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REMOVABLE SETS FOR HÖLDER CONTINUOUS p-HARMONIC FUNCTIONS ON METRIC MEASURE SPACES

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Abstract. We show that sets of weighted $(-p+\alpha(p-1))$-Hausdorff measure zero are removable for α-Hölder continuous Cheeger p-harmonic functions. The result is optimal for small α. Moreover, we obtain the optimal Hölder continuity of p-supersolutions in terms of the associated Riesz measures.

1. Introduction

Recently, there has been progress in the analysis on general metric measure spaces. The assumptions on the metric measure space are that it is equipped with a doubling measure and it supports a Poincaré inequality, see section 2. Under these assumptions, many important tools of the first-order calculus are available. We can conduct deep analysis of such a space in a wide range of topics. We can study, for example, Sobolev-type spaces, nonlinear potential theory and p-harmonic functions in metric space setting, see [BMS], [BBS1], [Ch], [HaK], [KM2], [Sh1] and [Sh2].

In this note, we study p-harmonic functions on complete metric spaces. We assume that the space is equipped with a doubling measure, see (5), and supporting a weak $(1,p)$-Poincaré inequality, see (8). To control the integrability of the derivative in metric space setting, we need a substitute for Sobolev space, which in this note is Newtonian space due to Shanmugalingam in [Sh1], denoted by $N^{1,p}$, see Definition 2.3. For the definition of p-harmonicity, we need a deep theorem due to Cheeger in [Ch]. Cheeger showed that under the assumptions above the metric space has a differentiable structure, with a fixed collection of coordinate functions, with which Lipschitz functions can be differentiated almost everywhere. This leads to the definition of p-harmonic functions with the Euler equation as follows.

We study the following equation for a function u in a domain Ω:

$$\int_\Omega |Du|^{p-2}Du\cdot D\varphi\,d\mu=0, \tag{1}$$

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where $1 < p < \infty$ is a number and D denotes the derivation operation, see Theorem 3.1. A continuous function u is (*Cheeger*) *p-harmonic* in a domain Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and (1) holds for all Lipschitz testing functions φ with compact support in Ω. A function $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-supersolution* in Ω if for every nonnegative Lipschitz functions φ with compact support in Ω, the inequality "$\geq$" holds in (1). For exact definitions, see section 3.

Many results for p-harmonic functions in Euclidean setting remain true, when moving into metric space setting. As an example, in [KS] it is shown that p-harmonic functions satisfy Harnack's inequality and are locally Hölder continuous. In the proof of Theorem 5.2 in [KS], it is shown that there exists $0 < \kappa \leq 1$ such that for every p-harmonic function h in Ω we have the local Hölder continuity estimate

$$\operatorname{osc}(h, B(x,r)) \leq C\left(\frac{r}{R}\right)^{\kappa} \operatorname{osc}(h, B(x,R)), \tag{2}$$

where $0 < r < R$, $B(x,2R) \subset\subset \Omega$, and C and κ are independent of r, R and h. In this paper, we study the removable sets for Hölder continuous p-harmonic functions.

We say that a compact set E is *removable* for α-Hölder continuous p-harmonic functions, if every α-Hölder continuous function $u\colon \Omega \to \mathbf{R}$, p-harmonic in $\Omega \setminus E$, is actually p-harmonic in Ω.

We state the main removability result in this paper. Weighted Hausdorff measure is defined in Definition 2.5.

Theorem 1.1. *Let X be a complete metric measure space with a doubling measure μ supporting a weak $(1,p)$-Poincaré inequality. Let $\Omega \subset X$ be open and bounded, and let $0 < \alpha < \kappa$, where κ is from (2). A closed set $E \subset \Omega$ is removable for α-Hölder continuous p-harmonic functions if and only if E is of weighted $(-p+\alpha(p-1))$-Hausdorff measure zero.*

When the measure is (*Ahlfors*) *Q-regular*, that is, there exist an exponent $Q > 0$ and a constant $C > 0$ such that $C^{-1}r^Q \leq \mu(B(x,r)) \leq Cr^Q$, for all balls $B(x,r) \subset X$, we get the following corollary.

Corollary 1.2. *Suppose that the assumptions in Theorem 1.1 hold, and in addition that μ is Q-regular. A closed set $E \subset \Omega$ is removable for α-Hölder continuous p-harmonic functions if and only if E is of $(Q-p+\alpha(p-1))$-Hausdorff measure zero.*

It was shown in [BMS] that there is one to one correspondence between p-supersolutions $u \in N^{1,p}_0(\Omega)$ and Radon measures ν in the dual $N^{1,p}_0(\Omega)^*$ given by

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu, \tag{3}$$

whenever $\varphi \in N^{1,p}_0(\Omega)$.

To prove Theorem 1.1, we study the Riesz measure of a p-supersolution. In the following theorem, we obtain the optimal Hölder continuity of p-supersolutions in terms of the associated Riesz measure. It has interest of its own.

Theorem 1.3. *Let $\Omega \subset X$ be open and bounded, and $0 < \alpha < \kappa$, where κ is as in (2). Assume that u is a p-supersolution in Ω and $\nu \in N_0^{1,p}(\Omega)^*$ is a Radon measure such that (3) holds. Then $u \in C_{\mathrm{loc}}^{0,\alpha}(\Omega)$ if and only if there is a constant $M > 0$ such that*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq M r^{-p+\alpha(p-1)}, \tag{4}$$

for all balls $B(x,4r) \subset \Omega$.

In Euclidean spaces Carleson [Ca] proved Theorem 1.1 for harmonic functions. For $\mathscr{A}$-harmonic functions in $\mathbf{R}^n$, where $\mathscr{A}$ is of p-Laplacian type, see [HKM, Chapter 3], the main results in this paper are proven in [KiZ].

In metric space setting, the necessary part in Theorem 1.3, that is, that $u \in C_{\mathrm{loc}}^{0,\alpha}(\Omega)$ implies (4), was obtained in [BMS]. In this paper, we prove that the growth condition (4) is sufficient. Removable singularities for bounded p-harmonic functions on metric spaces are studied in [B2].

Most of the theory of p-harmonic functions on metric spaces has been done for p-harmonic functions defined via upper gradient, referred to as p-minimizers, see e.g. [Sh2] and [KM2]. All those proofs go through for Cheeger p-harmonic functions as well. On the other hand, certain results for Cheeger p-harmonic functions do not apply for p-harmonic functions defined using the upper gradients. Major advantage of using Cheeger derivatives is that the differential equation (1) is available. Theory for Cheeger p-harmonic functions is studied, for example, in [BMS] and [BBS1].

This paper is organized as follows. In section 2, we discuss the necessary background such as the basic assumptions on the metric measure space, the definitions of Sobolev spaces on metric spaces and weighted Hausdorff measure. Also a few general theorems are introduced there. We study the theory of p-harmonic functions on metric spaces in section 3. Also a balayage in metric spaces is introduced there. In section 4, we show the connection between the Hölder continuous p-supersolution and the corresponding Radon measure and prove Theorem 1.3. Finally, in section 5 we study the removable sets for p-harmonic functions and prove Theorem 1.1.

2. Preliminaries

Throughout the paper we denote by $C > 0$ a constant, whose value may vary between each usage, even in the same line.

The triple (X, d, μ) denotes a complete metric measure space X and μ is assumed to be a Radon measure, which means that it is Borel regular and every compact set is of finite measure. We also assume that the measure of every nonempty open set is positive.

The ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B(x,r) = \{y \in X : d(y,x) < r\}.$$

We write

$$u_A = \frac{1}{\mu(A)} \int_A u \, d\mu = \fint_A u \, d\mu,$$

for a measurable $A \subset X$ and a measurable function $u\colon X \to [-\infty, \infty]$. The norm of v in $L^p(X,\mu) = L^p(X)$ is denoted by

$$\|v\|_p = \left(\int_X |v|^p d\mu \right)^{1/p}.$$

We denote the characteristic function of the set $E \subset X$ as χ_E.

Let $\alpha \in (0,1]$. A function $u\colon X \to \mathbf{R}$ is said to be *locally α-Hölder continuous*, that is, $u \in C^{0,\alpha}_{\mathrm{loc}}(X)$, if for some constant $C > 0$,

$$|u(x) - u(y)| \le C d(x,y)^\alpha$$

whenever $x, y \in X$ are such that $d(x,y) < 1$. The function u is *Lipschitz continuous*, $u \in \mathrm{Lip}(X)$, if $u \in C^{0,1}(X)$. We also use the notation $u \in \mathrm{Lip}_0(X)$ when the function u has compact support.

We make the following assumptions on the metric measure space (X, d, μ). First, we assume that the equipped measure μ is *doubling*, that is, there exists a constant $C_d \ge 1$ such that for all balls $B(x,r)$ in X,

$$\mu(B(x,2r)) \le C_d \mu(B(x,r)). \tag{5}$$

If the measure μ is doubling, then there exist constants c, $s > 0$ that depend only on the doubling constant of μ, such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge c \left(\frac{r}{R} \right)^s, \tag{6}$$

whenever $r < R$, $x \in X$ and $y \in B(x,R)$, see [He, pp. 103–104]. Usually we refer s to be the natural dimension of the space X and in this note we always assume $s > 1$.

The second assumption is a geometric condition on the space, which requires the space to be sufficiently regular. We assume that the metric measure space admits a Poincaré inequality. To define a Poincaré inequality, we need a notion, *upper gradient*, which is a substitute of Sobolev gradient in the setting of metric space.

Definition 2.1. Let $u\colon X \to [-\infty, \infty]$ be a function. A nonnegative measurable function $g\colon X \to [0, \infty]$ is said to be an *upper gradient* of u if for all compact rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \le \int_\gamma g \, ds. \tag{7}$$

If $u(x) = u(y) = \infty$ or $u(x) = u(y) = -\infty$, we define the left side of (7) to be ∞. See [HeK], [Ch] or [Sh1] for discussion on the upper gradients.

Definition 2.2. Let $1 \le p < \infty$. A metric measure space (X, d, μ) is said to admit a *weak* $(1,p)$*-Poincaré* (*or* p*-Poincaré*) *inequality* if there are constants $C_p > 0$ and $\tau \ge 1$ such that

$$\fint_{B(x_0,r)} |u - u_{B(x_0,r)}| \, d\mu \le C_p r \left(\fint_{B(x_0,\tau r)} g^p \, d\mu \right)^{1/p} \tag{8}$$

for all balls $B(x_0, r) \subset X$, for all integrable functions u in $B(x_0, r)$ and for all upper gradients g of u.

The above definition is due to Heinonen and Koskela [HeK]. There are various formulation for a Poincaré inequality on a metric measure space. When the space is complete and is equipped with a doubling Borel regular measure, many different definitions coincide.

The following Sobolev type spaces on metric spaces were introduced by Shanmugalingam in [Sh1].

Definition 2.3. Let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu \right)^{1/p} + \inf_g \left(\int_X g^p \, d\mu \right)^{1/p},$$

where infimum is taken over all upper gradients of u. The quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\} / \sim,$$

is the *Newtonian space* on X, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

For properties of the Newtonian spaces, we refer to [Sh1].

Definition 2.4. (i) The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all of functions u in $N^{1,p}(X)$, whose restriction to E is bounded below by 1. We say that a property regarding points in X holds *p-quasieverywhere*, denoted as p-q.e., if the set of points for which the property does not hold has p-capacity zero.

(ii) We define "Newtonian space with zero boundary values" $N_0^{1,p}(\Omega)$ for domain $\Omega \subset X$, to be the class of those Newtonian functions u for which $u\chi_{X\setminus\Omega} = 0$ p-quasieverywhere.

(iii) Let $\Omega \subset X$ be a domain. We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every compactly contained subdomain $\Omega' \subset\subset \Omega$, and for every cut-off function $\eta \in \mathrm{Lip}_0(\Omega)$ such that $\eta = 1$ in Ω', $\eta f \in N^{1,p}(X)$. Furthermore, $f_j \to f$ in $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $\eta f_j \to \eta f$ in $N^{1,p}(X)$, as $j \to \infty$, for every Ω' and every $\eta \in \mathrm{Lip}_0(\Omega)$.

(iv) The dual space of $N_0^{1,p}(\Omega)$ is denoted by $N_0^{1,p}(\Omega)^*$.

The space $N^{1,p}(X)$ is a Banach space [Sh1]. If X admits the $(1,p)$-Poincaré inequality and the measure is doubling, Lipschitz functions are dense in $N^{1,p}(X)$ [Sh1]. Moreover, if X is complete, the space $\mathrm{Lip}_0(\Omega)$ is dense in $N_0^{1,p}(\Omega)$. Here we defined Newtonian spaces as in Shanmugalingam [Sh1]. Cheeger [Ch] defines Sobolev spaces with upper gradients in a different way, yet his spaces coincide with corresponding Newtonian spaces when $p > 1$. This is proven in [Sh1]. The Sobolev type spaces introduced by Hajłasz [Ha] also coincide with these spaces under our assumptions.

Here we define a version of the weighted Hausdorff measure on the metric measure space, see e.g. [Mat].

Definition 2.5. Let (X, d, μ) be a metric measure space. Let $\alpha \in \mathbf{R}$, $0 < \delta \leq \infty$. For any function $f\colon X \to [0, \infty]$ we set

$$\mathscr{H}_\mu^{\alpha,\delta}(f) = \inf \sum_j c_j r_j^\alpha \mu(B_j),$$

where infimum is taken over all families $\{(B_j, c_j)\}$, where $0 < c_j < \infty$, $B_j = B(x_j, r_j) \subset X$ are balls such that $r_j \leq \delta$ and

$$f \leq \sum_j c_j \chi_{B_j}.$$

Then

$$\mathscr{H}_\mu^{\alpha}(f) = \sup_{\delta > 0} \mathscr{H}_\mu^{\alpha,\delta}(f).$$

For $E \subset X$ we define the *weighted* (α, δ)*-content of* E as $\mathscr{H}_\mu^{\alpha,\delta}(E) = \mathscr{H}_\mu^{\alpha,\delta}(\chi_E)$, and the *weighted* α*-dimensional Hausdorff measure* of E as $\mathscr{H}_\mu^{\alpha}(E) = \mathscr{H}_\mu^{\alpha}(\chi_E)$.

We need the following weighted version of Frostman's lemma. The proof is similar to that of Theorem 8.17 in [Mat].

Theorem 2.6. *Assume that μ is a doubling measure on X. Let $\alpha \in \mathbf{R}$ and $K \subset X$ be a compact set such that $\mathscr{H}_\mu^{\alpha}(K) > 0$. Then there exist $\delta > 0$ and a Radon measure ν in X such that ν is supported on K, $\nu(K) > 0$ and*

$$\nu(B(x, r)) \leq C r^\alpha \mu(B(x, r))$$

for all balls $B(x, r) \subset X$ with $0 < r \leq \delta$. Here the constant C depends only on the doubling constant of μ.

Proof. Choose $\delta > 0$ such that $\mathscr{H}_\mu^{\alpha,\delta}(K) > 0$. Define a function p on $C(K)$ by

$$p(f) = \inf \sum_i c_i r_i^\alpha \mu(B_i),$$

where infimum is taken over all families $\{(B_i, c_i)\}$, where $0 < c_i < \infty$, $B_i = B(x_i, r_i) \subset X$ are balls such that $r_i \leq \delta$ and

$$f \leq \sum_i c_i \chi_{B_i}.$$

For nonnegative $f \in C(K)$ we have $p(f) = \mathscr{H}_\mu^{\alpha,\delta}(f)$. Moreover, $p(tf) = tp(f)$ and $p(f+g) \leq p(f) + p(g)$, for all $f, g \in C(X)$ and $t \geq 0$. By the Hahn–Banach theorem [Rud, Theorem 3.2], we can extend the linear functional $c \mapsto cp(1)$, $c \in \mathbf{R}$, from the subspace of constant functions to a linear functional $L\colon C(K) \to \mathbf{R}$ satisfying $L(1) = p(1) = \mathscr{H}^{\alpha,\delta}_\mu(K)$ and $-p(-f) \le L(f) \le p(f)$ for $f \in C(K)$. When $f \ge 0$, then $p(-f) = 0$ and therefore $L(f) \ge 0$. By Riesz representation theorem, there exists a Radon measure ν on K such that $L(f) = \int_K f\,d\nu$ for $f \in C(K)$.

The measure ν is the desired measure in Theorem 2.6. Indeed, let x be a point in K and $r < \delta$. Choose a sequence of continuous functions f_i such that $0 \le f_i \le 1$, $f_i = 1$ on $B(x,r)$ and $\operatorname{spt} f_i \subset B(x, r + \frac{1}{i})$. Then

$$\begin{aligned}\nu(B(x,r)) &\le \lim_{i\to\infty} \int_X f_i\,d\nu \le \lim_{i\to\infty} \mathscr{H}^{\alpha,\delta}_\mu(B(x, r+\tfrac{1}{i}))\\ &\le \lim_{i\to\infty} (r+\tfrac{1}{i})^\alpha \mu(B(x,r+\tfrac{1}{i})) \le C_d \lim_{i\to\infty}(r+\tfrac{1}{i})^\alpha \mu(B(x,r))\\ &= C_d r^\alpha \mu(B(x,r)),\end{aligned}$$

where we used the doubling property of μ. □

The following lemma is a generalized version of Lemma 2.1 in [Gia], which is due to Campanato. It involves an additional weight function ω.

Lemma 2.7. *Let $\phi(t)$ and $\omega(t)$ be nonnegative and nondecreasing functions on $(0,R)$. Assume that there are constants $c_\omega > 0$ and $s > 0$ such that*

$$\frac{\omega(\lambda r)}{\omega(r)} \ge c_\omega \lambda^s, \tag{9}$$

for every $r > 0$ and $0 < \lambda \le 1$. Suppose that

$$\phi(\rho) \le A_1 \left[\frac{\omega(\rho)}{\omega(r)}\left(\frac{\rho}{r}\right)^{\beta+\delta} + \varepsilon\right]\phi(r) + A_2\omega(r)r^\beta$$

for all $0 < \rho \le r \le R$ and $\varepsilon > 0$, where A_1 and $A_2 = A_2(\varepsilon)$ are nonnegative constants, $\beta \in \mathbf{R}$ and $\delta > 0$. Here A_1, β and δ do not depend on ε. Then we have

$$\phi(\rho) \le c\left[\frac{\omega(\rho)}{\omega(r)}\left(\frac{\rho}{r}\right)^{\beta}\phi(r) + A_2\omega(\rho)\rho^\beta\right]$$

for all $0 < \rho \le r \le R$, where $c = c(\beta,\delta,A_1,s,c_\omega) > 0$.

Proof. For $\lambda \in (0,1)$ and $r < R$, we have

$$\phi(\lambda r) \le A_1\lambda^{\beta+\delta}\left[\frac{\omega(\lambda r)}{\omega(r)} + \varepsilon\lambda^{-(\beta+\delta)}\right]\phi(r) + A_2\omega(r)r^\beta.$$

We may assume $A_1 > 1$. Choose $\lambda < 1$ such that $2A_1\lambda^{\delta/2} = 1$, and $\varepsilon = c_\omega\lambda^{s+\beta+\delta}$, when we have by (9) that

$$\varepsilon_0\lambda^{-(\beta+\delta)} \le \frac{\omega(\lambda r)}{\omega(r)}.$$

Therefore, we have

$$\begin{aligned}\phi(\lambda r) &\le \lambda^{\beta+\delta/2}\frac{\omega(\lambda r)}{\omega(r)}\phi(r) + A_2\omega(r)r^\beta \\ &\le \lambda^{\beta+\delta/2}\frac{\omega(\lambda r)}{\omega(r)}\phi(r) + A_2c_\omega^{-1}\lambda^{-s}\omega(\lambda r)r^\beta,\end{aligned}$$

where we used (9). Thus, for all integers $k > 0$

$$\begin{aligned}\phi(\lambda^{k+1}r) &\le \lambda^{\beta+\delta/2}\frac{\omega(\lambda^{k+1}r)}{\omega(\lambda^k r)}\phi(\lambda^k r) + A_2c_\omega^{-1}\lambda^{-s}\omega(\lambda^{k+1}r)\lambda^{k\beta}r^\beta \\ &\le \lambda^{(k+1)(\beta+\delta/2)}\frac{\omega(\lambda^{k+1}r)}{\omega(r)}\phi(r) + A_2c_\omega^{-1}\lambda^{-s}r^\beta\lambda^{k\beta}\omega(\lambda^{k+1}r)\sum_{j=0}^{k}(\lambda^{\delta/2})^j \\ &\le \lambda^{(k+1)(\beta+\delta/2)}\frac{\omega(\lambda^{k+1}r)}{\omega(r)}\phi(r) + \frac{A_2c_\omega^{-1}\lambda^{k\beta-s}r^\beta\omega(\lambda^{k+1}r)}{1-\lambda^{\delta/2}}.\end{aligned}$$

Next we choose k so that $\lambda^{k+2}r < \rho \le \lambda^{k+1}r$. Then Lemma 2.7 follows from the last inequality and (9). □

The key tool for our proofs is the following Adams inequality in the setting of metric spaces. This is proven in [Mäk]. For the Adams inequality in Euclidean spaces, see e.g. [AH], [Ma], [Tu] and [Zi].

Theorem 2.8. *Let (X, d, μ) be a complete metric measure space such that it admits a weak $(1, t)$-Poincaré inequality for some $1 \le t < p$, and μ is a doubling Radon measure. Suppose that ν is a Radon measure on X satisfying*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \le Mr^{\alpha_0} \quad \text{with} \quad \alpha_0 = \frac{sq}{p} - s - \frac{q}{t},$$

for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$, where $1 < p < q < \infty$, $p/t < s$ and M is a positive constant. Here s is from (6). If $u \in \operatorname{Lip}_0(B_0)$ for some ball $B_0 = B(x_0, r_0) \subset X$, for which $r_0 < \operatorname{diam} X/10$, we have

$$\left(\int_{B_0} |u|^q \, d\nu\right)^{1/q} \le C\mu(B_0)^{1/q-1/p}\, r_0^{\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}} M^{1/q} \left(\int_{B_0} (\operatorname{Lip} u)^p \, d\mu\right)^{1/p},$$

where $C = C(p, q, s, t, C_d, C_p, \tau) > 0$.

The assumption, that the space admits a $(1, t)$-Poincaré inequality for some $t < p$, follows from the $(1, p)$-Poincaré inequality by the result in [KeZ]. Since we need t explicitly in the formulas, we make this assumption in the theorem above. Notice also that by the Hölder inequality, we may choose t such that $p/t < s$ if necessary, since we assume $s > 1$.

Throughout this paper we assume that the metric measure space (X, d, μ) is complete, doubling and it admits a weak p-Poincaré inequality. We also assume that $\Omega \subset X$ is a nonempty bounded open set in X such that $\operatorname{Cap}_p(X \setminus \Omega) > 0$, which is immediately true if X is unbounded.

3. p-harmonic functions

Cheeger [Ch] proved that a metric measure space which admits a Poincaré inequality with a doubling measure has a "differentiable structure" under which Lipschitz functions have derivatives almost everywhere, see Theorem 4.38 in [Ch].

Theorem 3.1. *Let (X,d,μ) be a metric measure space and μ a doubling Borel regular measure. Assume that X admits a weak $(1,p)$-Poincaré inequality for some $1<p<\infty$. Then there exists a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha=(X_1^\alpha,\dots,X_{k(\alpha)}^\alpha)\colon X\to \mathbf{R}^{k(\alpha)}$ such that $\mu(X\setminus\bigcup_\alpha U_\alpha)=0$ and for all α, the following holds:*

The functions $X_1^\alpha,\dots,X_{k(\alpha)}^\alpha$ are linearly independent on U_α and $1\le k(\alpha)\le N$, where N is a constant depending only on the doubling constant and the constants in the Poincaré inequality. If $f\colon X\to\mathbf{R}$ is Lipschitz, then there exist unique bounded vector-valued functions $d^\alpha f\colon U_\alpha\to\mathbf{R}^{k(\alpha)}$ such that for μ-a.e. $x_0\in U_\alpha$,

$$\lim_{r\to 0+}\sup_{x\in B(x_0,r)}\frac{|f(x)-f(x_0)-d^\alpha f(x_0)\cdot(X^\alpha(x)-X^\alpha(x_0))|}{r}=0.$$

We can assume that the sets U_α are pairwise disjoint and extend $d^\alpha f$ by zero outside U_α. We get a linear differential mapping $D\colon f\mapsto Df$ if we regard $d^\alpha f(x)$ as vectors in $\mathbf{R}^N$ and let $Df=\sum_\alpha d^\alpha f$. It is shown in [Ch] that for all Lipschitz functions and μ-a.e. $x\in X$,

$$|Df(x)|\approx g_f(x)=\inf_g \limsup_{r\to 0+} \fint_{B(x,r)} g\,d\mu, \tag{10}$$

where g_f is the minimal p-weak upper gradient of f and the infimum is taken over all upper gradients g of f. Cheeger also proved that the differential operator can be extended to all functions of the associated Sobolev space. In particular, this holds for Newtonian space $N^{1,p}(X)$, which coincides with the space considered by Cheeger, as mentioned before. One easily verifies that the "gradient" Du satisfies the product and chain rules, see [Ch]. Moreover, if u_i is a sequence in $N^{1,p}(X)$, then $u_i\to u$ in $N^{1,p}(X)$ if and only if $u_i\to u$ in $L^p(X,\mu)$ and $Du_i\to Du$ in $L^p(X,\mu;\mathbf{R}^n)$.

Now we can define p-harmonic functions by using the Cheeger gradient defined above.

Definition 3.2. A continuous function u is (Cheeger) *p-harmonic* in Ω if $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$ and it satisfies

$$\int_\Omega |Du|^{p-2}Du\cdot D\varphi\,d\mu=0, \tag{11}$$

for all $\varphi\in \mathrm{Lip}_0(\Omega)$.

In addition, since $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$, the identity (11) holds for all $\varphi\in N^{1,p}_0(\Omega)$, due to the density of $\mathrm{Lip}_0(\Omega)$ in $N^{1,p}_0(\Omega)$.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies (11) if and only if

$$\int_{\Omega'} |Du|^p \, d\mu \le \int_{\Omega'} |D(u+\varphi)|^p \, d\mu \tag{12}$$

for all $\Omega' \subset\subset \Omega$ and all $\varphi \in N^{1,p}_0(\Omega')$. Therefore it is a quasiminimizer in the sense of Kinnunen-Shanmugalingam [KS] and the results of [KS] apply to p-harmonic functions as well.

Note that in many papers p-harmonic functions are defined as continuous p-minimizers, and p-harmonic functions defined above are called Cheeger p-harmonic functions. In this paper we discuss only Cheeger p-harmonic functions, which from now on are called p-harmonic functions. Other results for p-harmonic functions defined here are studied e.g. in [BMS], [BBS1], [B3] and also in [KS2].

Definition 3.3. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-supersolution* in Ω if for every $0 \le \varphi \in N^{1,p}_0(\Omega)$ there holds

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\varphi \, d\mu \ge 0,$$

or equivalently,

$$\int_{\Omega} |Du|^p d\mu \le \int_{\Omega} |D(u+\varphi)|^p \, d\mu.$$

A function $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-subsolution* if $-v$ is p-supersolution.

From [BMS, Prop. 3.5, Remark 3.6] we recall that for every p-supersolution u in Ω, there exists a Radon measure $\nu \in N^{1,p}_0(\Omega)^*$ such that

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_{\Omega} \varphi \, d\nu, \tag{13}$$

for all $\varphi \in N^{1,p}_0(\Omega)$. We call the measure ν the *Riesz measure associated with* u in Ω. Moreover, it is shown in [BMS, Prop. 3.9] that if $\Omega \subset X$ is bounded and $\nu \in N^{1,p}_0(\Omega)^*$ is a Radon measure on Ω, there exists a unique $u \in N^{1,p}_0(\Omega)$ satisfying (13) for all $\varphi \in N^{1,p}_0(\Omega)$. Moreover, u is a p-supersolution in Ω.

Definition 3.4. A function $u\colon \Omega \to (-\infty, \infty]$ is *p-superharmonic* in Ω if

(i) u is lower semicontinuous and is not identically infinite on any component of Ω, and
(ii) for all nonempty open set $V \subset\subset \Omega$ with $V \ne \Omega$ and all functions $v \in C(\overline{V})$ such that v is p-harmonic in V and $v \le u$ on ∂V, we have $v \le u$ in V.

A function v is *p-subharmonic* in Ω, if $-v$ is p-superharmonic in Ω.

We define p-superharmonic functions as in [B1, 4.1(iii.g)]. For other equivalent definitions of p-superharmonic functions, see Theorem 6.1 in [B1]. In [KM2], it is proven, that for every p-supersolution u in Ω there is a p-superharmonic function v such that $u = v$ μ-a.e. in Ω.

Next we introduce some basic properties for p-harmonic functions.

We need the weak Harnack inequality from Remark 4.4 (2) in [KS]: For any p-harmonic function u in a domain $\Omega \subset X$, we have for all balls $B(x_0,\rho) \subset\subset B(x_0,R) \subset\subset \Omega$ and every $q>0$, that

$$\sup_{B(x_0,\rho)} |u| \le \frac{C}{(1-\rho/R)^{s/q}} \left(\fint_{B(x_0,R)} |u|^q \, d\mu \right)^{1/q}, \tag{14}$$

where $C>0$ is a constant independent of the ball and of the function u. Here s is as in (6).

Lemma 3.5. *Let $B(x_0,R) \subset\subset \Omega$. There exists a number $0<\kappa\le 1$ such that*

$$\fint_{B(x_0,r)} |Du|^p \, d\mu \le C \left(\frac{r}{R}\right)^{p\kappa - p} \fint_{B(x_0,R)} |Du|^p \, d\mu,$$

for each $0<r<R$ and for any p-harmonic function u in Ω.

Proof. We have the De Giorgi inequality: There exists $C=C(p)>0$ such that

$$\int_{B(x,r)} |Du(y)|^p \, d\mu(y) \le \frac{C}{(R-r)^p} \int_{B(x,R)} |u(y)-u_{B(x,R)}|^p \, d\mu(y), \tag{15}$$

whenever $B(x,r) \subset\subset B(x,R) \subset\subset \Omega$, see [KS, (3.2)].

Next the Lemma follows from the De Giorgi inequality (15), oscillation inequality (2) and weak Harnack inequality (14). □

We will need Morrey's Dirichlet growth theorem.

Theorem 3.6. *Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $\alpha \in (0,1)$. If*

$$\fint_{B(x,r)} |Du|^p \, d\mu \le C r^{p\alpha - p}, \tag{16}$$

for all balls $B(x,r) \subset \Omega$, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

Proof. For a.e. $x,y \in \Omega$ such that $B(x,4\tau d(x,y)) \cup B(y,4\tau d(x,y)) \subset \Omega$, we have

$$|u(x)-u(y)| \le C d(x,y)^\alpha \left(u^\sharp_{\alpha,4d(x,y)}(x) + u^\sharp_{\alpha,4d(x,y)}(y) \right),$$

where

$$u^\sharp_{\alpha,R}(z) = \sup_{0<r<R} r^{-\alpha} \fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu$$

is a fractional sharp maximal function and $C=C(\alpha,C_d)>0$, see Lemma 3.6 in [HKi]. By the Poincaré inequality and (16), we have $u^\sharp_{\alpha,R}(z) \le C_p$. Hence

$$|u(x)-u(y)| \le C d(x,y)^\alpha.$$

Thus we can choose a representative $\tilde{u} \in N^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ such that $\tilde{u}=u$ p-q.e. in Ω, see Corollary 3.3 in [Sh1]. □

Lemma 3.7. *If u is non-negative, continuous function on Ω, $I \subset \Omega$ is a closed set such that $u=0$ on I and u is a p-subsolution in $\Omega \setminus I$, then u is a p-subsolution in Ω.*

Proof. Let $\varphi \in N_0^{1,p}(\Omega)$ be non-negative. Let U be the support of φ; then U is relatively compact subset of Ω. We need to show that

$$\int_U |Du|^p \, d\mu \le \int_U |D(u-\varphi)|^p \, d\mu.$$

Since $u \ge 0$ on Ω and $u = 0$ on I, it follows that $(u-\varphi)^+ \le u$ on Ω with $u-(u-\varphi)^+$ having support in U and that $(u-\varphi)^+ = 0$ on I. Observe that $\int_U |D(u-\varphi)|^p \, d\mu \ge \int_U |D(u-\varphi)^+|^p \, d\mu$. Since u is a p-subsolution in $\Omega \setminus I$ and by the above statement, $(u-\varphi)^+$ has support in $U \setminus I$, it follows that

$$\int_U |Du|^p \, d\mu \le \int_U |D(u-\varphi)^+|^p \, d\mu \le \int_U |D(u-\varphi)|^p \, d\mu. \qquad \square$$

The rest of the section is devoted to the theory of balayage. Balayage in metric measure spaces is studied in [BBMP] and the following theorems are proven there.

First, we recall that the *liminf-regularization* $\hat{u}$ of any function $u\colon \Omega \to [-\infty,\infty]$ is defined by

$$\hat{u}(x) = \liminf_{y \to x} u(y).$$

Then $\hat{u} \le u$. Moreover, if u is locally bounded below, then $\hat{u}$ is lower semicontinuous.

Definition 3.8. Let $\psi\colon \Omega \to (-\infty,\infty]$ be a function that is locally bounded below, and let

$$\Phi^\psi = \Phi^\psi(\Omega) = \{u : u \text{ is } p\text{-superharmonic in } \Omega \text{ and } u \ge \psi \text{ in } \Omega\}.$$

Then we define

$$R^\psi(x) = R^\psi(\Omega)(x) = \inf\{u(x) : u \in \Phi^\psi\}.$$

The liminf-regularization $\hat{R}^\psi(x) = \liminf_{y\to x} R^\psi(y)$ is called the *balayage* of ψ in Ω. To obtain a meaningful function $\hat{R}^\psi$, we need to assume the set Φ^ψ to be non-empty.

Notice that we can analogously define $\hat{\underline{R}}^\psi$. Indeed, we let $\underline{\Phi}^\psi$ be a set of all p-subharmonic functions, which are below ψ. In this case ψ is assumed to be locally bounded above. Then $\hat{\underline{R}}^\psi$ is defined by taking the upper semicontinuous regularization of supremum of $\underline{\Phi}^\psi$.

Theorem 3.9. *The balayage $\hat{R}^\psi(\Omega)$ is p-superharmonic in Ω.*

The following theorem is a metric space version of Theorem 8.14 in [HKM], see [BBMP].

Theorem 3.10. *If ψ is a continuous and bounded above in Ω, then $\hat{R}^\psi$ is continuous p-supersolution with $\hat{R}^\psi \ge \psi$. Moreover, $\hat{R}^\psi$ is p-harmonic in the open set $\{\hat{R}^\psi > \psi\}$.*

In metric spaces regular boundary points can be defined using Perron solution, as usually done in Euclidean setting. Equivalent definitions, e.g. in terms of barrier, are also available, see [BB1]. We say that a set is *regular* if every boundary point is regular. The property of regular sets used in this paper is that every nonempty open set $\Omega \subset X$, $\Omega \neq X$, can be exhausted by a sequence of regular sets.

We need a version of Theorem 9.26 in [HKM] in metric spaces, which is proven in [BBMP] as well.

Theorem 3.11. *Let ψ be continuous in a regular set D and $u = \hat{R}^{\psi}$. Then*

$$\lim_{y\to x} u(y) = \psi(x)$$

for all $x \in \partial D$.

4. Hölder continuous supersolutions and Radon measures, Proof of Theorem 1.3

First, we prove the sufficient part of Theorem 1.3.

Theorem 4.1. *Let κ be the number given by (2). Suppose that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu,$$

for all $\varphi \in N^{1,p}_0(\Omega)$, where $\nu \in N^{1,p}_0(\Omega)^*$ is a Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \le M r^{-p+\alpha(p-1)},$$

whenever $B(x,2r) \subset \Omega$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

To prove Theorem 4.1, we need the following Lemma.

Lemma 4.2. *Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu,$$

for all $\varphi \in N^{1,p}_0(\Omega)$, where $\nu \in N^{1,p}_0(\Omega)^*$ is a Radon measure. Let $B(x_0, 2R) \subset \Omega$ such that $R < \operatorname{diam} X/10$. Assume that there are constants $C_0 > 0$ and $0 < \alpha < \kappa$ such that

$$\frac{\nu(B(x_0,r))}{\mu(B(x_0,r))} \le C_0 r^{-p+\alpha(p-1)}, \tag{17}$$

for all $0 < r \le R$. Then for each $0 < r < R$ and $\varepsilon > 0$ we have

$$\int_{B(x_0,r)} |Du|^p \, d\mu \le C_1 \left(\frac{\mu(B(x_0,r))}{\mu(B(x_0,R))} \left(\frac{r}{R}\right)^{-p+p\kappa} + \varepsilon \right) \int_{B(x_0,R)} |Du|^p \, d\mu + C_2 \mu(B(x_0,R)) R^{-p+p\alpha},$$

where $C_1 = C_1(p) > 0$ and $C_2 = C_2(p,\alpha,C_0,\varepsilon) > 0$ are constants.

Proof. Without loss of generality we may assume that $r < R/2$. We denote $B_{\hat r} = B(x_0, \hat r)$. Let h be the p-harmonic function in B_R such that $u-h \in N_0^{1,p}(B_R)$. Then

$$\begin{aligned}\int_{B_r} |Du|^p\,d\mu &= \int_{B_r} \big[|Du|^{p-2}Du - |Dh|^{p-2}Dh\big]\cdot(Du-Dh)\,d\mu\\ &\quad + \int_{B_r} |Du|^{p-2}Du\cdot Dh\,d\mu + \int_{B_r} |Dh|^{p-2}Dh\cdot(Du-Dh)\,d\mu\\ &\le \int_{B_R} \big[|Du|^{p-2}Du - |Dh|^{p-2}Dh\big]\cdot(Du-Dh)\,d\mu\\ &\quad + \int_{B_r} |Du|^{p-1}|Dh|\,d\mu,\end{aligned} \tag{18}$$

where we used (11) for h. Since $u-h \in N_0^{1,p}(B_R)$, using it as a testing function for the equations for u and h, we obtain

$$\int_{B_R} \big[|Du|^{p-2}Du - |Dh|^{p-2}Dh\big]\cdot(Du-Dh)\,d\mu = \int_{B_R} (u-h)\,d\nu \tag{19}$$

We will estimate the right hand side of the above identity by Adams inequality, Theorem 2.8, which is formulated for Lipschitz functions. In our case $u-h \in N_0^{1,p}(B_R)$. Thus we need the following approximation argument. By [Sh2, Theorem 4.8] $\mathrm{Lip}_0(B_R)$ is dense in $N_0^{1,p}(B_R)$ and hence for $u-h \in N_0^{1,p}(B_R)$ there exist $\varphi_k \in \mathrm{Lip}_0(B_R)$ converging to $u-h$ both in $N_0^{1,p}(B_R)$ and p-q.e. in B_R, see [Sh1, Corollary 3.9]. Identity (13) yields that the functions φ_k form a Cauchy sequence in $L^1(B_R,\nu)$. Hence a subsequence of $\{\varphi_k\}$ converges to φ ν-a.e. in B_R, and by [BMS, Lemma 3.8] $\varphi = u-h$ ν-a.e. Thus in this case, Adams inequality, Theorem 2.8, can be applied as follows.

First, we need the result in [KeZ], that our space X admits a weak $(1,t)$-Poincaré inequality for some $1 \le t < p$. Now we choose

$$q = \frac{(s-p+\alpha(p-1))tp}{st-p}.$$

Then we get by Hölder inequality, (17) and Theorem 2.8, that

$$\begin{aligned}&\int_{B_R} (u-h)\,d\nu \le C[R^{-p+\alpha(p-1)}\mu(B_R)]^{\frac{q-1}{q}}\left(\int_{B_R}|u-h|^q\,d\nu\right)^{1/q}\\ &\le C\mu(B_R)^{1-1/p}R^{[-p+\alpha(p-1)]\frac{q-1}{q}+\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}}\left(\int_{B_R}|D(u-h)|^p\,d\mu\right)^{1/p}\\ &\le C\mu(B_R)^{1-1/p}R^{(-p+\alpha p)(1-1/p)}\left(\left(\int_{B_R}|Du|^p\,d\mu\right)^{1/p}+\left(\int_{B_R}|Dh|^p\,d\mu\right)^{1/p}\right)\end{aligned}$$

Thus by Young's inequality and the minimizing property of h, we obtain

$$\int_{B_R}(u-h)\,d\nu \le C\mu(B_R)R^{-p+\alpha p}+\varepsilon\int_{B_R}|Du|^p\,d\mu. \tag{20}$$

Next we estimate the last term in (18) by Young's inequality and Lemma 3.5

$$\begin{aligned}\int_{B_r}|Du|^{p-1}|Dh|\,d\mu &\le \frac{1}{2}\int_{B_r}|Du|^p\,d\mu + C\int_{B_r}|Dh|^p\,d\mu\\ &\le \frac{1}{2}\int_{B_r}|Du|^p\,d\mu + C\frac{\mu(B_r)}{\mu(B_R)}\left(\frac{r}{R}\right)^{p\kappa-p}\int_{B_R}|Dh|^p\,d\mu\\ &\le \frac{1}{2}\int_{B_r}|Du|^p\,d\mu + C\frac{\mu(B_r)}{\mu(B_R)}\left(\frac{r}{R}\right)^{p\kappa-p}\int_{B_R}|Du|^p\,d\mu.\end{aligned}$$

Now plugging the previous estimate, (19) and (20) into (18), we obtain

$$\int_{B_r}|Du|^p\,d\mu \le C_1\left(\frac{\mu(B_r)}{\mu(B_R)}\left(\frac{r}{R}\right)^{-p+p\kappa}+\varepsilon\right)\int_{B_R}|Du|^p\,d\mu + C_2\mu(B_R)R^{-p+p\alpha},$$

which proves the lemma. □

Proof of Theorem 4.1. Fix $B(x_0,2R)\subset\Omega$ such that $R<\operatorname{diam}X/10$. For any $0<r<R$ and $\varepsilon>0$, we have by Lemma 4.2, that

$$\begin{aligned}\int_{B(x_0,r)}|Du|^p\,d\mu \le C\left(\frac{\mu(B(x_0,r))}{\mu(B(x_0,R))}\left(\frac{r}{R}\right)^{-p+p\kappa}+\varepsilon\right)&\int_{B(x_0,R)}|Du|^p\,d\mu\\ &+C\mu(B(x_0,R))R^{-p+p\alpha}.\end{aligned}$$

Now we can choose ε small enough. Lemma 2.7 gives us

$$\int_{B(x_0,r)}|Du|^p\,d\mu \le C\mu(B(x_0,r))r^{-p+p\alpha},$$

where C is independent of u and r. Thus by Morrey's Dirichlet growth theorem, Theorem 3.6, $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. □

The necessary part of Theorem 1.3 is proved in [BMS]. For the sake of completeness, we write down the proof.

Theorem 4.3. *Let $\Omega\subset X$ and u be a p-supersolution in Ω. Assume that $\nu\in N^{1,p}_0(\Omega)^*$ is a Radon measure such that u is a solution of*

$$\int_\Omega |Du|^{p-2}Du\cdot D\varphi\,d\mu = \int_\Omega \varphi\,d\nu,$$

for all $\varphi\in N^{1,p}_0(\Omega)$. If $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $0<\alpha<1$, then there is a constant $M>0$ such that

$$\frac{\nu(B(x,r))}{\mu(B(x,r))}\le Mr^{-p+\alpha(p-1)},$$

whenever $B(x,4r)\subset\Omega$.

Proof. Fix any ball $B(x,r)$ such that $B(x,4r)\subset\Omega$. From Lemma 4.8 in [BMS] we get

$$r^p\frac{\nu(B(x,r))}{\mu(B(x,r))}\le C(\inf_{B(x,r)}u-\inf_{B(x,2r)}u)^{p-1}.$$

Since $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, there holds

$$\frac{\nu(B(x,r))}{\mu(B(x,r))}\le Mr^{-p+\alpha(p-1)}.$$

This finishes the proof of Theorem 4.3. □

5. Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into following lemmas.

Lemma 5.1. *Let $K\subset\Omega$ be a non-empty compact set. Suppose ψ is continuous with*

$$|\psi(x)-\psi(y)|\le C_\psi d(x,y)^\alpha$$

for all $x\in K$ and $y\in\Omega$, where $C_\psi>0$ and $\alpha>0$. Let $u=\hat{R}^\psi$ and ν be the Riesz measure associated with u, see (13) *and Theorem 3.10. Then*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))}\le Cr^{-p+\alpha(p-1)}$$

for all $r<r_0=\frac{1}{360}\operatorname{dist}(K,\partial\Omega)$ and $x\in K$, $C=C(p,M,\alpha,C_d,C_\psi)>0$.

Proof. Let $I:=\{x\in\Omega:\psi(x)=u(x)\}$. First, let $x_0\in I$. We may assume $u(x_0)=0=\psi(x_0)$. If $r\le\frac{1}{20}\operatorname{dist}(x_0,X\setminus\Omega)$ and $\gamma_0:=\operatorname{osc}(\psi,B(x_0,20r))$, then $(u-\gamma_0)^+$ is a subsolution by Lemma 3.7, and $u+\gamma_0$ is a nonnegative supersolution in $B(x_0,20r)$.

Hence by [KS, Theorem 4.2 with Remark 4.4] and [BMS, Lemma 4.5]

$$\begin{aligned}\sup_{B(x_0,r)}(u-\gamma_0)&\le C\left(\fint_{B(x_0,2r)}|(u-\gamma_0)^+|^{p-1}\,d\mu\right)^{\frac{1}{p-1}}\\&\le C\left(\fint_{B(x_0,2r)}(u+\gamma_0)^{p-1}\,d\mu\right)^{\frac{1}{p-1}}\\&\le C\inf_{B(x_0,4r)}(u+\gamma_0)\le C\gamma_0.\end{aligned}$$

Since $u\ge\psi\ge-\gamma_0$, we have

$$\operatorname{osc}(u,B(x_0,r))\le c\gamma_0=c\ \operatorname{osc}(\psi,B(x_0,20r)).\tag{21}$$

Let $r\le\frac{1}{160}\operatorname{dist}(x_0,\partial\Omega)$ and let $\eta\in\operatorname{Lip}_0(B(x_0,2r))$ be a nonnegative cut-off function with $\eta=1$ in $B(x_0,r)$ and $|D\eta|\le C/r$ in $B(x_0,2r)$. Then by (13) and by the Hölder inequality

$$\begin{aligned}\nu(B(x_0,r)) &\le \int_{B(x_0,2r)} \eta^p \, d\nu = \int_{B(x_0,2r)} |Du|^{p-2} Du \cdot D\eta^p \, d\mu \\ &\le p \int_{B(x_0,2r)} \eta^{p-1} |Du|^{p-1} |D\eta| \, d\mu \\ &\le C \left(\int_{B(x_0,2r)} |Du|^p \eta^p \, d\mu \right)^{\frac{p-1}{p}} \left(\int_{B(x_0,2r)} |D\eta|^p \, d\mu \right)^{1/p} \\ &\le C r^{-p} \mu(B(x_0,r)) \operatorname{osc}(u, B(x_0,4r))^{p-1} \\ &\le C r^{-p} \mu(B(x_0,r)) \operatorname{osc}(\psi, B(x_0,80r))^{p-1},\end{aligned}$$

where in the second last step we used the De Giorgi inequality (15), and in the last step (21).

Now, if $x_0 \in I$ is such that

$$\operatorname{dist}(x_0, K) \le r \le 2r_0,$$

we have

$$\frac{\nu(B(x_0,r))}{\mu(B(x_0,r))} \le C r^{-p+\alpha(p-1)}, \tag{22}$$

where $C = C(p, C_\psi, C_d) > 0$.

For $x_0 \in K$ and $r < r_0$, either $B(x_0,r) \cap I = \emptyset$ and thus $\nu(B(x_0,r)) = 0$, or there is $x \in B(x_0,r) \cap I$. In the latter case we have by (22)

$$\frac{\nu(B(x_0,r))}{\mu(B(x_0,r))} \le C(C_d) \frac{\nu(B(x,2r))}{\mu(B(x,2r))} \le C r^{-p+\alpha(p-1)}$$

and the lemma is proven. □

Lemma 5.2. *Let $E \subset \Omega$ be a closed set and $\beta \in (-p,-1]$. Suppose that u is a continuous function in Ω and p-harmonic in $\Omega \setminus E$ such that*

$$|u(x_0) - u(y)| \le C d(x_0,y)^{(\beta+p)/(p-1)} \tag{23}$$

for all $y \in \Omega$, $x_0 \in E$. If E is of weighted β-Hausdorff measure zero, then u is p-harmonic in Ω.

Proof. Fix a regular set $D \subset\subset \Omega$. Let $v = \hat{R}^u = \hat{R}^u(D)$ and let ν be a Riesz measure associated with v, see (13) and Theorem 3.10. Let $K \subset E \cap D$ be a compact set and $\alpha = (\beta+p)/(p-1)$. Now $\beta = -p + \alpha(p-1)$, so from (23) and Lemma 5.1 we infer

$$\nu(B(x,r)) \le C r^\beta \mu(B(x,r))$$

for all $r \le \frac{1}{360} \operatorname{dist}(K, \partial\Omega)$ and $x \in K$. Because $\mathscr{H}^\beta_\mu(K) = 0$, we may cover K by balls $B(x_j, r_j)$ so that

$$\nu(K) \le \sum_j \nu(B(x_j,r_j)) \le C \sum_j r_j^\beta \mu(B(x_j,r_j)) < \varepsilon,$$

where $\varepsilon > 0$ is any given number. It follows that $\nu(E \cap D) = 0$ and thus $\nu = 0$. Now $v \in N^{1,p}_{\mathrm{loc}}(D)$ is continuous by Theorem 3.10 and p-harmonic in D by (13).

Next let $w = \underline{\hat{R}}^u(D)$. Similarly we find that w is p-harmonic in D. Since $v = u = w$ on ∂D by Theorem 3.11, we have by the uniqueness of p-harmonic functions, Theorem 5.6 in [Sh2], that $v = w$ in D. Since

$$w \le u \le v = w,$$

u is p-harmonic in D and the result follows, since any bounded open set can be exhausted from inside by regular open sets, see [BB2]. □

Now we obtain the main results of this section.

Corollary 5.3. *Suppose that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, $0 < \alpha \le 1$, is p-harmonic in $\Omega \setminus E$. If E is a closed set of weighted $(-p + \alpha(p-1))$-Hausdorff measure zero, then u is p-harmonic in Ω.*

In the following theorem, we show that the Corollary 5.3 is sharp, when $0 < \alpha < \kappa$.

Theorem 5.4. *Let κ be as in (2) and $0 < \alpha < \kappa$. Suppose that $E \subset \Omega$ is a closed set with positive weighted $(-p+\alpha(p-1))$-Hausdorff measure. Then there is $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ which is p-harmonic in $\Omega \setminus E$, but does not have a p-harmonic extension to Ω.*

Proof. Let $K \subset E$ be compact with

$$\mathscr{H}_\mu^{-p+\alpha(p-1)}(K) > 0.$$

By Frostman's lemma, Lemma 2.6, there exist $\delta > 0$ and a nonnegative Radon measure ν living on K with $\nu(K) > 0$ such that

$$\nu(B(x,r)) \le C r^{-p+\alpha(p-1)} \mu(B(x,r)),$$

for all balls $B(x,r) \subset X$ with $0 < r \le \delta$. Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu, \tag{24}$$

for all $\varphi \in N^{1,p}_0(\Omega)$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ by Theorem 4.1 and it is p-harmonic function in $\Omega \setminus E$ by (24), since $\nu(\Omega \setminus E) = 0$. However u does not have a p-harmonic extension to Ω, since $\nu(E) > 0$. □

Proof of Theorem 1.1. Theorem 1.1 follows from Corollary 5.3 and Theorem 5.4. □

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