathbf{D}(\frac{1}{2},\frac{1}{2})) \to \mathbf{D}$ and as $g_1(\mathbf{D}(\frac{1}{2},\frac{1}{2}))$ is a Jordan domain φ has a homeomorphic extension to the boundary. Thus we can further choose φ so that the image of the arc $\{z : z = e^{i\theta} : \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\}$, lying in the boundary of $g_1(\mathbf{D}(\frac{1}{2},\frac{1}{2}))$, is the arc $\{z : z = e^{i\theta} : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$. We then define a homeomorphism $g\colon \mathbf{D} \to \mathbf{D}$ by

$$g(z) = \varphi \circ g_1\left(\frac{z+1}{2}\right). \tag{39}$$

Then $\mathbf{K}(z,g) = \mathbf{K}((z+1)/2, g_1)$ and so

$$\iint_{\mathbf{D}} \mathbf{K}(z,g)\,|dz|^2 = 4\iint_{\mathbf{D}(\frac{1}{2},\frac{1}{2})} \mathbf{K}(z,f)\,|dz|^2 < 2\sqrt{2}\pi.$$

However, the cluster set of the point $-1 \in \partial\mathbf{D}$ under g is the arc $\{z : z = e^{i\theta} : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$.

References

[1] Anderson, G. D., M. K. Vamanamurthy, and M. Vuorinen: Conformal invariants, inequalities, and quasiconformal maps. - Canad. Math. Soc. Ser. Monogr. Adv. Texts, John Wiley & Sons, New York, 1997.

[2] Astala, K., T. Iwaniec, P. Koskela, and G. J. Martin: Mappings of BMO–bounded distortion. - Math. Ann. 317, 2000, 703–726.

[3] Astala, K., T. Iwaniec, and G. J. Martin: Elliptic partial differential equations and quasiconformal mappings in the plane. - Princeton Univ. Press, 2009.

[4] Astala, K., T. Iwaniec, and G. J. Martin: Deformations with smallest mean distortion and the Nitsche conjecture. - Preprint.

[5] Astala, K., T. Iwaniec, G. J. Martin, and J. Onninen: Extremal mappings of finite distortion. - Proc. London Math. Soc. 91, 2005, 655–702.

[6] Astala, K., T. Iwaniec, G. J. Martin, and J. Onninen: Schottky's theorem on conformal mappings between annuli. - In: Proceedings of the Third International Conference on Complex Analysis and Dynamical Systems, held in honor of D. Aharonov, L. Aizenberg, S. Krushkal and U. Srebro in Nahariya, Israel, in January, 2006; Contemp. Math. 455, 2008, 35–40.

[7] Bonfert-Taylor, P., G. Martin, A. W. Reid, and E. C. Taylor: Teichmüller mappings, quasiconformal homogeneity, and non-amenable covers of Riemann surfaces. - Pure Appl. Math. Q. (to appear).

[8] Burckel, R.: An introdction to classical complex analysis. - Birkhäuser, 1979.

[9] Duren, P.: Harmonic mappings in the plane. - Cambridge Tracts in Math. 156, Cambridge Univ. Press, Cambridge, 2004.

[10] Gehring, F. W., and B. Palka: Quasiconformally homogeneous domains. - J. Anal. Math. 30, 1976, 172–199.

[11] Iwaniec, T., and G. J. Martin: Geometric function theory and nonlinear analysis. - Oxford Univ. Press, 2001.

[12] Iwaniec, T., and G. J. Martin: The Beltrami equation. - Mem. Amer. Math. Soc. 191, 2008.

[13] Lehto, O., and K. Virtanen: Quasiconformal mappings in the plane. - Springer-Verlag, 1973.

[14] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: On Q-homeomorphisms. - Ann. Acad. Sci. Fenn. Math. 30, 2005, 49–69.

[15] Nehari, Z.: Conformal mapping. - Dover, 1975.

[16] Teichmüller, O.: Ein Verschiebungssatz der quasikonformen Abbildung. - Deutsche Math. 7, 1944, 336–343.

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