Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 145–156

GROWTH ESTIMATES FOR SOLUTIONS OF NONHOMOGENEOUS LINEAR COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. Two pointwise growth estimates are established for the solutions of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = A_k(z),$$

where the coefficients $A_0(z), \ldots, A_k(z)$ are analytic in the disc $\{z : |z| < R\}$, $0 < R \leq \infty$. These pointwise estimates yield several growth estimates for the p-characteristic (generalized Nevanlinna proximity function) of the solutions. The sharpness of the results as well as some further consequences are discussed.

1. Introduction and main results

Two parallel studies [1, 8] on the growth of solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = A_k(z) \tag{1.1}$$

appeared in 2004. The main results in both papers are stated in the disc $D_R = \{z \in \mathbf{C} : |z| < R\}$, where $0 < R \leq \infty$. The advantage of the pointwise growth estimate due to Chiang and Hayman [1] is that it is valid (outside of an exceptional set) for solutions of equation (1.1) with meromorphic coefficients $A_0(z), \ldots, A_k(z)$. The results in [8], however, appear to give sharper growth estimates for the solutions of (1.1) in the special case $A_k(z) \equiv 0$, $R < \infty$, and the coefficients $A_0(z), \ldots, A_{k-1}(z)$ are analytic functions of finite order of growth. Moreover, the estimates in [8] are valid at any point in D_R.

The purpose of this paper is to generalize the pointwise growth estimates in [8] for the non-homogeneous equation (1.1), where the coefficients are analytic in D_R, $0 < R \leq \infty$. The main results are Theorems 1 and 2 below, which essentially reduce to the corresponding results in [8] if $A_k(z) \equiv 0$.

Theorem 1. *Let the coefficients $A_0(z), \ldots, A_k(z)$ of* (1.1) *be analytic in D_R, where $0 < R \leq \infty$, and let f be a solution of* (1.1).

2000 Mathematics Subject Classification: Primary 34M10; Secondary 30D35.

Key words: Linear differential equation, growth estimate, Herold's comparison theorem, p-characteristic function.

(a) *If $R < \infty$, then there exist a constant $C_1 > 0$ with $C_1 \le C\sum_{j=0}^{k-1} |f^{(j)}(0)|$, and a constant $C_2 > 0$, such that*

$$|f(re^{i\theta})| \le \left(C_1 \max\{1, R^{k-1}\} + \frac{1}{(k-1)!} \int_0^r |A_k(se^{i\theta})|(R-s)^{k-1}\, ds \right) \cdot \exp\left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_0^r |A_j^{(n)}(se^{i\theta})|(R-s)^{k-j+n-1}\, ds \right)$$

for all $\theta \in [0, 2\pi)$ and $r \in [0, R)$.

(b) *If $R = \infty$, then there exist a constant $C_1 > 0$ with $C_1 \le C\sum_{j=0}^{k-1} \max_{|\zeta|=1} |f^{(j)}(\zeta)|$, and a constant $C_2 > 0$, such that*

$$|f(re^{i\theta})| \le r^{k-1} \left(C_1 + \frac{1}{(k-1)!} \int_0^r |A_k(se^{i\theta})|\, ds \right) \cdot \exp\left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_0^r |A_j^{(n)}(se^{i\theta})| s^{k-j+n-1}\, ds \right)$$

for all $\theta \in [0, 2\pi)$ and $r \in [1, \infty)$.

The proof of Theorem 1 is based on a representation theorem for the solutions of (1.1), see Theorem 9 below.

Example. The sharpness of Theorem 1(a) is illustrated as follows. The functions

$$f(z) = \frac{C_1}{(R-z)^2} \exp\left(\frac{1}{R-z}\right) + \frac{C_2}{(R-z)^2} + \frac{1}{R-z}, \quad C_1, C_2 \in \mathbf{C},$$

form the general solution of the equation

$$f'' + A_1(z)f' + A_0(z)f = A_2(z), \tag{1.2}$$

where the coefficients

$$A_0(z) = \frac{6(R-z)+2}{(R-z)^3}, \; A_1(z) = \frac{6(z-R)-1}{(R-z)^2} \; \text{and} \; A_2(z) = \frac{2(R-z)+1}{(R-z)^4} \tag{1.3}$$

are analytic in the disc D_R, $0 < R < \infty$. The growth rate of the maximum modulus $M(r, f)$ of any solution f of (1.2) is at most

$$M(r, f) = O\left(\frac{1}{(R-r)^2} \exp\left(\frac{1}{R-r}\right) \right), \quad r \to R^-, \tag{1.4}$$

which is of the same magnitude as the upper bound given by Theorem 1.

Theorem 2. *Let the coefficients $A_0(z), \ldots, A_k(z)$ of (1.1) be analytic in D_R, where $0 < R \le \infty$, and let f be a solution of (1.1). Let $n_c \in \{1, \ldots, k\}$ be the number of nonzero coefficients $A_0(z), \ldots, A_k(z)$, and let $\theta \in [0, 2\pi)$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in D_R$ is such that $A_j(z_\theta) \ne 0$ for some $j = 0, \ldots, k-1$, then, for all $r \in (\nu, R)$,*

$$|f(re^{i\theta})| \le C\left(\max_{0 \le x \le r} |A_k(xe^{i\theta})| + 1 \right) \exp\left(\delta r + n_c \int_0^r \max_{0 \le j \le k-1} |A_j(se^{i\theta})|^{\frac{1}{k-j}}\, ds \right), \tag{1.5}$$

where

$$C \leq (1+\varepsilon)\max\left\{n_c, \max_{0\leq j\leq k-1}\left\{\frac{|f^{(j)}(z_\theta)|}{n_c^j \max\limits_{0\leq n\leq k-1}|A_n(z_\theta)|^{\frac{j}{k-n}}}\right\}\right\} \tag{1.6}$$

and

$$\delta = \begin{cases} 0, & \text{if } A_k(z)\equiv 0,\\ 1, & \text{otherwise.}\end{cases} \tag{1.7}$$

The proof of Theorem 2 is based on Herold's comparison theorem, the application of which in its full generality yields an estimate for the growth of the derivatives of the solutions, see Theorem 5 below.

Example. The sharpness of Theorem 2 in the case $0 < R < \infty$ is illustrated as follows. The functions

$$f(z) = \frac{C_1}{(R-z)^2}\exp\left(\frac{1}{R-z}\right) + \frac{C_2}{(R-z)^2} + R - z, \quad C_1, C_2 \in \mathbf{C},$$

form the general solution of equation (1.2), where the coefficients $A_0(z)$ and $A_1(z)$ are as in (1.3), but now

$$A_2(z) = \frac{12(R-z)+3}{(R-z)^2}.$$

The maximal growth rate of any solution f of (1.2) is again at most as in (1.4), which is of the same magnitude as the upper bound given by Theorem 2.

It is worth noting that Theorem 2 yields well known sharp growth estimates for entire solutions of (1.1) with polynomial coefficients. These estimates are originally due to Gundersen, Steinbart and Wang, see Theorem A below.

The remainder of this paper is organized in the following way. Consequences of Theorems 1 and 2 are discussed in Sections 2 and 3, respectively. In particular, a new unit disc analogue of Theorem A is given in Theorem 8 below. Finally, Theorems 1 and 2 are proved in Sections 4 and 5, respectively.

2. Consequences of Theorem 1

Let f be analytic in D_R, where $0 < R \leq \infty$. For $1 \leq p < \infty$, we define a *p-characteristic* of f as

$$m_p(r,f) := \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+|f(re^{i\theta})|\right)^p d\theta\right)^{1/p}, \quad 0 \leq r < R,$$

generalizing the Nevanlinna proximity function $m(r,f) = m_1(r,f)$. Moreover, the element of the Lebesgue area measure on D_R is denoted by $d\sigma_z$.

Corollaries 3 and 4 below are generalizations of [8, Corollary 4.2] to the non-homogeneous case $A_k(z) \not\equiv 0$. See also [9, Lemma E].

Corollary 3. *Let the coefficients $A_0(z), \ldots, A_k(z)$ of (1.1) be analytic in D_R, where $0 < R \leq \infty$, and let f be a solution of (1.1). Let $1 \leq p < \infty$, and denote*

$$\Phi_p(r) = \begin{cases} \displaystyle\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_{D_r} |A_j^{(n)}(z)|^p (R-|z|)^{p(k-j+n-1)}\, d\sigma_z, & \text{if } R<\infty,\\ \displaystyle\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_{D_r} |A_j^{(n)}(z)|^p |z|^{p(k-j+n-1)}\, d\sigma_z, & \text{if } R=\infty. \end{cases}$$

(a) *If $R<\infty$, then there exist a constant $C_1>0$, depending on p, f and R, and a constant $C_2>0$, depending on p, such that*

$$m_p(r,f)^p \le C_1 + C_2\left(\int_0^{2\pi}\left(\log^+\int_0^r |A_k(se^{i\theta})|(R-s)^{k-1}\,ds\right)^p d\theta + \Phi_p(r)\right)$$

for all $r\in[0,R)$.

(b) *If $R=\infty$, then there exist a constant $C_1>0$, depending on p and f, and a constant $C_2>0$, depending on p, such that*

$$m_p(r,f)^p \le C_1 + C_2\left(\int_0^{2\pi}\left(\log^+\int_0^r |A_k(se^{i\theta})|s^{k-1}\,ds\right)^p d\theta + \Phi_p(r)\right)$$

for all $r\in[1,\infty)$.

Proof. Theorem 1(a) yields

$$\begin{aligned}\log^+|f(re^{i\theta})| \le D_1 &+ \log^+\int_0^r |A_k(se^{i\theta})|(R-s)^{k-1}\,ds\\ &+ D_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r |A_j^{(n)}(se^{i\theta})|(R-s)^{k-j+n-1}\,ds,\end{aligned}$$

where $D_1>0$ depends on f and R, and $D_2>0$. Raising both sides to the power p, using the Hölder inequality (if $p>1$), and integrating with respect to θ, it follows that

$$\begin{aligned}m_p(r,f)^p \le C_1 + D_3\Bigg(&\int_0^{2\pi}\left(\log^+\int_0^r |A_k(se^{i\theta})|(R-s)^{k-1}\,ds\right)^p d\theta\\ &+\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^{2\pi}\int_0^r |A_j^{(n)}(se^{i\theta})|^p(R-s)^{p(k-j+n-1)}\,ds\,d\theta\Bigg),\end{aligned}\tag{2.1}$$

where $C_1>0$ is as in the assertion, and $D_3>0$ depends on p. The assertion (a) now follows by applying the proof of [7, Lemma 4.6] to (2.1). The proof of the assertion (b) follows similarly using Theorem 1(b). □

Next, the expressions involving the function $A_k(z)$ in Corollary 3 are estimated upwards by using classical maximal theorems due to Hayman [6] (the case $p=1$) and Hardy–Littlewood [5] (the case $p>1$).

Corollary 4. *Suppose the assumptions in Corollary 3 hold, and let $0<\rho<R$.*

(a) *If $p=1$, then there exist a constant $C_1>0$, depending on f and R, and a constant $C_2>0$, such that*

$$m(r,f) \le C_1 + C_2\left(\log^+ r + \left(1+\log\frac{\rho+r}{\rho-r}\right)m(\rho,A_k) + \Phi_1(r)\right)$$

for all $r \in [0, \rho)$.

(b) *If* $p > 1$, *then there exist a constant* $C_1 > 0$, *depending on* p, f *and* R, *and a constant* $C_2 > 0$, *depending on* p, *such that*

$$m_p(r, f)^p \leq C_1 + C_2\big((\log^+ r)^p + m_p(r, A_k)^p + \Phi_p(r)\big)$$

for all $r \in [0, R)$.

Proof. (a) Since the function $A_k(z)$ is of bounded characteristic in D_ρ, [6, Theorem 1] yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\int_0^r |A_k(se^{i\theta})|\,ds\right)d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\max_{0\leq s\leq r}|A_k(se^{i\theta})|\right)d\theta + \log^+ r\\ &\leq \left(1+\frac{1}{\pi}\log\frac{\rho+r}{\rho-r}\right)m(\rho, A_k) + \log^+ r\end{aligned}$$

for all $r \in [0, \rho)$. The desired estimate for $m(r, f)$ now follows by Corollary 3.

(b) By [5, Theorem 17],

$$\begin{aligned}&\int_0^{2\pi}\left(\log^+\int_0^r |A_k(se^{i\theta})|\,ds\right)^p d\theta\\ &\leq 2^{p-1}\int_0^{2\pi}\left(\log^+\max_{0\leq s\leq r}|A_k(se^{i\theta})|\right)^p d\theta + 2^p\pi(\log^+ r)^p\\ &= 2^{p-1}\int_0^{2\pi}\left(\log^+\max_{0\leq t\leq 1}|A_k(tre^{i\theta})|\right)^p d\theta + 2^p\pi(\log^+ r)^p\\ &\leq C(p)m_p(r, A_k)^p + 2^p\pi(\log^+ r)^p\end{aligned}$$

for all $r \in [0, R)$. The estimate for $m_p(r, f)^p$ now follows by Corollary 3. □

3. Consequences of Theorem 2

An application of Herold's comparison theorem in its full generality at the end of the proof of Theorem 2 yields the following pointwise growth estimate for the derivatives of solutions of (1.1).

Theorem 5. *Suppose the assumptions in Theorem 2 hold. Then, for all* $r \in (\nu, R)$ *and* $j = 0, \ldots, k-1$,

$$|f^{(j)}(re^{i\theta})| \leq CH_\theta(r)G_\theta(r)^j \exp\left(\delta r + n_c\int_0^r \max_{0\leq j\leq k-1}|A_j(se^{i\theta})|^{\frac{1}{k-j}}\,ds\right),$$

where

$$\begin{aligned}H_\theta(r) &= \max_{0\leq x\leq r}|A_k(xe^{i\theta})| + 1\\ G_\theta(r) &= \max_{\nu\leq t\leq r}\left\{1 + n_c\max_{0\leq n\leq k-1}|A_n(te^{i\theta})|^{\frac{1}{k-n}} + \frac{H_\theta^*(t)}{H_\theta(t)}\right\},\\ H_\theta^*(t) &= \lim_{s\to t^-} H'_\theta(s),\end{aligned}$$

and all the other expressions are as in Theorem 2.

By definition, $H_\theta^*(t) = H'_\theta(t)$ when the derivative H'_θ exists, and $H_\theta^*(t) = 0$ otherwise. If the derivative of H_θ exists at every point on the interval (ν, R), then the assertion in Theorem 5 follows directly by the proof of Theorem 2, and by Theorem B below. Noting that there can be at most finitely many points on $[\nu, r]$ at which the derivative of H_θ does not exist, the general assertion in Theorem 5 follows by continuity.

It is worth noting that if the coefficient functions $A_0(z), \dots, A_k(z)$ in (1.1) grow "slowly" in a finite disc, then Theorem 5 gives better growth estimates for the derivatives of solutions of (1.1) than what Cauchy's integral formula applied to the estimate in Theorem 2 would give.

Next, some further consequences of Theorem 2 will be obtained. Corollaries 6 and 7 below are generalizations of [8, Corollary 5.3] to the non-homogeneous case $A_k(z) \not\equiv 0$. See also [9, Lemma F].

Corollary 6. *Let the coefficients $A_0(z), \dots, A_k(z)$ of (1.1) be analytic in D_R, where $0 < R \le \infty$, and let f be a solution of (1.1). Let $1 \le p < \infty$, and denote*

$$\Psi_p(r) = \sum_{j=0}^{k-1} \int_{D_r} |A_j(z)|^{\frac{p}{k-j}} \, d\sigma_z.$$

Then there exist a constant $C_1 > 0$, depending on p and f, and a constant $C_2 > 0$, depending on p, such that, for all $r \in [0, R)$,

$$m_p(r,f)^p \le C_1 + C_2 \left(\delta r^p + \int_0^{2\pi} \left(\log^+ \max_{0 \le x \le r} |A_k(xe^{i\theta})| \right)^p d\theta + \Psi_p(r) \right),$$

where δ is the constant in (1.7).

Proof. Theorem 2 yields, for all $r \in [0, R)$,

$$\log^+ |f(re^{i\theta})| \le D_1 + \delta r + \log^+ \max_{0 \le x \le r} |A_k(xe^{i\theta})| + n_c \sum_{j=0}^{k-1} \int_0^r |A_j(se^{i\theta})|^{\frac{1}{k-j}} \, ds, \tag{3.1}$$

where $D_1 > 0$ depends on f. Raising both sides to the power p, using the Hölder inequality (if $p > 1$), and integrating with respect to θ, it follows, for all $r \in [0, R)$, that

$$\begin{aligned} m_p(r,f)^p \le C_1 + D_2 \Bigg(& \delta r^p + \int_0^{2\pi} \left(\log^+ \max_{0 \le x \le r} |A_k(xe^{i\theta})| \right)^p d\theta \\ & + \sum_{j=0}^{k-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{p}{k-j}} \, ds \, d\theta \Bigg), \end{aligned} \tag{3.2}$$

where $C_1 > 0$ is as in the assertion, and $D_2 > 0$ depends on p. The assertion now follows by applying the proof of [7, Lemma 4.6] to (3.2). $\square$

It is worth noting that the corresponding area integrals in $\Phi_p(r)$ and $\Psi_p(r)$ of Corollaries 3 and 6 are not of the same growth in general. This has been discussed in detail in the case of the unit disc in [9, Section 4].

Corollary 7. *Suppose the assumptions in Corollary 6 hold, and let $0 < \rho < R$. Then there exist a constant $C_1 > 0$, depending on p and f, and a constant $C_2 > 0$, depending on p, such that for all $r \in [0, \rho)$,*

$$m(r,f) \le C_1 + C_2 \left(\delta r + \left(1 + \log \frac{\rho + r}{\rho - r} \right) m(\rho, A_k) + \Psi_1(r) \right),$$

and for all $r \in [0, R)$,

$$m_p(r,f)^p \le C_1 + C_2\big(\delta r^p + m_p(r,A_k)^p + \Psi_p(r)\big), \quad p>1,$$

where δ *is the constant in* (1.7).

The proof of Corollary 7 is similar to that of Corollary 4 using the maximal theorems due to Hardy–Littlewood [5] and Hayman [6].

The next result is due to Gundersen, Steinbart and Wang, see [3, Lemma 6] and [4, Lemma 1].

Theorem A. *Let the coefficients* $A_0(z),\dots,A_{k-1}(z)$ *of* (1.1) *be polynomials, and let* $A_k(z)$ *be an entire function. Then all solutions* $f \not\equiv 0$ *of* (1.1) *satisfy*

$$\rho(A_k) \le \rho(f) \le \max\left\{\rho(A_k), \max_{0\le j\le k-1}\left\{\frac{\deg(A_j)}{k-j}\right\}+1\right\}, \tag{3.3}$$

where

$$\rho(g) = \limsup_{r\to\infty}\frac{\log m(r,g)}{\log r}$$

is the order of growth of an entire function g. *Moreover, there exists a solution* f_0 *of* (1.1) *satisfying*

$$\rho(f_0) = \max\left\{\rho(A_k), \max_{0\le j\le k-1}\left\{\frac{\deg(A_j)}{k-j}\right\}+1\right\}. \tag{3.4}$$

As noted in [3], the first inequality in (3.3) follows from an elementary order consideration on both sides of equation (1.1). An alternative proof for the second inequality in (3.3) can be obtained by applying either Corollary 6 or Corollary 7. The validity of (3.4) illustrates the sharpness of Corollaries 6 and 7 in the case when $R=\infty$.

The expression $\frac{\deg(A_j)}{k-j}+1$ in Theorem A can be written as

$$\alpha_j = \frac{\deg(A_j)}{k-j}+1 = \limsup_{r\to\infty}\frac{\log\left(r\displaystyle\int_0^{2\pi}|A_j(re^{i\theta})|^{\frac{1}{k-j}}\,d\theta\right)}{\log r}.$$

The statement (3.3), for example, then reads as

$$\rho(A_k) \le \rho(f) \le \max\left\{\rho(A_k), \max_{0\le j\le k-1}\{\alpha_j\}\right\}.$$

This gives rise to the following unit disc analogue of Theorem A.

Theorem 8. *Let the coefficients* $A_0(z),\dots,A_k(z)$ *of* (1.1) *be analytic in* D. *For each* $j=0,\dots,k-1$, *define*

$$\alpha_j = \limsup_{r\to 1^-}\frac{\log\left((1-r)\displaystyle\int_0^{2\pi}|A_j(re^{i\theta})|^{\frac{1}{k-j}}\,d\theta\right)}{-\log(1-r)},$$

and suppose that

$$\max_{0\le j\le k-1}\{\alpha_j\} < \infty. \tag{3.5}$$

Then all solutions $f \not\equiv 0$ *of* (1.1) *satisfy*

$$\rho(A_k) \le \rho(f) \le \max\left\{\rho(A_k),\ \max_{0\le j\le k-1}\{\alpha_j\}\right\}, \tag{3.6}$$

where

$$\rho(g) = \limsup_{r\to 1^-} \frac{\log^+ m(r,g)}{-\log(1-r)}$$

is the order of growth of a function g *analytic in* D*. Moreover, there exists a solution* f_0 *of* (1.1) *satisfying*

$$\rho(f_0) = \max\left\{\rho(A_k),\ \max_{0\le j\le k-1}\{\alpha_j\}\right\}. \tag{3.7}$$

Proof. The coefficients $A_0(z),\dots,A_{k-1}(z)$ are of order of growth zero by (3.5). Hence the first inequality in (3.6) follows from an elementary order consideration on both sides of equation (1.1). The second inequality in (3.6) follows by applying Corollary 7. Thus it remains to show that the statement (3.7) holds.

For convenience, set $\alpha = \max_{0\le j\le k-1}\{\alpha_j\}$. If $\rho(A_k) \ge \alpha$, then $\rho(f) = \rho(A_k)$ holds for any solution f of (1.1) by (3.6). Assume then that $\rho(A_k) < \alpha$. Let f be a solution of (1.1). If $\rho(f) = \alpha$, then the statement (3.7) holds. Noting that $\alpha > 0$, suppose now that f satisfies $\rho(f) < \alpha$. By [2, Theorem 1], the homogeneous equation corresponding to (1.1) possesses at least one solution g such that $\rho(g) = \alpha$. Then $f + g$ is a solution of (1.1), and $\rho(f + g) = \alpha$. Hence the statement (3.7) holds in this case also. □

Assumption (3.5) is natural since its plane analogue in Theorem A is that the polynomial coefficients are of finite degree. The validity of (3.7) illustrates the sharpness of Corollary 7 in the case when $R = 1$.

If the assumptions in Theorem A hold, then there always exists at least one solution f_0 of (1.1) such that

$$\rho(f_0) \ge \max_{0\le j\le k-1}\left\{\frac{\deg(A_j)}{k-j}\right\} + 1 \ge 1.$$

The lower bound 1 can be reached if all of the coefficients $A_0(z),\dots,A_{k-1}(z)$ are constant functions. If at least one of the coefficients $A_0(z),\dots,A_{k-1}(z)$ is transcendental entire, then it can be deduced via the classical theorem of Frei [11, Theorem 4.2] that there exists at least one solution f_0 of (1.1) of infinite order of growth. Therefore, the terms involving "$\log r$" in Corollary 4 can be considered as small error terms. An analogous observation holds true for the terms involving "r" in Corollaries 6 and 7.

4. Proof of Theorem 1

The proof of Theorem 1 is based on the following representation theorem for the solutions of (1.1).

Theorem 9. *Let the coefficients* $A_0(z),\dots,A_k(z)$ *of* (1.1) *be analytic in* D_R*, where* $0 < R \le \infty$*, and let* f *be a solution of* (1.1)*. Then, for any* $z, z_0 \in D_R$*,*

$$f(z) = \sum_{n=0}^{k-1} c_n (z-z_0)^n + \frac{1}{(k-1)!}\int_{z_0}^{z} A_k(\xi)(z-\xi)^{k-1}\, d\xi$$

$$+\sum_{j=0}^{k-1}\sum_{n=0}^{j} d_{j,n}\int_{z_0}^{z} A_j^{(n)}(\xi)f(\xi)(z-\xi)^{k-j+n-1}\,d\xi,$$

where the constants $c_n \in \mathbf{C}$ depend on the initial values $f(z_0), f'(z_0), \ldots, f^{(k-1)}(z_0)$, the constants $d_{j,n} \in \mathbf{Q}$, and the path of integration is a piecewise smooth curve in D_R joining z and z_0.

Theorem 9 can be proved by following the proof of [8, Theorem 3.1], and therefore the details are omitted.

The assertions in Theorem 1 are now proved separately.

(a) Theorem 9, in the case when $z_0 = 0$ and the path of integration is the line segment $[0, z]$, yields

$$\begin{aligned}|f(re^{i\theta})| &\le C_1\max\{1, R^{k-1}\} + \frac{1}{(k-1)!}\int_0^r |A_k(se^{i\theta})|(R-s)^{k-1}\,ds\\ &\quad+\int_0^r\left(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|(R-s)^{k-j+n-1}\right)|f(se^{i\theta})|\,ds,\end{aligned}\tag{4.1}$$

where $C_1 > 0$ satisfies

$$C_1 \le C\sum_{j=0}^{k-1}|f^{(j)}(0)|,$$

and $C_2 = \max\{|d_{j,n}|\} > 0$. The assertion (a) now follows by applying the Bellman inequality [12, Theorem 1.3.1] to (4.1).

(b) Similarly as above, with $z = re^{i\theta}$, $r > 1$, $z_0 = e^{i\theta}$ and the path of integration being the line segment $[e^{i\theta}, z]$, we obtain

$$\begin{aligned}|f(re^{i\theta})| &\le C_1 r^{k-1} + \frac{1}{(k-1)!}\int_1^r |A_k(se^{i\theta})|(r-s)^{k-1}\,ds\\ &\quad+\int_1^r\left(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|(r-s)^{k-j+n-1}\right)|f(se^{i\theta})|\,ds,\end{aligned}\tag{4.2}$$

where $C_1 > 0$ satisfies

$$C_1 \le C\sum_{j=0}^{k-1}|f^{(j)}(e^{i\theta})| \le C\sum_{j=0}^{k-1}\max_{|\zeta|=1}|f^{(j)}(\zeta)|,$$

and $C_2 = \max\{|d_{j,n}|\} > 0$. The estimate in (4.2) holds when $r = 1$ as well. Note that, for all $1 \le s \le r$, $j \in \{0, \ldots, k-1\}$ and $n \in \{0, \ldots, j\}$,

$$\frac{(r-s)^{k-j+n-1}}{r^{k-1}} \le \frac{1}{r^{j-n}} \le \frac{s^{k-j+n-1}}{s^{k-1}}.\tag{4.3}$$

Dividing (4.2) by r^{k-1} and using (4.3), it follows that

$$\begin{aligned}\frac{|f(re^{i\theta})|}{r^{k-1}} &\le C_1 + \frac{1}{(k-1)!}\int_1^r |A_k(se^{i\theta})|\,ds\\ &\quad+\int_1^r\left(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|s^{k-j+n-1}\right)\frac{|f(se^{i\theta})|}{s^{k-1}}\,ds.\end{aligned}\tag{4.4}$$

The assertion (b) now follows by applying the Bellman inequality [12, Theorem 1.3.1] to (4.4).

5. Proof of Theorem 2

The proof of Theorem 2 is based on the following version of Herold's comparison theorem [10, Satz 1] which can be verified by a careful examination of the original proof.

Theorem B. *Let $p_0(x), \dots, p_k(x)$ be complex valued functions defined on $[a,b)$, let $E \subset [a,b)$ be a finite set of points, and let $P_0(x), \dots, P_k(x)$ be real valued non-negative functions such that $|p_j(x)| \le P_j(x)$ for all $x \in [a,b)\setminus E$. Moreover, let $P_j(x)$ be continuous for all $x \in [a,b)\setminus E$. If $v(x)$ is a solution of the differential equation*

$$v^{(k)} - \sum_{j=1}^{k} p_{k-j}(x) v^{(k-j)} = p_k(x),$$

and $V(x)$ satisfies

$$V^{(k)} - \sum_{j=1}^{k} P_{k-j}(x) V^{(k-j)} = P_k(x)$$

on $[a,b)\setminus E$, where

$$|v^{(j)}(a)| \le V^{(j)}(a), \quad j = 0, \dots, k-1,$$

then

$$|v^{(j)}(x)| \le V^{(j)}(x), \quad j = 0, \dots, k-1,$$

for all $x \in [a,b)\setminus E$.

It is now proceeded to prove Theorem 2. By [8, Theorem 3.1] it can be assumed that $A_k(z) \not\equiv 0$. Denote

$$H_\theta(x) = \max_{0 \le r \le x} |A_k(re^{i\theta})| + 1, \quad 0 \le x < R.$$

Since $A_k(z)$ is analytic in D_R, it follows that H_θ is a non-decreasing continuous function which is differentiable with respect to x outside of the countable set E which satisfies $\#\{E \cap [0,r]\} < \infty$ for any $0 < r < R$.

Let first $r \in (\nu, R) \setminus E$, and denote

$$h_\theta(x) = \frac{1}{n_c} + \max_{0 \le j \le k-1} |A_j(xe^{i\theta})|^{\frac{1}{k-j}}, \quad 0 \le x < R. \tag{5.1}$$

Take ρ such that $r < \rho < R$, and let $\varepsilon_0 > 0$. Define $H_\theta^*(x) = H_\theta'(x)$ when $x \notin E$, and $H_\theta^*(x) = 0$ otherwise. Then the function $n_c h_\theta(x) + H_\theta^*(x)/H_\theta(x)$ is Riemann integrable on $[\nu, \rho]$. Thus there exists a partition $P = \{\nu = x_0, x_1, \dots, x_{n-1}, x_n = \rho\}$ of $[\nu, \rho]$ such that $E \subset P$, $x_j \ne r$ for all $j = 0, \dots, n$, and

$$U\left(P,\, n_c h_\theta + \frac{H_\theta^*}{H_\theta}\right) - \int_\nu^\rho \left(n_c h_\theta(s) + \frac{H_\theta^*(s)}{H_\theta(s)}\right) ds < \varepsilon_0, \tag{5.2}$$

where $U(P,\, n_c h_\theta + H_\theta^*/H_\theta)$ is the upper Riemann sum of the function $n_c h_\theta + H_\theta^*/H_\theta$, corresponding to the partition P. Define the auxiliary function $g_\theta : [\nu, \rho] \to \mathbf{R}_+$ by

$$g_\theta(t) = \sup_{x_j < x < x_{j+1}} \left\{ n_c h_\theta(x) + \frac{H_\theta^*(x)}{H_\theta(x)} \right\}, \quad x_j \le t < x_{j+1}, \quad j = 0, \dots, n-1.$$

Then $g_\theta(t)$ is a step function which satisfies

$$g_\theta(t) \geq n_c h_\theta(t) + \frac{H_\theta^*(t)}{H_\theta(t)} \geq 1, \quad \nu \leq t \leq \rho. \tag{5.3}$$

Moreover,

$$U\left(P,\, n_c h_\theta + \frac{H_\theta^*}{H_\theta}\right) = \int_\nu^\rho g_\theta(s)\,ds,$$

and so, by (5.2),

$$\int_\nu^r g_\theta(s)\,ds < \int_\nu^r \left(n_c h_\theta(s) + \frac{H_\theta^*(s)}{H_\theta(s)}\right) ds + \varepsilon_0. \tag{5.4}$$

By (5.3), the constant

$$C_0 = \max\left\{n_c, \max_{0\leq j\leq k-1}\left\{\frac{|f^{(j)}(z_\theta)|}{g_\theta(\nu)^j}\right\}\right\} \tag{5.5}$$

is well-defined, and it satisfies

$$C_0 \leq \max\left\{n_c, \max_{0\leq j\leq k-1}\left\{\frac{|f^{(j)}(z_\theta)|}{n_c^j \max\limits_{0\leq n\leq k-1} |A_n(z_\theta)|^{\frac{j}{k-n}}}\right\}\right\}. \tag{5.6}$$

Define the auxiliary function

$$V(t) = C_0 H_\theta(\nu)\exp\left(\int_\nu^t g_\theta(s)\,ds\right), \quad \nu \leq t < \rho,$$

and the constants

$$\delta_j = \begin{cases} 0, & \text{if } A_j(z) \equiv 0, \\ 1, & \text{otherwise,} \end{cases}$$

where $j = 0, \ldots, k-1$. Then, since $g_\theta^{(l)}(t) \equiv 0$ for all $t \in [\nu, \rho)\backslash P$ when $l \geq 1$, $V(t)$ satisfies the differential equation

$$V^{(k)} - \sum_{j=1}^{k} \frac{g_\theta(t)^j \delta_{k-j}}{n_c} V^{(k-j)} = \frac{C_0 H_\theta(\nu) g_\theta(t)^k}{n_c} \exp\left(\int_\nu^t g_\theta(s)\,ds\right)$$

on $[\nu, \rho) \setminus P$. Since

$$|f^{(j)}(\nu e^{i\theta})| = |f^{(j)}(z_\theta)| \leq C_0 g_\theta(\nu)^j, \quad j = 0, \ldots, k-1,$$

by (5.5), and since $H_\theta(\nu) \geq 1$, it follows that

$$|f^{(j)}(\nu e^{i\theta})| \leq V^{(j)}(\nu) = C_0 H_\theta(\nu) g_\theta(\nu)^j, \quad j = 0, \ldots, k-1.$$

Clearly, $v(t) = f(te^{i\theta})$ solves the equation

$$v^{(k)} + p_{k-1}(t)v^{(k-1)} + \cdots + p_0(t)v = p_k(t),$$

where

$$p_j(t) = \begin{cases} e^{i(k-j)\theta} A_j(te^{i\theta}), & j = 0, \ldots, k-1 \\ e^{ik\theta} A_k(te^{i\theta}), & j = k. \end{cases}$$

Using (5.1), (5.3) and (5.5), the coefficients $p_j(t)$ satisfy

$$|p_j(t)| = |A_j(te^{i\theta})| \leq \frac{g_\theta(t)^{k-j}\delta_j}{n_c}, \quad j = 0, \ldots, k-1,$$

and

$$\begin{aligned}|p_k(t)| = |A_k(te^{i\theta})| &\le H_\theta(t) = H_\theta(\nu)\exp\left(\int_\nu^t \frac{H_\theta^*(s)}{H_\theta(s)}\,ds\right)\\ &\le \frac{C_0 H_\theta(\nu) g_\theta(t)^k}{n_c}\exp\left(\int_\nu^t g_\theta(s)\,ds\right).\end{aligned}$$

Moreover,

$$|v^{(j)}(\nu)| \le V^{(j)}(\nu), \quad j = 0,\dots,k-1.$$

It now follows, by Theorem B and (5.4), that

$$\begin{aligned}|f(re^{i\theta})| = |v(r)| \le V(r) &= C_0 H_\theta(\nu)\exp\left(\int_\nu^r g_\theta(s)\,ds\right)\\ &\le C_0 H_\theta(\nu)\exp\left(\int_\nu^r \left(n_c h_\theta(s) + \frac{H_\theta^*(s)}{H_\theta(s)}\right) ds + \varepsilon_0\right)\\ &\le C H_\theta(\nu)\exp\left(\int_\nu^r \frac{H_\theta^*(s)}{H_\theta(s)}\,ds\right)\exp\left(n_c\int_\nu^r h_\theta(s)\,ds\right)\\ &= C H_\theta(r)\exp\left(n_c\int_\nu^r h_\theta(s)\,ds\right), \quad r \in (\nu, R)\setminus E,\end{aligned}$$

where, by (5.6) and choosing ε_0 to be sufficiently small, C satisfies (1.6). If $r \in E$ the assertion follows by continuity.

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Received 28 November 2007