Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 33, 2008, 549–560

OMEGA RESULT FOR THE ERROR TERM IN THE MEAN SQUARE FORMULA FOR DIRICHLET L-FUNCTIONS

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Abstract. Let q be a positive integer and let $E(q,x)$ denote the error term in the asymptotic formula for the mean value $\sum_{\chi \bmod q} \int_0^x |L(1/2+it,\chi)|^2\,dt$. We obtain in this paper an Ω-result for $E(q,x)$, which is an extension of the corresponding Ω-result for the Riemann zeta-function.

1. Introduction

In 1949, Atkinson[1] discovered an explicit formula for the error term $E(t)$ in the mean square formula of the Riemann zeta-function $\zeta(s)$,

$$\int_0^t |\zeta(\frac{1}{2}+iu)|^2\,du = t\log\frac{t}{2\pi} + (2\gamma-1)t + E(t) \quad (t>0).$$

His result is not merely a refinement of the mean square formula established by Littlewood [9], but has important applications as well. For instance, Heath-Brown [5] applied Atkinson's formula, amongst other tools, to establish an estimate for the twelfth power moment of $\zeta(s)$. In addition, based on this formula more analogous properties of $E(t)$, $\Delta(t)$ and $P(t)$ are explored, which are by no means obvious from their definitions. ($\Delta(t)$ and $P(t)$ denote the error terms in the Dirichlet divisor problem and the circle problem respectively.) Nowadays we have the (unsettled) conjecture that all of $E(t)$, $\Delta(t)$ and $P(t)$ are $O_\varepsilon(t^{1/4+\varepsilon})$. The opposite direction of this problem, that is the Ω-results, was recently advanced by Soundararajan [12] for $\Delta(t)$ and $P(t)$, and by the authors [8] for $E(t)$, superseding the respective records in [3] and [4]. These Ω-results are believed to be sharp up to the $\log\log$-factor.

In this paper, we are concerned with the error term in the mean square formula of Dirichlet L-functions. Let $q\in\mathbf{N}$ and define

$$E(q,x) := \sum_{\chi \bmod q}\int_0^x |L(1/2+it,\chi)|^2\,dt - \frac{\phi(q)^2}{q}x\big(\log\frac{qx}{2\pi} + \sum_{p|q}\frac{\log p}{p-1} + 2\gamma - 1\big)$$

2000 Mathematics Subject Classification: Primary 11M06.
Key words: Mean-square formula, Atkinson's formula, omega-results.

where χ is a Dirichlet character mod q. Meurman [11] generalized Atkinson's formula to the case of $E(q,x)$ and in particular, proved that

$$E(q,x) \ll_\varepsilon \begin{cases} (qx)^{1/3+\varepsilon} + q^{1+\varepsilon} & \text{if } q \ll x, \\ (qx)^{1/2+\varepsilon} + qx^{-1} & \text{if } q \gg x. \end{cases}$$

From this, it follows a mean estimate of $\sum_{\chi \bmod q} |L(1/2+it,\chi)|^2$ over a short interval $[T, T+H]$, and a subconvex pointwise estimate by Heath-Brown's trick in [5]. (The readers are referred to [11] for details, and [2] for an alternative approach with some refinements.) However it is not expected to get a very strong subconvexity estimate along this line of argument. Indeed for the case $q = 1$ which reduces to the situation of the Riemann zeta-function, we have proved in [8] that

$$E(1,t) = \Omega\bigg((t\log t)^{\frac14} (\log_2 t)^{\frac34 (2^{4/3}-1)} (\log_3 t)^{-\frac58} \bigg).$$

On the other hand, the Ω-result for $E(q,t)$ with $q > 1$ is not present in the literature. Our purpose here is to establish the following.

Theorem 1. *There are absolute constants $c_0 > 0$ and $0 < \theta < 10^{-3}$ such that for all sufficiently large X, and all integers $1 \le q \le X^\theta$, there exists an $x \in [X, X^3]$ for which*

$$|E(q,x)| \ge c_0 e^{-2\sum_{p|q} p^{-1/4}} \cdot (qx\log x)^{\frac14} (\log_2 x)^{\frac34(2^{4/3}-1)} (\log_3 x)^{-\frac58}.$$

Here $\log_j$ denotes the j-th iterated logarithm, i.e. $\log_2 = \log\log$, $\log_3 = \log\log\log$ etc.

Remarks. (i) As in the case of $E(t)$ (i.e. $q = 1$), we expect the result is sharp (in the x-aspect) up to a factor of $(\log_2 x)^{o(1)}$.

(ii) When $\omega(q) \ll (\log_3 x)^{4/3} (\log_4 x)^{1/3}$ where $\omega(q)$ denotes the number of distinct prime factors of q, the factor $e^{2\sum_{p|q} p^{-1/4}}$ is of size $(\log_2 x)^{o(1)}$.

(iii) For each *fixed* q, we have

$$E(q,x) = \Omega\big((x\log x)^{\frac14} (\log_2 x)^{\frac34(2^{4/3}-1)} (\log_3 x)^{-\frac34} \big) \qquad (\text{as } x \to \infty),$$

which extends the result for $E(t)$.

To prove our theorem, we apply Soundararajan's ingenious method to series of the form $\sum_n f(n)\cos(2\pi\lambda_n x + \beta)$, where the coefficients $f(n)$ are non-negative. But like the case with $E(t)$, the method cannot be applied directly. First, $E(q,x)$ is not of this form, and this can be remedied by convolving with a kernel function. Second, the coefficients of the resulting series are not of constant sign. (A crucial point of Soundararajan's method is the non-negativity of $f(n)$.) We shall proceed as in [8] to get around the difficulty. However, the oscillating factors in this case are more subtle, see (2.1), which will be resolved by employing an averaging process (in (3.3)) with the möbius function. This sifting process does not entirely compromise with the admissible range in the convolution. It turns out that extra terms come up and we need to invoke a good upper estimate of $\Delta(x)$ to control these.

2. Preliminaries

We begin with the following result of Meurman [11, Theorem 1]: for $x > 3$ and $N \asymp x$,

$$E(q,x) = \frac{\phi(q)}{q} \sum_{k|q} k\mu(\frac{q}{k}) \Big\{ \Sigma_1(k,x) + \Sigma_2(k,x) \Big\} + O(\frac{d(q)\phi(q)}{\sqrt{q}} \log^2 x + \frac{q}{x}) \tag{2.1}$$

where

$$\Sigma_1(k,x) := \Big(\frac{2x}{\pi}\Big)^{1/4} \sum_{n \le N} (-1)^{kn} \frac{d(n)}{(kn)^{3/4}} e(x,kn) \cos f(x,kn)$$

and

$$\Sigma_2(k,x) := -2 \sum_{n \le kB(x,\sqrt{kN})} \frac{d(n)}{\sqrt{kn}} \Big(\log \frac{kx}{2\pi n} \Big)^{-1} \cos g(x, \frac{n}{k}).$$

Here, we have used the following notation:

$$\begin{aligned}
d(n) &:= \sum_{k|n} 1, \text{ the divisor function,} \\
e(x,n) &:= \Big(1 + \frac{\pi n}{2x}\Big)^{-1/4} \Big(\frac{\pi n}{2x}\Big)^{1/2} \left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2x}}\right)^{-1}, \\
f(x,n) &:= 2x \operatorname{arsinh}\sqrt{\frac{\pi n}{2x}} + (\pi^2 n^2 + 2\pi n x)^{1/2} - \frac{\pi}{4}, \\
g(x,X) &:= x \log \frac{x}{2\pi X} - x + 2\pi X + \frac{\pi}{4}, \\
B(x,X) &:= \frac{x}{2\pi} + \frac{X^2}{2} - X \left(\frac{x}{2\pi} + \frac{X^2}{4}\right)^{1/2} = \left\{ \sqrt{\frac{x}{2\pi} + \left(\frac{X}{2}\right)^2} - \frac{X}{2} \right\}^2.
\end{aligned}$$

Recall that $\operatorname{arsinh}(x) = \log(x + \sqrt{x^2+1})$.

Define

$$E_1(q,x) := (2x)^{-\frac{1}{2}} E(q, 2\pi x^2) \quad \text{for} \quad x \ge 1, \tag{2.2}$$

and consider its convolution with the Fejér kernel

$$K(u) := (\pi u)^{-2} \sin^2(\pi u).$$

Lemma 1. *Let x be any sufficiently large number and let $m \geq 1$ be any given integer. We have for all $1 \leq z \leq x^{\frac{2}{7}(1-1/m)}$,*

$$\int_{-z}^{z} E_1(q, x+uz^{-1})e(-u)K(u)\,du$$
$$= \frac{1}{2}e(-\frac{1}{8})\frac{\phi(q)}{q}\sum_{k|q} k^{1/4}\mu(\frac{q}{k})\sum_{n\leq z^2/k}(-1)^{kn}d(n)n^{-\frac{3}{4}}e(2\sqrt{nkx})\widehat{K}(1-2z^{-1}\sqrt{nk})$$
$$+\frac{d(q)\phi(q)}{q}O_m\left(qx^{-2/7}+x^{-1/(15m)}+z^{-1/2}\log z\right)+O(qx^{-5/2})$$

where $\widehat{K}(y)=\max(0,1-|y|)$ is the Fourier transform of $K(u)$, and the O-constants are absolute or depend on m only. Here and in the sequel $e(y) := e^{2\pi iy}$.

Proof. The argument is essentially the same as that in [7, Section 4], but here we require a wider range of z. In view of (2.1) and (2.2), we need to evaluate

$$J_i := \int_{-z}^{z} \Sigma_i^*(k, x+uz^{-1})e(-u)K(u)\,du \quad (i=1,2),$$

where $\Sigma_i^*(k,x) = (2x)^{-1/2}\Sigma_i(k, 2\pi x^2)$.

Consider the case $i = 2$. We have

$$J_2 = -\sqrt{\frac{2}{k}}\sum_{n\leq kB}\frac{d(n)}{\sqrt{n}}\int_F^z H(x+uz^{-1}, n/k)\cos g(2\pi(x+uz^{-1})^2, n/k)e(-u)K(u)\,du$$

where $B = B(2\pi(x+1)^2, \sqrt{kN})$, $F = \max\left(-z, z(\sqrt{n/k+\sqrt{nN}}-x)\right)$, and

$$H(x+uz^{-1}, n/k) = (x+uz^{-1})^{-1/2}\Big(\log\frac{(x+uz^{-1})^2}{n/k}\Big)^{-1},$$

which is monotonically decreasing in u. Take $N = 2[x^2]$. We express $K(u)$ in terms of the Fourier transform $\widehat{K}(u)$ and write the cosine function into a combination of exponential functions. The integral $\int_F^z$ becomes half of the sum of

$$\int_{-1}^{1}(1-|y|)\int_F^z H(x+uz^{-1}, n/k)e^{i(\pm g(2\pi(x+uz^{-1})^2, n/k)+2\pi(y-1)u)}\,du\,dy.$$

Notice that $\pm\partial_u g + 2\pi(y-1) \gg xz^{-1}$ for $n \leq kB \leq 0.4kx^2$ and $|y| \leq 1$. Thus, by the first derivative test, the above inner integral over u is $\ll \max_u H(u)(x/z)^{-1} \ll zx^{-3/2}$. Thus,

$$J_2 \ll \frac{1}{\sqrt{k}}\sum_{n\leq kB}\frac{d(n)}{\sqrt{n}}zx^{-3/2} \ll \frac{z\log x}{\sqrt{kx}},$$

since $B \sim x^2/k$. The contribution of this to the integral in the lemma is hence

$$\ll \frac{d(q)\phi(q)}{\sqrt{q}}\frac{z\log x}{\sqrt{x}}$$

which is absorbed in $d(q)\phi(q)q^{-1}O(qx^{-2/7}+x^{-1/(15m)})$.

Next, we evaluate J_1. As in [7, p. 59], we have

(i) $$e(2\pi(x+uz^{-1})^2,kn)=\begin{cases}1+O(knx^{-2}) & \text{for } 1\le kn\ll x^2,\\ O((kn)^{1/4}x^{-1/2}) & \text{for } kn\gg x^2;\end{cases}$$

$$\partial_u e(2\pi(x+uz^{-1})^2,kn)\ll\begin{cases}knz^{-1}x^{-3} & \text{for } 1\le kn\ll x^2,\\ (kn)^{1/4}z^{-1}x^{-3/2} & \text{for } kn\gg x^2.\end{cases}$$

(ii) $f(2\pi(x+uz^{-1})^2,kn)=4\pi\sqrt{kn}(x+uz^{-1})-\pi/4+O((kn)^{3/2}x^{-1})$ for $kn\ll x^2$. Furthermore

$$\partial_u f(2\pi(x+uz^{-1})^2,kn)\gg\begin{cases}\sqrt{kn}z^{-1} & \text{for } 1\le kn\ll x^2,\\ z^{-1}x\log(knx^{-2}) & \text{for } kn\gg x^2;\end{cases}$$

$$\partial_u^2 f(2\pi(x+uz^{-1})^2,kn)\ll\begin{cases}(kn)^{3/2}z^{-2}x^{-3} & \text{for } 1\le kn\ll x^2,\\ z^{-2}\log(knx^{-2}) & \text{for } kn\gg x^2.\end{cases}$$

We interchange the summation and integration in J_1, then apply partial integration to the u-integral in

$$\begin{aligned}J_1(x,n)&:=\int_{-z}^{z}e(2\pi(x+uz^{-1})^2,kn)\cos f(2\pi(x+uz^{-1})^2,kn)e(-u)K(u)\,du\\ &=\frac{1}{2}\sum_{\pm}\int_{-1}^{1}(1-|y|)\int_{-z}^{z}e(2\pi(x+uz^{-1})^2,kn)e^{i(\pm f(2\pi(x+uz^{-1})^2,kn)+2\pi(y-1)u)}\,du\,dy\end{aligned}$$

for $kn\gg x^{\frac{4}{7}(1-1/(2m))}$. But this time we need a higher order of approximation,

$$\frac{1}{\pm\partial_u f+2\pi(y-1)}=\frac{1}{\pm\partial_u f-2\pi}\sum_{0\le r<m}\frac{(2\pi y)^r}{(\mp\partial_u f+2\pi)^r}+O(|\pm\partial_u f-2\pi|^{m+1}).$$

Suppose $kn\ll x^2$. We note that $\int_{-1}^{1}(2\pi iy)^r(1-|y|)e(\pm zy)\,dy=K^{(r)}(z)\ll_r z^{-2}$ for $|z|\gg 1$. By the above estimates in (ii), $\partial_u f\pm 2\pi\gg\sqrt{kn}z^{-1}\gg x^{1/(7m)}$ when x is greater than a suitable absolute constant. The remaining part of the partial integration causes a term of $O(x^{-3}z\sqrt{kn})$. It follows that

$$\begin{aligned}J_1(x,n)&\ll\sum_{0\le r<m}\left(\frac{z}{\sqrt{kn}}\right)^{r+1}K^{(r)}(z)+\left(\frac{z}{\sqrt{kn}}\right)^{m+1}+x^{-3}z\sqrt{kn}\\ &\ll\left(\frac{z}{\sqrt{kn}}\right)^{m+1}+\frac{1}{z\sqrt{kn}}\end{aligned}$$

for these n's. For $x^2\ll kn\le kN$, the same argument shows that

$$J_1(x,n)\ll\big((zx^{-1})^{m+1}+(zx)^{-1}\big)(kn)^{1/4}x^{-1/2}.$$

Thus,

$$\begin{aligned}&\sum_{x^{\frac{4}{7}(1-1/(2m))}\ll kn}(-1)^{kn}\frac{d(n)}{(kn)^{3/4}}J_1(x,n)\\&\ll\sum_{x^{\frac{4}{7}(1-1/(2m))}\ll kn\le x^2}\frac{d(n)}{(kn)^{3/4}}\Big(\Big(\frac{z}{\sqrt{kn}}\Big)^{m+1}+\frac{1}{z\sqrt{kn}}\Big)\\&\quad+\sum_{x^2/k\le n\le N}\frac{d(n)}{(kn)^{3/4}}((zx)^{-1}+(zx^{-1})^{m+1})(kn)^{1/4}x^{-1/2}\\&\ll\big((kzx^{\frac{1}{7}(1-1/(2m))})^{-1}+(z\sqrt{kx})^{-1}+zk^{-1}x^{-\frac{1}{7}(1-1/(2m))}(zx^{-\frac{2}{7}(1-1/(2m))})^m\big)\log x.\end{aligned}$$

Thus J_1 is reduced to

$$\sum_{n\ll k^{-1}x^{\frac{4}{7}(1-1/(2m))}}(-1)^{kn}\frac{d(n)}{(kn)^{3/4}}J_1(x,n)+O\big(k^{-1}x^{-1/(15m)}+(z\sqrt{kx})^{-1}\log x\big). \tag{2.3}$$

For $n\ll k^{-1}x^{\frac{4}{7}(1-1/(2m))}$, we use the approximate formulae in (i) and (ii) together with partial integration to show that

$$\begin{aligned}J_1(x,n)&=\int_{-z}^{z}\cos(4\pi\sqrt{kn}(x+uz^{-1})-\pi/4)e(-u)K(u)\,du+O((kn)^{3/2}x^{-1})\\&=e(2\sqrt{kn}x-\frac{1}{8})\widehat{K}(1-2z^{-1}\sqrt{kn})+O\big(\min(z^{-1},(kn)^{-1/2})+(kn)^{3/2}x^{-1}\big).\end{aligned}$$

Immediately the first summand yields the main term, and plainly

$$\begin{aligned}&\sum_{n\ll k^{-1}x^{\frac{4}{7}(1-1/(2m))}}\frac{d(n)}{(kn)^{3/4}}\big(\min(z^{-1},(kn)^{-1/2})+(kn)^{3/2}x^{-1}\big)\\&\ll k^{-1}z^{-1/2}\log z+k^{-1}x^{-1/(15m)}.\end{aligned}$$

So this part with the O-term in (2.3) yields at most $q^{-1}\phi(q)d(q)\cdot O\big(q^{1/2}x^{-1/2}\log x+x^{-1/(15m)}+z^{-1/2}\log z\big)$. Our proof is thus complete. □

Define for $\nu=0,1,2,\ldots$,

$$\Phi_\nu(x,T)=\sum_n(-1)^{\nu n}\frac{d(n)}{n^{3/4}}e(2\sqrt{n}x)\widehat{K}(1-2T^{-1}\sqrt{n}).$$

Lemma 2. *Let $T\ge 2$ and $\nu=0$ or 1. For all $T^{-1}<x\le 200$, we have*

$$\Phi_\nu(x,T)\ll x^{-1/2}\log T.$$

Let $\Delta(x)=\sum_{n\le x}d(n)-x(\log x+2\gamma-1)$ for $x\ge 1$. Suppose the number $\kappa\in(1/4,1/2]$ satisfies $\Delta(x)\ll_\kappa x^\kappa$ for all $x\ge 1$. Then, $\Phi_\nu(x,T)\ll_\kappa x^{2\kappa-1/2}$ for all $x\ge 200$.

Proof. Consider the case $\nu = 0$. To deal with small x, we apply Stieltjes integration to express

$$
\begin{aligned}
\Phi_0(x,T) &= \int_1^{T^2} (\log t + 2\gamma) t^{-3/4} e(2\sqrt{tx}) \widehat{K}(1 - 2T^{-1}\sqrt{t})\, dt \\
&\quad + \widehat{K}(1 - 2T^{-1}\sqrt{t}) e(2\sqrt{tx}) t^{-3/4} \Delta(t) \Big|_1^{T^2} \\
&\quad - \int_1^{T^2} \Delta(t) \cdot \frac{d}{dt}\Big(\frac{e(2\sqrt{tx})}{t^{3/4}} \widehat{K}(1 - 2T^{-1}\sqrt{t}) \Big)\, dt.
\end{aligned} \tag{2.4}
$$

Note that $\widehat{K}(1-2T^{-1}\sqrt{t}) = 2\sqrt{t}/T$ if $t \le (T/2)^2$ and $2(1-T^{-1}\sqrt{t})$ if $(T/2)^2 \le t \le T^2$. We split the first integral into two pieces by dividing the range of integration at $t = Y$. The integral over $[1, Y]$ is imposed the trivial bound $O(Y^{1/4}\log Y)$. For the other part (over $[Y, T^2]$), a partial integration yields

$$
\begin{aligned}
&\int_Y^{T^2} (\log t + 2\gamma) t^{-1/4} \widehat{K}(1 - 2T^{-1}\sqrt{t})\, d\frac{e(2\sqrt{tx})}{2\pi i x} \\
&\ll x^{-1} Y^{-1/4} \log Y + x^{-1} \int_Y^{T^2} \frac{\log t}{t^{1/4}} \Big| \frac{d}{dt} \widehat{K}(1 - 2T^{-1}\sqrt{t}) \Big|\, dt \\
&\ll x^{-1} Y^{-1/4} \log Y + x^{-1} T^{-1/2} \log T.
\end{aligned}
$$

Optimizing with the choice $Y = x^{-2}$ ($\le T^2$), the first term in (2.4) is $\ll x^{-1/2}\log T$. The second term in (2.4) is clearly $\ll 1$ as $\Delta(t) \ll t^{1/2}$. Furthermore, it is known that

$$
\int_1^X |\Delta(t)|\, dt \ll X^{5/4},
$$

whence the last term in (2.4) is

$$
\ll \int_1^{T^2} (x t^{-5/4} + t^{-7/4}) |\Delta(t)|\, dt \ll x^{-1/2} \log T,
$$

for $x \ll x^{-1/2}$. This finishes the proof of this case.

When x is large, we express $\Phi_0(x,T)$ in terms of $\Delta(x)$. Recall the truncated Voronoi series (see [10])

$$
\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le M} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + R_M(x) \tag{2.5}
$$

where $M \ge x^6$ and $R_M(x) \ll x^{-1/4}$ if $\|x\| \gg x^{5/2} M^{-1/2}$ and $R_M(x) \ll x^{\varepsilon}$ otherwise. Then, by taking $M = \max(x^6, T^2)$,

$$
\Phi_0(x,T) = 2\sqrt{2}\pi e(\frac{1}{8}) \int_{-T}^{T} (x + uT^{-1})^{-1/2} \Delta((x + uT^{-1})^2) e(-u) K(u)\, du + O(\log x).
$$

From $\Delta(x) \ll x^{\kappa}$, our result follows as the L^1-norm of $K(u)$ is bounded.

Next we consider the case $\nu = 1$. To this end, we introduce the function

$$\Delta^*(x) := \frac{1}{2} \sum_{n \le 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1).$$

Then we have (see [6])

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x),$$

hence $\Delta^*(x) \ll x^\kappa$ and $\int_1^X |\Delta^*(t)|\, dt \ll X^{5/4}$. It is easily verified that

$$-d(n) + 4d(\frac{n}{2}) - 2d(\frac{n}{4}) = (-1)^n d(n)$$

holds for each integer n (here $d(y) := 0$ if y is not a positive integer). Hence, by (2.5) with $M \ge x^6$,

$$\Delta^*(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le M} (-1)^n \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(|R_M(x)|).$$

Thus, the whole argument for the case $\nu = 0$ can be carried over to $\Phi_1(x, T)$. $\square$

Lemma 3. *Let $0 < \alpha < 1/(2\kappa)$ be arbitrary but fixed where κ is defined as in Lemma 2. Then for any $2 \le z \le x$ and $\nu = 0, 1$, we have*

$$\frac{1}{d^{1/4}} \Phi_\nu(\frac{x}{\sqrt{d}}, \sqrt{d}z) \ll_{\alpha,\kappa} x^{-\alpha\kappa}$$

uniformly for $x^{2-1/(2\kappa)+\alpha} \le d \le \exp(x^{\frac{1}{2}-\alpha\kappa})$. The implied constant depends on κ and α only.

Proof. When $x/\sqrt{d} \ge 200$, the latter part of Lemma 2 implies that

$$\frac{1}{d^{1/4}} \Phi_\nu(\frac{x}{\sqrt{d}}, \sqrt{d}z) \ll_\kappa \left(\frac{x^{2-1/(2\kappa)}}{d} \right)^\kappa$$

and hence the result. As $x/\sqrt{d} \ge 1/(\sqrt{d}z)$, we infer by Lemma 2 again that

$$\frac{1}{d^{1/4}} \Phi_\nu(\frac{x}{\sqrt{d}}, \sqrt{d}z) \ll \frac{1}{d^{1/4}} \left(\frac{x}{\sqrt{d}} \right)^{-1/2} \log(dz) \ll x^{-1/2} \log(dx) \ll_{\alpha,\kappa} x^{-\alpha\kappa}. \quad \square$$

3. Proof of Theorem

Let X be any sufficiently large number and $x \in [X, X^{3/2}]$. Let τ be a parameter satisfying $1 \le \tau \le \log X$ which will be specified later. Also we let $q = \prod_{r=1}^{\omega(q)} p_r^{v_r}$ and assume $1 \le q \le x^\theta$ with $\theta < 8/21$. (Further restriction on the size of θ will be imposed below.)

In view of Lemma 1, it suffices to study the omega result of the finite series there. For simplicity, we denote

$$I(x, z) := 2e(\frac{1}{8}) \frac{q^{3/4}}{\phi(q)} \int_{-z}^{z} E_1(q, x + uz^{-1}) e(-u) K(u)\, du.$$

It is not hard to show by Lemma 1 the following:

(i) if q is odd, then

$$I(\frac{x}{\sqrt{q}}, z\sqrt{q}) = \sum_{k|q} \frac{\mu(k)}{k^{1/4}} \Phi_1(\frac{x}{\sqrt{k}}, \sqrt{k}z) + O(1);$$

(ii) if $q = 2q'$ for some odd q', then

$$I(\frac{x}{\sqrt{q}}, z\sqrt{q}) = \sum_{k|q'} \frac{\mu(k)}{k^{1/4}} \Phi'(\frac{x}{\sqrt{k}}, \sqrt{k}z) + O(1)$$

where $\Phi'(x, z) = \Phi_0(x, z) - 2^{-1/4}\Phi_1(2^{-1/2}x, 2^{1/2}z)$;

(iii) if $q = 2^s q'$ where $s \geq 2$, then

$$I(\frac{x}{\sqrt{q}}, z\sqrt{q}) = \sum_{k|q} \frac{\mu(k)}{k^{1/4}} \Phi_0(\frac{x}{\sqrt{k}}, \sqrt{k}z) + O(1).$$

The treatment of Cases (i) and (iii) are the same, but Case (ii) involves more technicality. In all the three cases, the presence of the oscillatory factors $\mu(k)$ causes the main difficulty. We shall apply an averaging process, in Step 1 below, to remove them. To make the process effective, we work on a much longer range of z (than allowed in Lemma 1). Step 2 treats the excessive terms with Lemma 3. Then we are in a position to complete the proof with the known omega result from [8] or [12], which is our Step 3.

To fix ideas, we consider first Case (i) and then explain at the end the treatment of the other two cases.

Case (i). *Step 1.* Suppose that q is an (odd) integer satisfying $1 \leq q \leq x^\theta$ with $0 < \theta < \kappa/4$. For each p_r dividing q, we find the smallest positive integer D_r such that

$$x^{2-1/(2\kappa)+3/4} \leq p_r^{D_r}, \qquad \text{thus} \quad \prod_{r=1}^{\omega(q)} p_r^{D_r+1} \ll \exp(4(\log x)^2), \tag{3.1}$$

since $\omega(q) \ll \log q/\log_2 q$. Recall that κ takes the value assumed in Lemma 2.

To remove the Möbius function, we need the following fact

$$\sum_{kl=p^\alpha,\, l|p^D} \mu(k) = \begin{cases} 1 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha = D+1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Let us consider, for $j = 1, \ldots, \omega(q)$, the sum

$$\Psi_j(x, z) := \sum_{l|\prod_{r=j}^{\omega(q)} p_r^{D_r}} \frac{1}{l^{1/4}} \sum_{k|\prod_{r=j}^{\omega(q)} p_r} \frac{\mu(k)}{k^{1/4}} \Phi_1(\frac{x}{\sqrt{kl}}, \sqrt{kl}z). \tag{3.3}$$

By virtue of (3.2), we see that

$$\Psi_1(x,\tau) = \Psi_2(x,\tau) - \frac{1}{p_1^{(D_1+1)/4}}\Psi_2\Big(\frac{x}{\sqrt{p_1^{D_1+1}}}, \sqrt{p_1^{D_1+1}}\tau\Big).$$

Besides, in view of (3.1) we can apply Lemma 3 (with $d = klp_1^{D_1+1}$) to each Φ_1 in the second Ψ_2. After summing over l and k, this part is

$$\ll \prod_{r=2}^{\omega(q)}(D_r+1)\cdot 2^{\omega(q)}x^{-3\kappa/4} \ll x^{-\kappa/4}$$

because $D_r \le 3\log x$ and $\omega(q) < 2\theta\log x/(\log_2 x)$ for all large x. Thus,

$$\Psi_1(x,\tau) = \Psi_2(x,\tau) + O(x^{-\kappa/4}).$$

Repeating this argument to $\Psi_r(x,\tau)$ with a successive use of Lemma 3, we deduce that

$$\Psi_1(x,\tau) = \Psi_{\omega(q)}(x,\tau) + O(\omega(q)x^{-\kappa/4}) = \Phi_1(x,\tau) + O(1). \tag{3.4}$$

Step 2. We select a number ρ so that

$$4\kappa - 1 < \rho < \frac{2}{7}\Big(1-\frac{1}{m}\Big)$$

for some $m \ge 1$. To see the legitimacy, we make use of the known upper bound $\Delta(x) \ll x^{7/22}$ to set $\kappa = 7/22$, whence we need

$$\frac{3}{11} < \frac{2}{7}\Big(1-\frac{1}{m}\Big)$$

which is satisfied if $m > 22$.[1] Hence we choose $m = 23$ and $\rho = 2\kappa - (5m+2)/(14m) = \frac{967}{3542}$.

Moreover, we take $\Delta = \frac{1}{2\kappa} - \frac{2}{\rho+1} > 0$ and assume $(0 <)\ \theta < \min\{\Theta, \Delta\kappa/4\}$ where

$$\Theta := \frac{\rho}{\rho+1} - \frac{\frac{2}{7}(1-\frac{1}{m})}{\frac{2}{7}(1-\frac{1}{m})+1} < 10^{-3}.$$

Thus for any $l \le x^{2\rho/(\rho+1)}$ and $k < x^{\Theta}$, we have, as $\tau \le 2\log x$,

$$\sqrt{kl}\,\tau < \left(\frac{x}{\sqrt{kl}}\right)^{\frac{2}{7}(1-\frac{1}{m})}.$$

We divide $\Psi_1(x,\tau)$ (see (3.3)) into

$$\Psi_1(x,\tau) = \sum_{l\le x^{2\rho/(\rho+1)}} + \sum_{l> x^{2\rho/(\rho+1)}} \tag{3.5}$$

[1] Our argument will fail if κ cannot take a value less than $9/28$.

so that Lemma 1 is applicable in the first summand. It follows that

$$\sum_{l\le x^{2\rho/(\rho+1)}} = \sum_{\substack{l\mid\prod_{r=1}^{\omega(q)}p_r^{D_r}\\ l\le x^{2\rho/(\rho+1)}}} \frac{1}{l^{1/4}}\Big\{I(\frac{x}{\sqrt{ql}},\sqrt{ql}\tau)+O(1)\Big\}. \tag{3.6}$$

In the second summand, we remark that $2\rho/(\rho+1) = 2-1/(2\kappa)+\Delta$. Thus we bound each $\Phi_1(x/\sqrt{kl},\sqrt{kl}\tau)$ with Lemma 3, so

$$\sum_{l>x^{2\rho/(\rho+1)}} \ll \prod_{r=1}^{\omega(q)}(D_r+1)\cdot 2^{\omega(q)}x^{-\Delta\kappa} \ll x^{3\theta-\Delta\kappa} \ll x^{-\Delta\kappa/4}. \tag{3.7}$$

Step 3. Set $\tau \asymp (\log X)^{\frac12}(\log_2 X)^{\frac12(1-\lambda+\lambda\log\lambda)}(\log_3 X)^{-\frac14}$ with $\lambda = 2^{\frac43}$, we see that as $\Re e\,\Phi_1(x,\tau) = P(x,\tau)$ in [8, (2.5)], there is a constant $c>0$ and $x\in[X^{1/2},X^{3/2}]$ with

$$\Re e\,\Phi_1(x,\tau) > c(\log X)^{\frac14}(\log_2 X)^{\frac34(2^{4/3}-1)}(\log_3 X)^{-\frac58} \tag{3.8}$$

by [8, Lemma 2.3]. Each $I(x/\sqrt{ql},\sqrt{ql}\tau)$ on the right-side of (3.6) is

$$\ll \frac{q^{3/4}}{\phi(q)}\sup_{x\in[X^{1/2},X^{3/2}]}|E_1(q,x)|.$$

Together with (3.4)–(3.7), we conclude that there is an absolute constant $c'>0$ and some $x\in[X,X^3]$ for which

$$|E(q,x)| > c'\frac{\phi(q)}{q}\Big(\sum_{l\mid\prod_{r=1}^{\omega(q)}p_r^{D_r}} l^{-1/4}\Big)^{-1}\cdot(qx\log x)^{\frac14}(\log_2 x)^{\frac34(2^{4/3}-1)}(\log_3 x)^{-\frac58}. \tag{3.9}$$

Now since

$$\frac{\phi(q)}{q}\Big(\sum_{l\mid\prod_{r=1}^{\omega(q)}p_r^{D_r}} l^{-1/4}\Big)^{-1} \ge \prod_{p\mid q}(1-p^{-1})(1-p^{-1/4}) \gg e^{-2\sum_{p\mid q}p^{-1/4}},$$

the desired Ω-result for $E(q,x)$ in Case (i) follows from (3.9).

Cases (ii) and (iii). The above argument clearly works for Case (iii) with an apparent modification of Φ_1 into Φ_0. Note that we then need an omega result of $Q(x,\tau)$ (in [8]) instead.

Finally we consider Case (ii). Lemma 3 clearly holds for Φ' in view of its definition (in (ii)). After applying the arguments in Steps 1 and 2 with q' in place of q, it remains to show that $\Re e\,\Phi'(x,\tau)$ attains an Ω-result of the same order as the right hand side of (3.8). To this end, we need to retrieve the method of Soundararajan [12] in which a key ingredient is the non-negativity of the coefficients. This method will still apply if Φ' can be expressed as a series with positive coefficients. We begin with the observation that the summands of the even n's in $\Phi_1(2^{-1/2}x,2^{1/2}\tau)$ constitute

$$\sum_{n \text{ even}} \frac{d(n)}{n^{3/4}} e(2\sqrt{\frac{n}{2}}x)\widehat{K}(1-2\tau^{-1}\sqrt{\frac{n}{2}}) = 2^{-3/4}\sum_n \frac{d(2n)}{n^{3/4}} e(2\sqrt{n}x)\widehat{K}(1-2\tau^{-1}\sqrt{n}).$$

Hence we deduce that

$$\begin{aligned}\Phi'(x,\tau) &= \sum_n \big(d(n) - \frac{1}{2}d(2n)\big)n^{-3/4}e(2\sqrt{n}x)\widehat{K}(1-2\tau^{-1}\sqrt{n}) \\ &\quad + 2^{-1/4}\sum_{n \text{ odd}} \frac{d(n)}{n^{3/4}} e(\sqrt{2n}x)\widehat{K}(1-\tau^{-1}\sqrt{2n})\end{aligned}$$

possesses the desired property, for $d(n) - d(2n)/2$ is nonnegative. Then we apply the argument of [12, Lemma 3] with the choice of the set $\mathscr{M}$ of integers having exactly $[2^{4/3}\log_2 X]$ distinct odd prime factors. Thus the omega result (3.8) holds true for $\Re e\,\Phi'(x,\tau)$.

Acknowledgements. The authors wish to thank the referee for careful reading of the manuscript and for suggestion which simplifies the proof of Lemma 1. The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (HKU 7042/04P).

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Received 5 October 2007