Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 215–232

DIRECTED POROSITY ON CONFORMAL ITERATED FUNCTION SYSTEMS AND WEAK CONVERGENCE OF SINGULAR INTEGRALS

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Abstract. The aim of the present paper is twofold. We study directed porosity in connection with conformal iterated function systems (CIFS) and with singular integrals. We prove that limit sets of finite CIFS are porous in a stronger sense than already known. Furthermore we use directed porosity to establish that truncated singular integral operators, with respect to general Radon measures μ and kernels K, converge weakly in some dense subspaces of $L^2(\mu)$ when the support of μ belongs to a broad family of sets. This class contains many fractal sets like CIFS's limit sets.

1. Introduction

A set $E \subset \mathbf{R}^n$ is called porous, or uniformly lower porous, if there exists a constant $c > 0$ so that for each $x \in E$ and $0 < r < d(E)$ there exists $y \in B(x, r)$ satisfying

$$B(y, cr) \subset B(x, r) \setminus E.$$

Here $B(x, r)$ is the closed ball centered at x with radius r and $d(\cdot)$ denotes diameter. Dimensional properties of porous sets were studied by Mattila in [M1]. Motivated by his work different aspects of porosity have been investigated widely in relation with dimensional estimates and densities. See, e.g., [S], [KS1], [KS2] and [JJKS]. Some other applications of porosities related with the boundary behavior of quasiconformal mappings can be found in [KR], [MVu] and [Vä].

Questions regarding porosities arise naturally in fractal geometry. This can be understood heuristically since many familiar self similar sets in $\mathbf{R}^n$ are constructed by removing pieces out of some n-dimensional set in every step of the iteration process. The theory of conformal iterated function systems (CIFS), where the limit set is generated by uniformly contracting conformal maps, was studied systematically by Mauldin and Urbański in [MU]. This theory extends previous results and allows one to analyze many more limit sets than the ones emerging from the usual similitude iterated function systems. The precise assumptions on CIFS are given in Section 2.

Over the past several years many authors have studied the dynamic and geometric properties of such limits sets, porosity being one of them. See, e.g., [MMU], [MayU], [U] and [K]. In [U], Urbański gave necessary and sufficient conditions for the limit set of a CIFS on $\mathbf{R}^n$ to be porous. As a consequence if the CIFS is finite and its limit set has Hausdorff dimension less than n, it is also porous. Furthermore in the aforementioned paper some interesting applications of porosities in continued fractions were established.

2000 Mathematics Subject Classification: Primary 28A80, 42B20.
Key words: CIFS, porosity, singular integrals.
The author is supported by the Finnish Graduate School in Mathematical Analysis.

If one considers typical examples of $(n-1)$-dimensional CIFS's limit sets, for example very simple self similar sets like the four corners Cantor set in the plane, intuitively one expects to find holes spread in many directions. Motivated by this simple observation we introduce the notion of directed porous sets. For $m \in \mathbf{N},\ 0 < m < n$, we denote by $G(n,m)$ the set of all m-dimensional planes in $\mathbf{R}^n$ crossing the origin.

Definition 1.1. Suppose $V \in G(n,m)$. A set $E \subset \mathbf{R}^n$ will be called *V-directed porous at $x \in E$*, if there exists a constant $c(V)_x > 0$, such that for all $r > 0$ we can find $y \in V + x$ satisfying

$$B(y, c(V)_x r) \subset B(x,r) \setminus E.$$

If E is V-directed porous at every $x \in E$, and $c(V) = \inf\{\sup c(V)_x : x \in E\} > 0$, it will be called *$V$-directed porous.*

Recall that a set $E \subset \mathbf{R}^n$ will be called m-rectifiable for $m = 1, \dots, n$, if there exist m-dimensional C^1-submanifolds M_i, $i \in \mathbf{N}$, such that

$$\mathscr{H}^m\Big(E \setminus \bigcup_{i=1}^{\infty} M_i\Big) = 0.$$

Here $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure. Sets intersecting m-rectifiable sets in a set of zero $\mathscr{H}^m$ measure are called m-purely unrectifiable. More information about rectifiability and related topics can be found in [M2].

In Section 2, we show that limit sets of finite CIFS have very strong porosity properties, extending Urbański's result in the following sense.

Theorem 1.2. *Let $E \subset \mathbf{R}^n$, $n \geq 2$, be the limit set of a given finite CIFS. If E is m-purely unrectifiable then it is V-directed porous for all $V \in G(n,m)$.*

In [K], Käenmäki studied the geometric structure of CIFS's limit sets. He proved that if E is a limit set of a given CIFS with $\dim_{\mathscr{H}} E = t$, where $\dim_{\mathscr{H}}$ stands for Hausdorff dimension, and $l \in \mathbf{N},\ 0 < l < n$, then either

(i) $\mathscr{H}^t(E \cap M) = 0$ for every l-dimensional C^1-submanifold of $\mathbf{R}^n$, or,
(ii) E lies in some l-dimensional affine subspace or l-dimensional geometric sphere when $n > 2$, and in some analytic curve when $n = 2$.

Combining the previous rigidity result with Theorem 1.2 we derive the following corollary.

Corollary 1.3. *Let $E \subset \mathbf{R}^n, n \geq 2$, be the limit set of a given finite CIFS. If $\dim_{\mathscr{H}} E \leq m$ where $m \in \mathbf{N},\ 0 < m < n$, then E is V-directed porous at every $x \in E$ for all, except at most one, $V \in G(n,m)$.*

The motivation for this paper comes from the theory of singular integral operators with respect to general measures. Given a Radon measure μ on $\mathbf{R}^n$ and a μ-measurable kernel $K\colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ that satisfies the antisymmetry condition

$$K(-x) = -K(x) \text{ for all } x \in \mathbf{R}^n,$$

the singular integral operator T associated with K and μ is formally given by

$$T^{\mu,K}(f)(x) = \int K(x-y) f(y)\, d\mu y.$$

Since the above integral does not usually exist when $x \in \operatorname{spt}\mu$, the truncated singular integral operators $T_\varepsilon^{\mu,K}$, $\varepsilon > 0$;

$$T_\varepsilon^{\mu,K}(f)(x) = \int_{|x-y|>\varepsilon} K(x-y) f(y)\, d\mu y,$$

are considered. Often for simplicity we will denote $T_\varepsilon^{\mu,K}$ by T_ε. Using this convention one defines the maximal operator T^*,

$$T^*(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|,$$

and the principal values of $T(f)$ at every $x \in \mathbf{R}^n$ which, if they exist, are given by

$$\text{p. v.}\, T(f)(x) = \lim_{\varepsilon\to 0} T_\varepsilon(f)(x).$$

In the classical setting, when $\mu = \mathscr{L}^n$, the Lebesgue measure in $\mathbf{R}^n$, and K is a standard Calderón–Zygmund kernel, cancelations and the denseness of smooth functions in L^1 force the principal values to exist almost everywhere for L^1-functions. One could naturally ask if the $L^2(\mu)$-boundedness of T^*, which means that there exists a constant $C > 0$ such that for all $f \in L^2(\mu)$,

$$\int T^*(f)^2\, d\mu \le C \int |f|^2\, d\mu,$$

forces the principal values to exist. The answer to the above question is not always positive, see, e.g., [D] and [C]. Interestingly enough even when μ is an m-dimensional Ahlfors–David (AD) regular measure in $\mathbf{R}^n$:

$$C^{-1} r^m \le \mu(B(x,r)) \le C r^m \text{ for } x \in \operatorname{spt}\mu,\ 0 < r < \operatorname{diam}(\operatorname{spt}\mu),$$

and K is any of the coordinate Riesz kernels:

$$R_i^m(x) = \frac{x_i}{|x|^{m+1}} \text{ for } i = 1, \dots, n,$$

the question remains open for $m > 1$. For $m = 1$, it has positive answer by Tolsa, see [T1], even for more general measures. Previous results by Mattila, Melnikov and Verdera, see [MM] and [MMV], dealt with the affirmative in the case of AD-regular measures.

Recently, in [MV], Mattila and Verdera proved that, for general measures and kernels, the $L^2(\mu)$-boundedness of T^* implies that the operators T_ε converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T\colon L^2(\mu) \to L^2(\mu)$ such that for all $f, g \in L^2(\mu)$,

$$\lim_{\varepsilon\to 0} \int T_\varepsilon(f)(x) g(x)\, d\mu x = \int T(f)(x) g(x)\, d\mu x. \tag{1.1}$$

Furthermore they showed that

$$T(f)(z) = \lim_{r\to 0} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} \int_{\mathbf{R}^n \setminus B(z,r)} K(x-y) f(y)\, d\mu y\, d\mu x \tag{1.2}$$

for μ a.e. z. One of the main points in their proof is that $L^2(\mu)$-boundedness forces the limits

$$\lim_{\varepsilon\to 0} \int T_\varepsilon(f)(x) g(x)\, d\mu \tag{1.3}$$

to exist when f, g are finite linear combinations of characteristic functions of balls. We will denote this dense subspace of $L^2(\mu)$ by $\mathscr{X}_B(\mathbf{R}^n)$.

Recall that if $E \subset \mathbf{R}^n$ is $\mathscr{H}^m$-measurable with $\mathscr{H}^m(E) < \infty$ and $\mu = \mathscr{H}^m \lfloor E$, the restriction of $\mathscr{H}^m$ on E, by the works of Mattila and Preiss [MP], Mattila and Melnikov [MM], Verdera [Ve] and Tolsa [T2], the principal values

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^n \setminus B(x,\varepsilon)} R_i^m(x-y)\, d\mu y$$

exist μ almost everywhere if and only if the set E is m-rectifiable.

With the last two paragraphs in mind one might ask if weak limits like in (1.3) might exist if we remove the strong L^2-boundedness assumption even when the measures are supported in some purely unrectifiable sets. Before stating the main results of Section 3 we give some basic notation. Let

$$Q(\mathbf{R}^n) = \{A(x,r) : x \in \mathbf{R}^n,\ r > 0 \text{ and } A(x,r) = \prod_{i=1}^{n} [x^i - r/2, x^i + r/2)\} \tag{1.4}$$

and denote by $\mathscr{X}_Q(\mathbf{R}^n)$ the dense subspace of $L^2(\mu)$, in the same manner as $\mathscr{X}_B(\mathbf{R}^n)$, while instead of balls we take cubes from $Q(\mathbf{R}^n)$.

Theorem 1.4. *Let μ be a finite Radon measure on $\mathbf{R}^n, n \geq 2$, satisfying*

$$\mu(B(x,r)) \leq C r^{n-1} \text{ for all } x \in \operatorname{spt}\mu \text{ and } r > 0. \tag{1.5}$$

Let $K\colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ be an antisymmetric kernel, satisfying for all $x \in \mathbf{R}^n$,

$$|K(x)| \leq C_K\, |x|^{-(n-1)}, \tag{1.6}$$

where C_K is a constant depending on the kernel K.

(i) *If $\operatorname{spt}\mu$ is V^i-directed porous for $i = 1, \dots, n$, where $V^i = \{x \in \mathbf{R}^n : x^i = 0\}$ are the usual coordinate planes of $\mathbf{R}^n$, the truncated singular integral operators $T_\varepsilon^{\mu,K}$ converge weakly in $\mathscr{X}_Q(\mathbf{R}^n)$.*
(ii) *If $\operatorname{spt}\mu$ is V-directed porous for all $V \in G(n, n-1)$, the truncated singular integral operators $T_\varepsilon^{\mu,K}$ converge weakly also in $\mathscr{X}_B(\mathbf{R}^n)$.*

As an immediate consequence of Theorems 1.2 and 1.4 we obtain the following corollary.

Corollary 1.5. *Let $E \subset \mathbf{R}^n$, $n \geq 2$, be a $(n-1)$-purely unrectifiable limit set of a given finite CIFS. If $\mu = \mathscr{H}^{n-1} \lfloor E$ and $K\colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ is a kernel as in Theorem 1.4, the limits*

$$\lim_{\varepsilon \to 0} \int T_\varepsilon(f)(x) g(x)\, d\mu$$

exist for $f, g \in \mathscr{X}_Q(\mathbf{R}^n)$ and $f, g \in \mathscr{X}_B(\mathbf{R}^n)$.

We conclude the introductory part with the following two remarks.

Remark 1.6. The kernels satisfying the assumptions of Theorem 1.4 belong to a quite broad class; $(n-1)$-dimensional Riesz kernels being one representative. Notice that we do not even require them to be continuous. In [CM], it was proved, with different techniques, that weak convergence in $\mathscr{X}_Q(\mathbf{R}^n)$ and in $\mathscr{X}_B(\mathbf{R}^n)$ holds for much more general measures if we restrict the kernels to a smaller but still large and widely used family.

Remark 1.7. One cannot hope of replacing the function spaces $\mathscr{X}_B(\mathbf{R}^n)$ and $\mathscr{X}_Q(\mathbf{R}^n)$ with $L^2(\mu)$ in Theorem 1.4. This follows because as it was remarked in [MV], by the Banach–Steinhaus Theorem, the weak convergence in $L^2(\mu)$ implies that the operators T_ε are uniformly bounded in $L^2(\mu)$ and singular integral operators associated with 1-dimensional Riesz kernels and 1-purely unrectifiable measures are not bounded in $L^2(\mu)$.

2. Directed porosity on conformal iterated function systems

We begin by describing the setting of CIFS, as introduced in [MU]. Let I be a countable set with at least two elements and let

$$I^* = \bigcup_{m\geq 1} I^m \text{ and } I^\infty = I^{\mathbf{N}}.$$

If $w = (i_1, i_2, \ldots) \in I^* \cup I^\infty$ and $n \in \mathbf{N}$, does not exceed $|w|$, the length of w, we denote $w|_n = (i_1, \ldots, i_n)$.

Choose Ω to be some open, bounded and connected subset of $\mathbf{R}^n$ and let $\{\varphi_i\}_{i\in I}$, $\varphi_i\colon \Omega \to \Omega$, be a family of injective maps such that for every $i \in I$ there exists some $0 < s_i < 1$ such that

$$|\varphi_i(x) - \varphi_i(y)| \leq s_i \, |x - y| \, . \tag{2.1}$$

Functions satisfying (2.1) are called contractive. We will further assume that the mappings φ_i are uniformly contractive, that is, $s = \sup\{s_i : i \in I\} < 1$, and conformal. Conformality here stands for $|\varphi_i'|^n = |J\varphi_i|$, where J is the Jacobian and the norm in the left side is the usual "sup-norm" for linear mappings. This definition is usually referred as 1-quasiconformality, see, e.g., [Vä]. By Theorem 4.1 of [R] conformal maps on subsets of $\mathbf{R}^n$, $n \geq 2$, are C^∞. Assume also that there exists a compact set $X \subset \Omega$ such that $\text{int}(X) \neq \emptyset$ with the property that $\varphi_i(X) \subset X$ for all $i \in I$. Notice that for $\Omega = \mathbf{R}^n$, $n \geq 3$, conformal, contractive mappings are similitudes, which means that equality holds in (2.1). We will call a family of functions $\{\varphi_i\}_{i\in I}$, as described above, a *conformal iterated function system* (CIFS) if it satisfies the following property.

Open set condition (OSC). There exists a non-empty open set $U \subset X$ (in the relative $X-$topology) such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i \neq j \in I$.

For $w = (i_1, \ldots, i_m) \in I^m$, denote $\varphi_w = \varphi_{i_1} \circ \ldots \circ \varphi_{i_m}$ and notice that

$$d(\varphi_w(X)) \leq s^m d(X).$$

Now define the mapping $\pi\colon I^\infty \to X$ such that

$$\pi(w) = \bigcap_{m\geq 1} \varphi_{w|_m}(X).$$

The *limit set* of the CIFS is defined as,

$$E = \pi(I^\infty) = \bigcup_{w\in I^\infty} \bigcap_{m\geq 1} \varphi_{w|_m}(X).$$

We will be interested in finite CIFS, where $\Omega \subset \mathbf{R}^n$, $n \geq 2$. The following important property of these function systems follows from smoothness of the mappings φ_i, for a proof see [MU], Lemma 2.2.

Bounded distortion property (BDP). There exists some $K \geq 1$ such that

$$|\varphi_w^{'}(x)| \leq K|\varphi_w^{'}(y)| \text{ for } w \in I^* \text{ and } x, y \in \Omega,$$

Finally we state two properties of CIFS that are going to be used often in the proofs. In both properties constants depend only on the initial CIFS parameters. The first one is a direct consequence of BDP and the connectedness of Ω. Since finite CIFS are controlled Moran constructions, it follows by [KV] that (CIFS 2) is equivalent to the OSC.

(CIFS 1). There exists some constant $D \geq 1$ such that

$$D^{-1}\|\varphi_w^{'}\| \leq d(\varphi_w(E)) \leq D\|\varphi_w^{'}\| \text{ for } w \in I^*.$$

Here $\|\varphi_w'\| = \sup_{x \in \Omega} |\varphi_w'(x)|$.

(CIFS 2). Denote

$$I(x,r) = \{w \in I^* : \varphi_w(E) \cap B(x,r) \neq 0 \text{ and } d(\varphi_w(E)) \leq r < d(\varphi_{w|_{|w|-1}}(E))\},$$

where $\varphi_0 = \mathrm{id}$. There exist a positive number $N \in \mathbf{N}$ and a constant $C > 0$, such that for every $x \in \mathbf{R}^n$ and every $0 < r \leq 1$

(i) $\mathrm{card}(I(x,r)) \leq N$, where $\mathrm{card}(\cdot)$ denotes cardinality,
(ii) $Cr \leq d(\varphi_w(E)) \leq r$ for $w \in I(x,r)$,
(iii) $E \cap B(x,r) \subset \bigcup_{w \in I(x,r)} \varphi_w(E)$.

The main result of this section reads as follows.

Theorem 2.1. *Let $E \subset \mathbf{R}^n$, $n \geq 2$, be the limit set of a given finite CIFS such that every conformal map $F\colon \Omega \to \mathbf{R}^n$ satisfies*

$$F(\Omega \cap B(x,r) \cap (V + x)) \cap E^c \neq \emptyset \text{ for all } x \in \mathbf{R}^n,\ r > 0 \text{ and } V \in G(n,m). \tag{2.2}$$

Then E is V-directed porous for all $V \in G(n,m)$.

Notice that Theorem 1.2 follows immediately from Theorem 2.1 since m-purely unrectifiable sets satisfy (2.2). The main step in proving Theorem 2.1 is the following Lemma.

Lemma 2.2. *Let $E \subset \mathbf{R}^n$ be the limit set of a given CIFS such that (2.2) holds for every conformal map $F\colon \Omega \to \mathbf{R}^n$. Then for every $V \in G(n,m)$ and every $\beta > 0$ there exists some $a(\beta) > 0$ such that for every $x \in \mathbf{R}^n$, $0 < r \leq 1$, $w \in I(x,r)$, $y \in x + V$ and $s \geq \beta d(\varphi_w(E))$ satisfying*

$$B(y,s) \subset B(x,r),$$

there exists $z \in x + V$ and $l \geq a(\beta)s$ such that

$$B(z,l) \subset B(y,s) \setminus \varphi_w(E).$$

Proof. Without loss of generality assume that $E \subset B(0,1)$. We will prove Lemma 2.2 in the case where V is some m-coordinate plane, say $V = \{x \in \mathbf{R}^n : x^i = 0$ for $i = m+1, \ldots, n\}$. The general statement follows after appropriate rotations of the set E. Let $V_x = x + V$ for $x \in \mathbf{R}^n$. By way of contradiction, suppose that Lemma 2.2

does not hold. Then there exists some constant $\beta > 0$ such that for every $j \in \mathbf{N}$ there exist sequences

$$\begin{aligned}
&\{x_j\}_{j\in\mathbf{N}} \in B(0,1),\\
&\{r_j\}_{j\in\mathbf{N}} \in (0,1],\\
&\{w_j\}_{j\in\mathbf{N}} \in I^* \text{ such that } w_j \in I(x_j,r_j) \text{ for every } j \in \mathbf{N},\\
&\{y_j\}_{j\in\mathbf{N}} \in B(0,1)\cap V_{x_j},\\
&\{s_j\}_{j\in\mathbf{N}} \in (0,1],
\end{aligned}$$

satisfying for all $j \in \mathbf{N}$ the following three conditions.

(C1) $B(y_j,s_j) \subset B(x_j,r_j)$.

(C2) $s_j \geq \beta d(\varphi_{w_j}(E))$.

(C3) For every $z \in V_{x_j}$ the condition

$$B(z,l) \subset B(y_j,s_j)\backslash\varphi_{w_j}(E)$$

implies $l < \frac{1}{j}s_j$.

By passing to an appropriate subsequence, if necessary, we find $y \in B(0,1)$ such that

$$y_j \to y.$$

$>$From now on we will denote $V_{x_j} = V_{y_j}$ by V_j. Let $\Psi_j\colon \mathbf{R}^n \to \mathbf{R}^n$ be defined for $z \in \mathbf{R}^n$ as,

$$\Psi_j(z) = \|\varphi_{w_j}^{'}\|^{-1}(z-y_j)+y_j.$$

We are going to use the following properties of Ψ_j:

(Ψ1) For all pairs $z, w \in \mathbf{R}^n$

$$|\Psi_j(w)-\Psi_j(z)| = \|\varphi_{w_j}^{'}\|^{-1}|w-z|.$$

(Ψ2) For every $\delta > 0$, and $V_j(\delta) = \{x \in \mathbf{R}^n : d(x,V_j) < \delta\}$,

$$\Psi_j(V_j) = V_j \text{ and } \Psi_j(V_j(\delta)) = V_j(\delta\|\varphi_{w_j}^{'}\|^{-1}).$$

(Ψ3) For every $r > 0$ and every $z \in V_j$,

$$\Psi_j(B(z,r)) = B(\Psi_j(z), \|\varphi_{w_j}^{'}\|^{-1}r).$$

Denote for $j \in \mathbf{N}$,

$$P_j = V_j(2s_jj^{-1}) \cap \varphi_{w_j}(E) \cap B(y_j,s_j) \tag{2.3}$$

and

$$T_j = \Psi_j(P_j). \tag{2.4}$$

By (C3), for every $z \in V_j \cap B(y_j,s_j)$

$$B(y_j,s_j)\cap B(z,2s_jj^{-1})\cap\varphi_{w_j}(E) \neq \emptyset. \tag{2.5}$$

Using (2.5) we can also show that for all $q \in V_j \cap B(y_j, \|\varphi_{w_j}^{'}\|^{-1}s_j)$ and every $r \geq 2\|\varphi_{w_j}^{'}\|^{-1}j^{-1}s_j$,

$$B(q,r)\cap T_j \neq \emptyset. \tag{2.6}$$

To see this, let

$$\widetilde{q} = (\|\varphi_{w_j}^{'}\|(q^1-y_j^1)+y_j^1,\ldots,\|\varphi_{w_j}^{'}\|(q^m-y_j^m)+y_j^m, y_j^{m+1},\ldots,y_j^n),$$

where $q=(q^1,\dots,q^m,y_j^{m+1},\dots,y_j^n)\in V_j\cap B(y_j,\|\varphi_{w_j}^{'}\|^{-1}s_j)$. Then $\Psi_j(\widetilde{q})=q$ and for $i=1,\dots,m$,

$$|\widetilde{q}^i-y_j^i|=\|\varphi_{w_j}^{'}\||q^i-y_j^i|\le\|\varphi_{w_j}^{'}\|\|\varphi_{w_j}^{'}\|^{-1}s_j.$$

This implies that $\widetilde{q}\in V_j\cap B(y_j,s_j)$. Therefore, by (2.5), we get

$$B(y_j,s_j)\cap B(\widetilde{q},2s_jj^{-1})\cap\varphi_{w_j}(E)\neq\emptyset.$$

Consequently

$$\Psi_j(V_j(2s_jj^{-1})\cap B(y_j,s_j)\cap\varphi_{w_j}(E)\cap B(\widetilde{q},2s_jj^{-1})))\neq\emptyset$$

and by $(\Psi 3)$

$$B(q,2\|\varphi_{w_j}^{'}\|^{-1}s_jj^{-1})\cap\Psi_j(P_j)\neq\emptyset.$$

Hence

$$B(q,r)\cap T_j\neq\emptyset\text{ for }r\ge 2\|\varphi_{w_j}^{'}\|^{-1}j^{-1}s_j.$$

Next we will show that there exists some constant $B>0$ such that for every $j\in\mathbf{N}$, large enough,

$$B^{-1}\le d(T_j)\le B. \tag{2.7}$$

To prove (2.7) let $p_j,q_j\in V_j\cap B(y_j,s_j)$ such that

$$p_j=(y_j^1-(s_j-s_jj^{-1}),y_j^2,\dots,y_j^n)$$

and

$$q_j=(y_j^1+(s_j-s_jj^{-1}),y_j^2,\dots,y_j^n).$$

Recalling (2.5) we notice that for every

$$e\in B(y_j,s_j)\cap B(p_j,2s_jj^{-1})\cap\varphi_{w_j}(E)$$

and

$$d\in B(y_j,s_j)\cap B(q_j,2s_jj^{-1})\cap\varphi_{w_j}(E),$$

we have

$$|e-d|\ge|p_j-q_j|-|p_j-e|-|q_j-d|\ge 2s_j-6s_jj^{-1}\ge\frac{s_j}{2},$$

for $j\ge 4$. Hence

$$d(P_j)=d(V_j(2s_jj^{-1})\cap\varphi_{w_j}(E)\cap B(y_j,s_j))\ge\frac{s_j}{2}\text{ where }j\ge 4.$$

By (C2) we also deduce that

$$d(P_j)\le d(\varphi_{w_j}(E))\le\beta^{-1}s_j.$$

Combining the two previous estimates we derive

$$\frac{s_j}{2}\le d(P_j)\le\beta^{-1}s_j. \tag{2.8}$$

Now by (2.8), (C2) and (CIFS 1) it follows that

$$\begin{aligned}d(T_j)&=d(\Psi_j(P_j))=\|\varphi_{w_j}^{'}\|^{-1}d(P_j)\\&\ge\|\varphi_{w_j}^{'}\|^{-1}\frac{s_j}{2}\ge\frac{\beta}{2}\|\varphi_{w_j}^{'}\|^{-1}d(\varphi_{w_j}(E))\\&\ge\frac{\beta}{2}D^{-1}\|\varphi_{w_j}^{'}\|^{-1}\|\varphi_{w_j}^{'}\|\end{aligned}$$

and, by (CIFS 1),

$$d(T_j) = \|\varphi'_{w_j}\|^{-1} d(P_j) \le \|\varphi'_{w_j}\|^{-1} d(\varphi_{w_j}(E)) \le D.$$

Therefore for all $j \in \mathbf{N},\ j \ge 4$,

$$B^{-1} \le d(T_j) \le B$$

where $B = \min\{D, 2\beta^{-1}D\}$. The following fact follows immediately from (CIFS 1), (C2) and (2.8), since $P_j \subset \varphi_{w_j}(E)$. We state it separately for the convenience of the reader. For all $j \in \mathbf{N},\ j \ge 4$,

$$\beta D^{-1} \|\varphi'_{w_j}\| \le s_j \le 2D \|\varphi'_{w_j}\|. \tag{2.9}$$

For every $j \in \mathbf{N}$ the functions $F_j \colon \Omega \to \mathbf{R}^n$ are defined as

$$F_j := \Psi_j \circ \varphi_{w_j}.$$

Observe that for all $j \in \mathbf{N}$

(F1) F_j are conformal,

(F2) F_j are bi-Lipschitz with constants not depending on j.

Property (F2) follows from BDP and the mean value theorem. To see this, for all $z, w \in \Omega$,

$$\begin{aligned} K^{-1}|z-w| &\le \|\varphi'_{w_j}\|^{-1} \|(\varphi^{-1}_{w_j})'\|^{-1} |z-w| \le \|\varphi'_{w_j}\|^{-1} |\varphi_{w_j}(z) - \varphi_{w_j}(w)| \\ &= |F_j(z) - F_j(w)| \le |z-w|. \end{aligned}$$

Using the Ascoli–Arzela theorem we are now able to find some uniformly convergent subsequence of F_j, which for the sake of simplicity we will keep on denoting by F_j, such that

$$F_j \to F \text{ and } F \colon \Omega \to \mathbf{R}^n \text{ is conformal and bi-Lipschitz.}$$

Notice that by standard complex analysis when $n = 2$, and basic properties of Möbius maps for $n \ge 3$, it follows that the map $F^{-1} \colon \mathbf{R}^n \to \Omega$ is also conformal.

Now define

$$\mathscr{G} = \{\alpha : \mathbf{N} \to \bigcup_{j=1}^{\infty} T_j \text{ such that } \alpha(j) \in T_j \text{ for all } j \in \mathbf{N}\}$$

and

$$\begin{aligned} T = \{t \in \mathbf{R}^n : \text{ there exist increasing } k \colon \mathbf{N} \to \mathbf{N} \text{ and} \\ \alpha \in \mathscr{G} \text{ such that } \alpha(k(j)) \to t\}. \end{aligned}$$

The set T has the following properties:

(T1) $y \in T$.

Recall that y is the limit of the sequence y_j. By (2.6),

$$B(y_j, 2\|\varphi'_{w_j}\|^{-1} s_j j^{-1}) \cap T_j \ne \emptyset \text{ for all } j \in \mathbf{N}.$$

Therefore, by (2.9), there exists some sequence $\{t_j\}_{j \ge 4}$ such that for all $j \in \mathbf{N},\ j \ge 4$,

$$t_j \in T_j \cap B(y_j, 4Dj^{-1}).$$

Since $y_j \to y$, we also get $t_j \to y$ and consequently $y \in T$.

(T2) $B(y, D^{-1}\frac{\beta}{100}) \cap V_y \subset T$.

Suppose that there exists some $a \in B(y, D^{-1}\frac{\beta}{100}) \cap V_y$ such that $a \notin T$. Then there exist $r_0 < D^{-1}\frac{\beta}{100}$ and $j_0 \in \mathbf{N}$ such that for all $j \geq j_0$,

$$B(a, r_0) \cap T_j = \emptyset.$$

Now choose some $j_1 \in \mathbf{N}$ such that for all $j \geq j_1$,

$$|y_j - y| \leq D^{-1}\frac{\beta}{100}.$$

Then for all such j,

$$B(a, r_0) \subset B(y_j, \|\varphi'_{w_j}\|^{-1} s_j).$$

To see this, take $b \in B(a, r_0)$. By (2.9),

$$|b - y_j| \leq |b - a| + |a - y| + |y - y_j| \leq 3D^{-1}\frac{\beta}{100} \leq \|\varphi'_{w_j}\|^{-1} s_j.$$

Choose $j_2 \in \mathbf{N},\ j_2 \geq j_1$, such that for all $j \geq j_2$,

$$|y_j - y| \leq \frac{r_0}{2}.$$

If $a = (a^1, \ldots, a^m, y^{m+1}, \ldots, y^n) \in V_y$ let $\widetilde{a}_j = (a^1, \ldots, a^m, y_j^{m+1}, \ldots, y_j^n) \in V_j$ and notice that

$$|\widetilde{a}_j - a| \leq |y - y_j|.$$

Then for $j \geq j_2$ and $r_1 = \frac{r_0}{2}$, by triangle inequality,

$$\widetilde{a}_j \in B(y_j, \|\varphi'_{w_j}\|^{-1} s_j) \tag{2.10}$$

and

$$B(\widetilde{a}_j, r_1) \subset B(a, r_0). \tag{2.11}$$

Hence for $j_* \in \mathbf{N}$ big enough satisfying

$$j_* \geq \max\{j_0, j_2\} \text{ and } 2\|\varphi'_{w_{j_*}}\|^{-1}\frac{s_{j_*}}{j_*} \leq r_1$$

we get,

(i) $B(a, r_0) \cap T_{j_*} = \emptyset$,
(ii) $\widetilde{a}_{j_*} \in V_{j_*} \cap B(y_{j_*}, \|\varphi'_{w_{j_*}}\|^{-1} s_{j_*})$,
(iii) $B(\widetilde{a}_{j_*}, r_1) \subset B(a, r_0)$.

Consequently

$$B(\widetilde{a}_{j_*}, 2\|\varphi'_{w_{j_*}}\|^{-1}\frac{s_{j_*}}{j_*}) \cap T_{j_*} = \emptyset$$

which contradicts (2.6).

(T3) $T \subset F(E)$.

Let $t \in T$, then there exist some increasing function $k(j)\colon \mathbf{N} \to \mathbf{N}$ and some $\alpha \in \mathscr{G}$ such that

$$\alpha(k(j)) \in T_{k(j)} \subset \Psi_{k(j)}(\varphi_{w_{k(j)}}(E)) = F_{k(j)}(E) \text{ and } \alpha(k(j)) \to t.$$

Therefore there exists a sequence $\{e_j\}_{j=1}^{\infty} \in E$ such that $F_{k(j)}(e_j) = \alpha(k(j))$. Since the limit set E is compact there exists some subsequence of $\{e_j\}_{j=1}^{\infty}$ converging to some point $e \in E$. To simplify notation assume that $e_j \to e$. Finally because the convergence $F_{k(j)} \to F$ is uniform, we also deduce that

$$\alpha(k(j)) = F_{k(j)}(e_j) \to F(e),$$

which implies that $t = F(e)$.

Properties (T2) and (T3) imply

$$F^{-1}(B(y, D^{-1}\frac{\beta}{100}) \cap V_y) \subset F^{-1}(T) \subset E.$$

Since F^{-1} is conformal, this contradicts (2.2), finishing the proof of Lemma 2.2. □

Proof of Theorem 2.1. Let $x \in \mathbf{R}^n$ and $0 < r < 1$. For $I(x,r) \subset I^*$, $N \in \mathbf{N}$ as in (CIFS 2) we get

$$I(x,r) = \{w_1, \ldots, w_m\} \text{ for some } m \le N \text{ and } d(\varphi_{w_i}(E)) \le r \text{ for } i = 1, \ldots, m.$$

Applying Lemma 2.2 for $b = 1$, as $r \ge d(\varphi_{w_1}(E))$, there exist $z_1 \in V_x$ and $l_1 \ge 0$ such that

$$B(z_1, l_1) \subset B(x,r) \backslash \varphi_{w_1}(E) \text{ and } l_1 \ge a(1)r.$$

As

$$r \ge d(\varphi_{w_2}(E))$$

we also get

$$l_1 \ge a(1)d(\varphi_{w_2}(E)).$$

Denote $a_1 := a(1)$. Again Lemma 2.2 implies that there exist $z_2 \in V_x$ and $l_2 \ge 0$ satisfying

$$B(z_2, l_2) \subset B(z_1, l_1) \backslash \varphi_{w_2}(E) \subset B(x,r) \text{ and } l_2 \ge a(a_1)l_1.$$

As before

$$l_2 \ge a(a_1)a(1)r \ge a(a_1)a_1 d(\varphi_{w_3}(E)).$$

In the same manner denote $a_2 := a(a_1)a_1$. There exist $z_3 \in V_x$ and $l_3 \ge 0$ such that

$$B(z_3, l_3) \subset B(z_2, l_2) \backslash \varphi_{w_3}(E)$$

and

$$l_3 \ge a(a_2)l_2 \ge a(a_2)a(a_1)a_1 r = a(a_2)a_2 d(\varphi_{w_4}(E)).$$

Repeating the same arguments, after m steps, we finally get that there exist some $z_m \in V_x \cap B(x,r), l_m > 0$ such that

$$B(z_m, l_m) \subset B(z_{m-1}, l_{m-1}) \backslash \varphi_{w_m}(E)$$

and

$$l_m \ge a(a_{m-1})l_{m-1} \ge a(a_{m-1}) \cdots a(a_1)a_1 r.$$

Therefore

$$B(z_m, C^* r) \subset B(x,r) \backslash \bigcup_{w \in I(x,r)} \varphi_w(E) = B(x,r) \backslash E$$

where $C^* = a(a_{m-1})a_{m-1} = a(a_{m-1}) \cdots a(a_1)a_1$ is a constant depending only on the CIFS's initial parameters. □

3. Geometric criteria for weak convergence

We begin this section with an auxiliary result necessary to prove Theorem 1.4.

Theorem 3.1. *Let μ be a finite Radon measure in $\mathbf{R}^n$ and $K\colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ an antisymmetric kernel satisfying (1.5) and (1.6) respectively.*

(i) *The truncated singular integral operators T_ε associated to μ and K converge weakly in $\mathscr{X}_Q(\mathbf{R}^n)$ if for any $V \in TA(n,n-1) = \{V_w^i : i = 1,\dots,n$ and $w \in \mathbf{R}^n\}$,*
  (a) $\mu(V) = 0$,
  (b) *there exists some positive number $a_V < 1$ such that*

$$\sum_{k=0}^{\infty} \mu(S_k(a_V, V))k < \infty, \tag{3.1}$$

where $S_k(a_V,V) = \{x \in \mathbf{R}^n : \sum_{j=k+1}^{\infty} a_V^j \le d(x,V) < \sum_{j=k}^{\infty} a_V^j\}$.

(ii) *The truncated singular integral operators T_ε, associated to μ and K converge weakly in $\mathscr{X}_B(\mathbf{R}^n)$ if for any sphere $C = S_x^R$, centered at x of radius R,*
  (a) $\mu(C) = 0$,
  (b) *there exists some positive number $a_C < \min\{1, R\}$ such that*

$$\sum_{k=0}^{\infty} \mu(S_k(a_C, C))k < \infty, \tag{3.2}$$

where $S_k(a_C,C) = \{x \in B(x,R) : \sum_{j=k+1}^{\infty} a_C^j \le d(x,C) < \sum_{j=k}^{\infty} a_C^j\}$.

Proof. We give the proof only for (i) since the proof of (ii) is almost identical. Denote $E = \operatorname{spt}\mu$ and without loss of generality assume that $E \subset B(0,1/2)$ and $\mu(E) \le 1$. Let

$$f = \sum_{i=1}^{l} a_i \chi_{Q_i} \text{ and } g = \sum_{j=1}^{m} b_j \chi_{P_j}$$

where $a_i, b_j \in \mathbf{R}$ and $Q_i, P_j \in Q(\mathbf{R}^n)$. For $0 < \delta < \varepsilon$,

$$\begin{aligned}
&\left| \int T_\varepsilon(f)(x) g(x)\, d\mu x - \int T_\delta(f)(x) g(x)\, d\mu x \right| \\
&= \left| \int (T_\varepsilon(f)(x) - T_\delta(f)(x)) \sum_{j=1}^{m} b_j \chi_{P_j}(x)\, d\mu x \right| \\
&= \left| \sum_{j=1}^{m} b_j \int_{P_j} \int_{B(x,\varepsilon)\setminus B(x,\delta)} K(x-y) f(y)\, d\mu y\, d\mu x \right| \\
&\le \sum_{j=1}^{m} \sum_{i=1}^{l} |b_j a_i| \left| \int_{P_j} \int_{Q_i} K(x-y)\, d\mu y\, d\mu x \right|_{\delta<|x-y|<\varepsilon}.
\end{aligned}$$

By the antisymmetry of K and Fubini's Theorem we have

$$\begin{aligned}
&\left| \int_{P_j} \int_{Q_i}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu x \right| \\
&= \left| \int_{P_j} \int_{Q_i \cap P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu x + \int_{P_j} \int_{Q_i \setminus P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu x \right| \\
&\le \left| \int_{P_j \cap Q_i} \int_{Q_i \cap P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu x \right| + \left| \int_{P_j \setminus Q_i} \int_{Q_i \cap P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu x \right| \\
&\quad + \left| \int_{P_j \setminus Q_i} \int_{Q_i \setminus P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu \right| + \left| \int_{P_j \cap Q_i} \int_{Q_i \setminus P_j}_{\delta<|x-y|<\varepsilon} K(x-y)\, d\mu y\, d\mu \right| \\
&\le \int_{Q_i} \int_{Q_i^c}_{\delta<|x-y|<\varepsilon} |K(x-y)|\, d\mu y\, d\mu x + 2 \int_{P_j} \int_{P_j^c}_{\delta<|x-y|<\varepsilon} |K(x-y)|\, d\mu y\, d\mu x.
\end{aligned}$$

Therefore it is enough to show that for every $A \in Q(\mathbf{R}^n)$

$$\int_A \int_{A^c} |K(x-y)|\, d\mu y\, d\mu x < \infty. \tag{3.3}$$

Since $\mu(V) = 0$ for every $V \in TA(n, n-1)$ instead of (3.3) it suffices to prove that

$$\int_{A^\circ} \int_{A^c} |K(x-y)|\, d\mu y\, d\mu x < \infty, \tag{3.4}$$

for all $A \in Q(\mathbf{R}^n)$. Let $G_i \in TA(n, n-1)$, $i = 1, \ldots, 2n$, be the hyperplanes that contain the $2n$ sides of A. For any $x \in A^\circ \cap E$ and any $i = 1, \ldots, 2n$ define the following distance functions

$$d_i(x) = d(x, G_i).$$

Let $N_i(x) > 0$, $i = 1, \ldots, 2n$, be such that

$$2^{N_i(x)} d_i(x) = 1.$$

Hence if $\lfloor N_i(x) \rfloor$ is the smallest integer greater than $N_i(x)$

$$\lfloor N_i(x) \rfloor \le (\log 2)^{-1} \log d_i(x)^{-1} + 1.$$

Therefore

$$E \setminus A \subset \bigcup_{i=1}^{2n} \bigcup_{j=1}^{\lfloor N_i(x) \rfloor} B(x, 2^j d_i(x)) \setminus B(x, 2^{j-1} d_i(x)),$$

and for all $x \in A^\circ \cap E$,

$$\begin{aligned}\int_{A^c} |K(x-y)|\, d\mu y &\leq C_K \int_{\bigcup_{i=1}^{2n} \bigcup_{j=1}^{\lfloor N_i(x) \rfloor} B(x,2^j d_i(x)) \setminus B(x,2^{j-1} d_i(x))} |x-y|^{-(n-1)}\, d\mu y \\ &= C_K \sum_{i=1}^{2n} \sum_{j=1}^{\lfloor N_i(x) \rfloor} \int_{B(x,2^j d_i(x)) \setminus B(x,2^{j-1} d_i(x))} |x-y|^{-(n-1)}\, d\mu y \\ &\leq C_K \sum_{i=1}^{2n} \sum_{j=1}^{\lfloor N_i(x) \rfloor} \frac{\mu\left(B(x, 2^j d_i(x))\right)}{2^{-(n-1)} d_i(x)^{n-1} 2^{j(n-1)}} \\ &\leq C_K \sum_{i=1}^{2n} \sum_{j=1}^{\lfloor N_i(x) \rfloor} \frac{C d_i(x)^{n-1} 2^{j(n-1)}}{2^{-(n-1)} d_i(x)^{n-1} 2^{j(n-1)}} \\ &\leq C_K C 2^{(n-1)} (\log 2)^{-1} \Big(\sum_{i=1}^{2n} \log d_i(x)^{-1} + 2n \Big).\end{aligned}$$

This leads to the following estimate

$$\int_{A^\circ} \int_{A^c} |K(x-y)|\, d\mu y\, d\mu x \leq \frac{C_K C 2^{(n-1)}}{\log 2} \Big(\sum_{i=1}^{2n} \int_{A^\circ} \log d_i(x)^{-1}\, d\mu x + 2n \Big). \tag{3.5}$$

Notice that for $i = 1, \ldots, 2n$, A° can be decomposed as

$$A \subset \bigcup_{k=0}^{\infty} S_k(a_i, G_i) \cup A_i',$$

where $a_i = a_{G_i}$ and $A_i' = \{x \in A : d_i(x) > s_i = \sum_{j=0}^{\infty} a_i^j\}$. Therefore

$$\int_{A^\circ} \log d_i(x)^{-1}\, d\mu x \leq \sum_{k=0}^{\infty} \int_{S_k(a_i, G_i)} \log d_i(x)^{-1}\, d\mu x + \log s_i^{-1}.$$

For $x \in S_k(a_{G_i}, G_i)$

$$d_i(x) > \sum_{j=k+1}^{\infty} a_i^j = a_i^{k+1} \frac{1}{1-a_i}$$

and

$$\log \frac{1}{d_i(x)} \leq \log \left(\frac{1-a_i}{a_i^{k+1}} \right) = k \log \frac{1}{a_i} + \log \frac{1-a_i}{a_i}.$$

Hence

$$\int_{A^\circ} \log \frac{1}{d_i(x)}\, d\mu x \leq \log \frac{1}{a_i} \sum_{k=0}^{\infty} \mu(S_k(a_i, G_i)) k + \log \frac{1-a_i}{a_i s_i}. \tag{3.6}$$

Using (3.5) and (3.6) we can estimate

$$\begin{aligned}&\int_{A^\circ} \int_{A^c} |K(x-y)|\, d\mu y\, d\mu x \\ &\leq \frac{C_K C 2^{(n-1)}}{\log 2} \left(\sum_{i=1}^{2n} \log \frac{1}{a_i} \sum_{k=0}^{\infty} \mu(S_k(a_i, G_i)) k + \sum_{i=1}^{2n} \log \frac{1-a_i}{a_i s_i} + 2n \right).\end{aligned}$$

Since, by (3.1), for $i = 1, \dots, 2n$

$$\sum_{k=0}^{\infty} \mu(S_k(a_i, G_i))k < \infty,$$

we have shown (3.4) and the proof of Theorem 3.1(i) is complete. □

We can now proceed in the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\operatorname{spt} \mu = E$ and without loss of generality assume that $E \subset B(0, 1/2)$. We start by proving (i). For $x \in \mathbf{R}^n,\ r > 0, i \in \{1, \dots, n\},\ q \in \mathbf{N}$ define the following grids,

$$\begin{aligned} Gr(x, r, i, q) = \{g \in A(x, r) : g^i = x^i \text{ and for } 1 \leq j \leq n,\ j \neq i, \\ g^j = (x^j - \frac{r}{2}) + \frac{r}{2q}(2k-1) \text{ for some } k = 1, \dots, q\}. \end{aligned}$$

Since E is V^i-directed porous for $i = 1, \dots, n$, as an immediate corollary of Definition 1.1 there exists some $N \in \mathbf{N},\ N \geq 2$, such that for every $x \in \mathbf{R}^n$ and every $r > 0$ there exists some $y \in V_x^i \cap A(x, r)$ satisfying

$$A(y, rN^{-1}) \subset A(x, r) \ \setminus E. \tag{3.7}$$

>From (3.7) we also deduce that there exist some $M \in \mathbf{N},\ M \geq 4$, in fact we can even choose $M = 2N$, such that for every $x \in \mathbf{R}^n$, every $r > 0$ and every $i = 1, \dots, n$ there exists some $g_{(x,r,i)} \in Gr(x, r, i, M)$ such that

$$A(g_{(x,r,i)}, rM^{-1}) \subset A(x, r) \setminus E. \tag{3.8}$$

By Theorem 3.1 it is enough to show that for every $x \in \mathbf{R}^n$ and every $i = 1, \dots, n$

$$\sum_{k=0}^{\infty} \mu(S_k(M^{-1}, V_x^i))k < \infty.$$

Thus we need to estimate the measure μ of the strips $V_x^i(2^{-1}M^{-k})$. The idea is to cover $V_x^i(2^{-1}M^{-k}) \cap E \cap A(x, 1)$ with cubes from $Q(\mathbf{R}^n)$ of sidelength M^{-k} with their centers in $Gr(x, 1, i, M^k)$. The use of the specific grids allows us to count the covering cubes easily. Note that in order to cover $V_x^i(2^{-1}M^{-k}) \cap A(x, 1)$ with cubes in $Q(\mathbf{R}^n)$, of sidelength M^{-k} and with centers in V_x^i we first cover $V_x^i \cap A(x, 1)$ with cubes $\{Q_j\}_{j \in J}$ in $Q(\mathbf{R}^{n-1})$. Then the required cubes needed to cover $V_x^i(2^{-1}M^{-k}) \cap A(x, 1)$ will be

$$\begin{aligned} P_j = \{(y^1, \dots, y^i, ..y^n) \in \mathbf{R}^n : (y^1, \dots, y^{i-1}, y^{i+1}, \dots, y^n) \in Q_j \text{ and} \\ y_i \in [x^i - 2^{-1}M^{-k}, x^i + 2^{-1}M^{-k})\}. \end{aligned}$$

See Figures A and B for an illustration.

For $x \in \mathbf{R}^n,\ r > 0$ and $i = 1, \dots, n$, denote

$$Gr^*(x, r, i, M) = Gr(x, r, i, M) \setminus \{g_{(x,r,i)}\}.$$

Fix some $x \in \mathbf{R}^n,\ r > 0$ and $i = 1, \dots, n$, then by (3.8)

$$V_i^x(r(2M)^{-1}) \cap E \cap A(x, r) \subset \bigcup_{y \in Gr^*(x,r,i,M)} A(y, rM^{-1})$$

and

$$\operatorname{card}(Gr^*(x, r, i, M)) = M^{n-1} - 1.$$

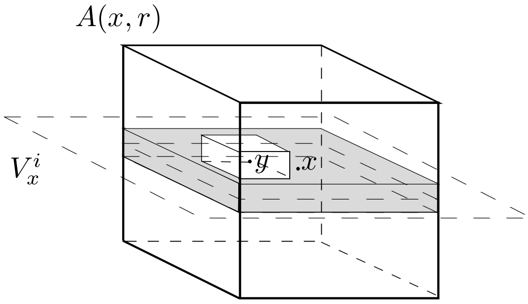

Figure A.

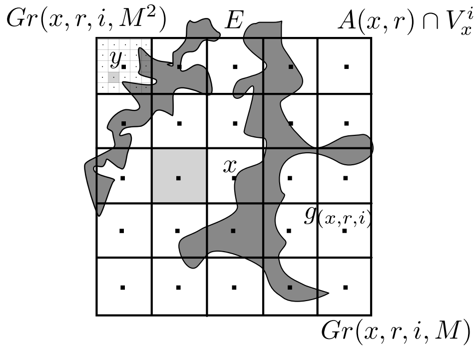

Figure B.

Notice that the cardinality of the grid $Gr^*(x,r,i,M)$ depends only on its thickness, i.e., only on M.

In the same manner for $y \in Gr^*(x,r,i,M)$ the cubes $A(y, rM^{-1})$ satisfy

$$V_i^x(r2^{-1}M^{-2}) \cap E \cap A(y, rM^{-1}) \subset \bigcup_{h \in Gr^*(y, rM^{-1}, i, M)} A(h, rM^{-2}).$$

Therefore

$$V_i^x(r2^{-1}M^{-2}) \cap E \cap A(x,r) \subset \bigcup_{\{h \in Gr^*(y, rM^{-1}, i, M) : y \in Gr^*(x,r,i,M)\}} A(h, rM^{-2})$$

and

$$\operatorname{card}(\{h \in Gr^*(y, rM^{-1}, i, M) : y \in Gr^*(x,r,i,M)\}) = (M^{n-1}-1)^2.$$

Notice that

$$\{h \in Gr^*(y, rM^{-1}, i, M) : y \in Gr^*(x,r,i,M)\} \subset Gr(x,r,i,M^2).$$

Inductively we conclude that for all $x \in \mathbf{R}^n,\ r > 0,\ i \in \{1,\dots,n\}$ and $k \in \mathbf{N}$ there exist sets of cubes

$$Q^k(x,r,i) \subset Q(\mathbf{R}^n),$$

consisting of cubes $A(g, \frac{r}{M^k})$ with $g \in Gr(x,r,i,M^k)$ satisfying

(i) $V_i^x(r2^{-1}M^{-k}) \cap E \cap A(x,r) \subset \bigcup\{Q : Q \in Q^k(x,r,i)\}$,
(ii) $\operatorname{card}(Q^k(x,r,i)) = (M^{n-1}-1)^k$.

Properties (i) and (ii) imply that for all $x \in \mathbf{R}^n, r > 0,\ i = 1,\dots,n$ and $k \in \mathbf{N}$

$$\begin{aligned}\mu(V_x^i(2^{-1}M^{-k}) \cap A(x,1)) &\le \sum_{Q \in Q^k(x,1,i)} \mu(Q)\\ &\le \operatorname{card}(Q^k(x,1,i))C(\sqrt{n}M^{-k})^{n-1}\\ &= C(\sqrt{n})^{n-1}(1-M^{1-n})^k.\end{aligned}$$

For every $x \in \mathbf{R}^n$ and every $i = 1,\dots,n$ there exist $y^1_{(x,i)}$ and $y^2_{(x,i)}$ such that

$$S_k(M^{-1}, V_x^i) = V^i_{y^1_{(x,i)}}(2^{-1}M^{-k}) \cup V^i_{y^2_{(x,i)}}(2^{-1}M^{-k})$$

and

$$S_k(M^{-1}, V_x^i) \cap E \subset A(y^1_{(x,i)}, 1) \cup A(y^2_{(x,i)}, 1).$$

Therefore we deduce that

$$\begin{aligned}\sum_{k=0}^{\infty} \mu(S_k(M^{-1}, V_x^i))k &= \sum_{k=0}^{\infty} \mu(V^i_{y^1_{(x,i)}}(2^{-1}M^{-k}) \cap A(y^1_{(x,i)}, 1))k \\ &\quad + \sum_{k=0}^{\infty} \mu(V^i_{y^2_{(x,i)}}(2^{-1}M^{-k}) \cap A(y^2_{(x,i)}, 1))k \\ &\leq 2C(\sqrt{n})^{n-1} \sum_{k=0}^{\infty} (1 - M^{1-n})^k k.\end{aligned}$$

This concludes the proof of (i) since

$$\sum_{k=0}^{\infty} (1 - M^{1-n})^k k < \infty.$$

For the proof of (ii) notice that since E is V-directed porous for all $V \in G(n, n-1)$ we can define the function, $\Theta\colon G(n, n-1) \to (0, 1)$, as

$$\Theta(V) = c(V)$$

where $c(V)$ are the numbers appearing in Definition 1.1. By compactness of $G(n, n-1)$, see, e.g. [M2], and continuity of Θ, we deduce that that Θ attains some minimal value c depending only on the set E. Using this observation, Theorem 3.1 (ii) and exactly the same arguments as in (i), adapted to spheres, we obtain (ii). □

Acknowledgement. I express my gratitude to my advisor, Professor Pertti Mattila, for many ideas, discussions, suggestions and for repeated proof reading. I would also like to thank Antti Käenmäki, for many valuable comments and for pointing out that the open set condition is the only extra assumption needed for the CIFS discussed in Section 2.

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Received 19 March 2008