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AN APPLICATION OF THE TOPOLOGICAL RIGIDITY OF THE SINE FAMILY

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Abstract. By using a result of Domínguez and Sienra on the topological rigidity of the Sine family, we give a different proof of a result in [8] which says that, for any bounded type irrational number $0 < \theta < 1$, the boundary of the Siegel disk of $e^{2\pi i\theta}\sin(z)$ is a quasi-circle passing through exactly two critical points $\pi/2$ and $-\pi/2$.

1. Introduction

Let $f\colon \mathbf{C} \to \mathbf{C}$ and $g\colon \mathbf{C} \to \mathbf{C}$ be two continuous maps. We say that f and g are topologically equivalent to each other if there are two homeomorphisms $\theta_1, \theta_2\colon \mathbf{C} \to \mathbf{C}$ such that $f = \theta_1^{-1} \circ g \circ \theta_2$. The following lemma on the topological rigidity of the Sine family was proved by Domínguez and Sienra(Lemma 1, [3]).

Lemma 1.1. *Let f be an entire function. If $f(z)$ is topologically equivalent to $\sin(z)$, then $f(z) = a + b\sin(cz + d)$ where $a, b, c, d \in \mathbf{C}$, and $b, c \neq 0$.*

The main purpose of this paper is to use this lemma to give a new but simpler proof of the following result, which was previously proved in [8].

Theorem. Let $0 < \theta < 1$ be a bounded type irrational number. Then the boundary of the Siegel disk of $f_\theta(z) = e^{2\pi i\theta}\sin(z)$ is a quasi-circle which passes through exactly two critical points $\pi/2$ and $-\pi/2$.

Here is the idea of the new proof. Following the idea of Cheritat [2], we first construct a Ghys-like model $G_\theta(z)$ for the map $e^{2\pi i\theta}\sin(z)$. Next we do a standard quasi-conformal surgery on the model map $G_\theta(z)$ and get an entire function $g_\theta(z)$. We then derive the theorem from Lemma 1.1 and the fact that g_θ and f_θ are topologically equivalent to each other.

It is interesting to contrast the proof here with the one in [8]. The proof in [8] is based on a non-symmetric model map $\widetilde{f_\theta}(z)$. One of the most important characteristics of this model map is its periodicity which plays a key role in the proof there. In this paper, we use the symmetric model map G_θ, which does not have the periodicity. A priori, the resulted entire map g_θ may not be periodic either.

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But because of the topological rigidity of the Sine family, the map g_θ turns out to be equal to f_θ, and this implies the theorem.

I would like to mention that by using trans-quasiconformal surgery introduced in [7] and the techniques in this paper, the theorem was recently extended to David type Siegel disks of the Sine family[9].

2. A Ghys-like model

We use the idea of Cheritat in the following construction (see [2]). Let Δ be the unit disk and $T(z) = \sin(z)$. It follows that the map $T(z)$ has exactly two critical values 1 and -1. Let D be the component of $T^{-1}(\Delta)$ which contains the origin. The following lemma is obvious and we leave the proof to the reader.

Lemma 2.1. *D is a Jordan domain which is symmetric about the origin. Moreover, ∂D passes through exactly two critical points $\pi/2$ and $-\pi/2$, and the map $T|\partial D\colon \partial D \to \partial\Delta$ is a homeomorphism.*

For $k \in \mathbf{Z}$, let $D_k = \{z + k\pi \mid z \in D\}$. It follows that $D_0 = D$. Note that for any two distinct integers k and j, if $\partial D_k \cap \partial D_j \neq \emptyset$, then any point in $\partial D_k \cap \partial D_j$ must be a critical point of $\sin(z)$. This, together with Lemma 2.1, implies

Lemma 2.2. *For any $k \in \mathbf{Z}$, $\partial D_k \cap \partial D_{k+1} = \{k\pi + \pi/2\}$. For any two distinct integers k and j with $|k - j| \neq 1$, $\partial D_k \cap \partial D_j = \emptyset$. In particular, if $\Lambda \subset \mathbf{Z}$ contains infinitely many elements but $\Lambda \neq \mathbf{Z}$, then $\bigcup_{k\in\Lambda} \partial D_k$ is disconnected.*

Let $\phi\colon \widehat{\mathbf{C}} - \overline{\Delta} \to \widehat{\mathbf{C}} - \overline{D}$ be the Riemann map such that $\phi(\infty) = \infty$ and $\phi(1) = \pi/2$. Since Δ and D are both symmetric about the origin, we have

Lemma 2.3. *ϕ is odd.*

For $z \in \mathbf{C}$, let z^* denote the symmetric image of z about the unit circle. Define

$$G(z) = \begin{cases} T \circ \phi(z) & \text{for } z \in \mathbf{C} - \Delta, \\ (T \circ \phi(z^*))^* & \text{for } z \in \Delta - \{0\}. \end{cases} \tag{1}$$

From the construction of $G(z)$, we have

Lemma 2.4. *$G(z)$ is holomorphic in $\mathbf{C} - \{0\}$ and is symmetric about the unit circle. Moreover, $G(z)$ is odd.*

By Lemma 2.1, we see that $G|\partial\Delta\colon \partial\Delta \to \partial\Delta$ is a critical circle homeomorphism. By Proposition 11.1.9 of [5], we get

Lemma 2.5. *There exists a unique $t \in [0, 1)$ such that $e^{2\pi i t} G|\partial\Delta\colon \partial\Delta \to \partial\Delta$ is a critical circle homeomorphism of rotation number θ.*

Let $t \in [0, 1)$ be the number given in Lemma 2.5. Let us denote $e^{2\pi i t}G(z)$ by $G_\theta(z)$. By Herman–Swiatek's quasi-symmetric linearization theorem on critical circle mappings [6], it follows that there is a quasi-symmetric homeomorphism $h\colon \partial\Delta \to \partial\Delta$ such that $h(1) = 1$ and $G_\theta|\partial\Delta = h^{-1} \circ R_\theta \circ h$, where R_θ is the rigid rotation given by θ.

Lemma 2.6. *G_θ and h are both odd.*

Proof. The assertion that G_θ is odd follows from that $G(z)$ is odd (Lemma 2.4). Now let us prove that h is odd. First let us show that $h(-1) = -1$. Let $U(N)$ be the number of the points in $\{G_\theta^k(1) \,|\, k = 1, \cdots, N\}$ which lie in the upper half circle. Let $L(N)$ be the number of the points in $\{G_\theta^k(-1) \,|\, k = 1, \cdots, N\}$ which lie in the lower half circle. Note that G_θ is odd, it follows that $U(N) = L(N)$. Since the angle length of the image of the upper half circle under h is equal to the limit of $2\pi U(N)/N$ as $N \to \infty$, and the angle length of the image of the lower half circle under h is equal to the limit of $2\pi L(N)/N$ as $N \to \infty$, it follows that the angle length of the images of the upper half circle and the lower half circle under h are equal to each other. This implies that $h(-1) = -1$.

To show that h is odd, let $t(z) = -h(-z)$. We have $t(1) = 1 = h(1)$. Since

$$t^{-1} \circ R_\theta \circ t(z) = -G_\theta|\partial\Delta(-z) = G_\theta|\partial\Delta(z),$$

it follows that $t = h$. This proves Lemma 2.6. □

Let $H\colon \overline{\Delta} \to \overline{\Delta}$ be the Douady–Earle extension of h. We refer the reader to [4] for the definition and properties of Douady–Earle extension. It follows that H is odd also. In particular, $H(0) = 0$. Define

$$\widetilde{G}_\theta(z) = \begin{cases} G_\theta(z) & \text{for } z \in \mathbf{C} - \Delta, \\ H^{-1} \circ R_\theta \circ H(z) & \text{for } z \in \Delta. \end{cases} \tag{2}$$

For $k \in \mathbf{Z}$, let $\Delta_k = \phi^{-1}(D_k)$. Note that $\Delta_0 = \Delta$.

Lemma 2.7. *$\widetilde{G}_\theta$ is odd. The critical set of $\widetilde{G}_\theta$ is contained in the set $\widetilde{G}_\theta^{-1}(\partial\Delta) = \bigcup_{k\in\mathbf{Z}} \partial\Delta_k$, and moreover, if $\Lambda \subset \mathbf{Z}$ contains infinitely many elements but $\Lambda \neq \mathbf{Z}$, then the set $\bigcup_{k\in\Lambda} \partial\Delta_k$ is disconnected.*

Proof. The first assertion follows from the construction of $\widetilde{G}_\theta$. The second one follows from Lemma 2.2. □

Let ν_0 be the complex structure in Δ which is the pull back of the standard complex structure by H. Since H is odd, we have

Lemma 2.8. *$\nu_0(-z) = \nu_0(z)$.*

Now we can define a $\widetilde{G}_\theta(z)$-invariant complex structure ν on the complex plane. The procedure is standard. For $z \in \Delta$, define $\nu = \nu_0$. For $z \notin \Delta$, there are two cases. In the first case, there is some integer $m \geq 1$ such that $\widetilde{G}_\theta^m(z) \in \Delta$. In this case, define $\nu(z)$ to be the pull back of $\nu_0(\widetilde{G}_\theta^m(z))$ by $\widetilde{G}_\theta^m$. In the second case, the forward orbit of z does not intersect the unit disk. In this case, define $\nu(z) = 0$. Since $\widetilde{G}_\theta$ is odd, by Lemma 2.8, we have

Lemma 2.9. *$\nu(-z) = \nu(z)$.*

Now by Ahlfors–Bers's theorem [1], there is a unique quasi-conformal homeomorphism ψ of the Riemann sphere which solves the Beltrami equation given by ν, and which fixes 0 and the infinity, and maps 1 to $\pi/2$.

Lemma 2.10. *ψ is odd.*

Proof. Let $r(z) = -\psi(-z)$. Let ν_r and ν_ψ denote the dilations of r and ψ, respectively. By Lemma 2.9, it follows that $\nu_r(z) = \nu_\psi(z)$. Since $r(0) = \psi(0) = 0$ and $r(\infty) = \psi(\infty) = \infty$, we get that $r(z) = a\psi(z)$ for some constant a. This implies that $\psi(z) = \psi(-(-z)) = -a\psi(-z) = a^2\psi(z)$. It follows that $a^2 = 1$. We then have $a = 1$ or $a = -1$. If $a = -1$, we get $\psi(-z) = \psi(z)$ for all z. This is impossible since $\psi(z)$ is a homeomorphism. Therefore $a = 1$. The Lemma follows. $\square$

Let $g_\theta(z) = \psi \circ \widetilde{G}_\theta \circ \psi^{-1}(z)$ and let $\Omega = \psi(\Delta)$. By the construction, we get

Lemma 2.11. *$g_\theta(z)$ is an odd entire function which has a Siegel disk Ω centered at the origin with rotation number θ. Moreover, Ω is symmetric about the origin, and $\partial\Omega$ is a quasi-circle passing through exactly two critical points $\pi/2$ and $-\pi/2$.*

For $k \in \mathbf{Z}$, let $\Omega_k = \psi(\Delta_k)$. It follows that $\Omega = \Omega_0$ and each Ω_k is a component of $g_\theta^{-1}(\Omega_0)$. By Lemma 2.7, we get

Lemma 2.12. *The critical set of g_θ is contained in the set $g_\theta^{-1}(\partial\Omega_0) = \bigcup_{k\in\mathbf{Z}} \partial\Omega_k$. Moreover, if $\Lambda \subset \mathbf{Z}$ contains infinitely many elements but $\Lambda \neq \mathbf{Z}$, then the set $\bigcup_{k\in\Lambda} \partial\Omega_k$ is disconnected.*

3. Topological equivalence

Lemma 3.1. *Let $f\colon \mathbf{C} \to \mathbf{C}$ and $g\colon \mathbf{C} \to \mathbf{C}$ be two continuous maps such that $f = g$ on the outside of the unit disk. If in addition, $f\colon \overline{\Delta} \to \overline{\Delta}$ and $g\colon \overline{\Delta} \to \overline{\Delta}$ are both homeomorphisms, then f and g are topologically equivalent to each other.*

Proof. Define $\theta_2(z) = z$ for $z \notin \Delta$ and $\theta_2(z) = g^{-1} \circ f(z)$ for $z \in \Delta$. It follows that $\theta_2\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism. Let $\theta_1 = \mathrm{id}$. Then $f = \theta_1^{-1} \circ g \circ \theta_2$. The Lemma follows. $\square$

Let $\phi\colon \widehat{\mathbf{C}} - \overline{\Delta} \to \widehat{\mathbf{C}} - \overline{D}$ be map in the definition of $G(z)$. Let $\eta\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a homeomorphic extension of ϕ. As before let $T(z) = \sin(z)$. It follows that $T(z)$ is topologically equivalent to $T \circ \eta$. Let $t \in [0, 1)$ be the number in Lemma 2.5. Let $S(z) = e^{2\pi i t} T \circ \eta(z)$. It follows that $S(z)$ is topologically equivalent to $T(z)$. By Lemma 3.1, we have

Lemma 3.2. *$S(z)$ is topologically equivalent to $\widetilde{G}_\theta(z)$.*

Lemma 3.3. *$g_\theta(z)$ is topologically equivalent to $T(z)$.*

Proof. By the construction of g_θ, it follows that g_θ is topologically equivalent to $\widetilde{G}_\theta$. The Lemma then follows from Lemma 3.2. $\square$

Proof of the Theorem. By Lemma 1.1, it follows that $g_\theta(z) = a + b\sin(cz + d)$ where $a, b, c, d \in \mathbf{C}$ and $b, c \neq 0$. Since $g_\theta(-z) = -g_\theta(z)$ by Lemma 2.11, by differentiating on both sides of the equation, we get

$$\cos(cz + d) = \cos(-cz + d)$$

for all z. It follows that

$$\sin(d)\sin(cz) = 0$$

for all z. Since $c \neq 0$, it follows that $d = k\pi$ for some integer k. Therefore, we may assume that $g_\theta(z) = a + b\sin(cz)$. Since $g_\theta(0) = 0$, it follows that $a = 0$. This implies that $g_\theta(z) = b\sin(cz)$.

Since $g'_\theta(\pi/2) = 0$, it follows that c is some odd integer. By changing the sign of b, we may assume that c is positive. Suppose $c = 2l + 1$ for some integer $l \geq 0$. Recall that $\Omega_0 = \Omega$ is the Siegel disk of g_θ centered at the origin. For $k \in \mathbf{Z}$, let $E_k = \{z + k\pi \mid z \in \Omega_0\}$. Since Ω_0 is symmetric about the origin, it follows that every E_k is a component of $g_\theta^{-1}(\Omega_0)$. Since $\partial\Omega_0$ passes through $\pi/2$ and $-\pi/2$ by Lemma 2.11, it follows that for every $k \in \mathbf{Z}$, $\pi/2 + k\pi \in \partial E_k \cap \partial E_{k+1}$, and hence that the set $\bigcup_{k\in\mathbf{Z}} \partial E_k$ is connected. By Lemma 2.12, we get $g_\theta^{-1}(\partial\Omega_0) = \bigcup_{k\in\mathbf{Z}} \partial E_k$. By Lemma 2.12 again, it follows that the critical set of g_θ is contained in $\bigcup_{k\in\mathbf{Z}} \partial E_k$. Since $\partial E_0 = \partial\Omega_0$ passes through exactly two critical points $\pi/2$ and $-\pi/2$ of $g_\theta(z)$ and since $g'_\theta(z) = (-1)^k g'_\theta(z + k\pi)$, it follows that every critical point of g_θ has the form $\pi/2 + k\pi$ where $k \in \mathbf{Z}$ is some integer. This implies that $c = 1$. It follows that $b = e^{2\pi i\theta}$ and therefore $g_\theta(z) = f_\theta(z)$. This completes the proof of the theorem. □

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