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UNIVERSAL BOUNDS FOR EIGENVALUES OF SCHRÖDINGER OPERATOR ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we consider eigenvalues of Schrödinger operator with a weight on compact Riemannian manifolds with boundary (possibly empty) and prove a general inequality for them. By using this inequality, we study eigenvalues of Schrödinger operator with a weight on compact domains in a unit sphere, a complex projective space and a minimal submanifold in a Euclidean space. We also study the same problem on closed minimal submanifolds in a sphere, compact homogeneous space and closed complex hypersurfaces in a complex projective space. We give explict bound for the $(k+1)$-th eigenvalue of the Schrödinger operator on such objects in terms of its first k eigenvalues. Our results generalize many previous estimates on eigenvalues of the Laplacian.

1. Introduction

Let $\overline{M}$ be a compact Riemannian manifold with or without boundary and let Δ be the Laplace operator acting on functions on $\overline{M}$. The study of the spectrum of Δ is an important topic and many works have been done in this area during the past years (see, e.g., [A], [Ch], [SY] and the references therein). When $\overline{M} = \overline{\Omega}$, where Ω is connected bounded domain with smooth boundary in the n-dimensional Euclidean space $\mathbf{R}^n$. The so called *Dirichlet eigenvalue problem* or the *fixed membrane problem* is stated as:

$$\Delta u = -\lambda u \ \text{ in } \ \Omega, \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

Let

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots,$$

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denote the successive eigenvalues of (1.1). Here each eigenvalue is repeated according to its multiplicity. In 1955 and 1956, Payne, Pólya and Weinberger [PPW1], [PPW2] proved that

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for} \quad \Omega \subset \mathbf{R}^2$$

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{\text{disk}}$$

with equality if and only if Ω is a disk. For $n \geq 2$, the analogous statements are

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for} \quad \Omega \subset \mathbf{R}^n,$$

and the *PPW conjecture*

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{n\text{-ball}},$$

with equality if and only if Ω is an n-ball. This important *PPW conjecture* was solved by Ashbaugh and Benguria in their excellent papers [AB1], [AB2], [AB3].

In [PPW2], Payne, Pólya and Weinberger also proved the bound

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, \tag{1.2}$$

for $\Omega \subset \mathbf{R}^2$. This result easily extends to $\Omega \subset \mathbf{R}^n$ as

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, \tag{1.3}$$

Two main advances in extending (1.3) were made by Hile–Protter in [HP] and Yang [Y], respectively. Namely, in 1980, Hile and Protter proved

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for } k = 1, 2, \cdots. \tag{1.4}$$

In 1991, Yang proved the following much stonger inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_{k+1} - \left(1 + \frac{4}{n}\right)\lambda_i\right) \leq 0, \quad \text{for } k = 1, 2, \ldots. \tag{1.5}$$

By elementary calculations, one can show that Yang's inequality (1.5) is sharper than the inequality (1.4) of Hile–Protter and that (1.4) is sharper than the inequality (1.3) of Payne–Pólya–Weinberger (see [A1] and [A2]). In [A2], generalizing Yang's inequality (1.5), Ashbaugh [A2] considered eigenvalues of Schrödinger operators with weight on bounded domains in $\mathbf{R}^n$ and obtained universal bounds for them.

The inequalities on the higher eigenvalues of the Laplacian on a connected bounded domain in $\mathbf{R}^n$ obtained by Payne–Pólya–Weinberger, Hile–Protter, Yang have also been extended to some Riemannian manifolds (cf. [CY1], [CY2], [H1], [HS],

[HM1], [HM2], [HS], [Leu], [Li], [YY]). In [CY1], Cheng and Yang studied eigenvalues of the Laplacian on either a bounded connected domain in an n-dimensional unit sphere $S^n(1)$, or a compact homogeneous Riemannain manifold, or an n-dimensional compact minimal submanifold in a unit sphere and obtained important bounds of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues. Recently, Cheng–Yang [CY2] obtained a general inequality for the eigenvalues of the Laplacian on compact manifolds with boundary (possibly empty) and used it to obtain universal bounds on eigenvalues of the Laplacian on compact domains or closed complex hypersurfaces in a complex projective space. In this paper, we obtain a general inequality for eigenvalues of Schrödinger operator with weight on compact Riemannian manifolds with boundary (possibly empty). By using this inequality, we obtain explict bound for the $(k+1)$-th eigenvalue in terms of its first k eigenvalues of the Schrödinger operator with weight on compact domains in a unit sphere, a complex projective space and a minimal submanifold in a Euclidean space. We also prove similar results for closed minimal submanifolds in a sphere, compact homogeneous space and closed complex hypersurfaces in a complex projective space.

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2. A general inequality for eigenvalues of Schrödinger operator on compact Riemannian manifolds

In this section, we will prove a general result for eigenvalues of Schrödinger operator with weight on compact manifolds. Namely, we have

Theorem 2.1. *Let $(\overline{M}, \langle,\rangle)$ be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty). Let V a nonnegative continuous function on $\overline{M}$, and ρ a weight function which is positive and continuous on $\overline{M}$. Denote by Δ the Laplacian of $\overline{M}$ and consider the eigenvalue problem*

$$-\Delta u + Vu = \lambda\rho u \quad \text{in } M, \quad u|_{\partial M} = 0. \tag{2.1}$$

Let λ_i be the i-th eigenvalue of (2.1) and u_i be the orthonormal eigenfunction corresponding to λ_i, that is, u_i satisfies

$$-\Delta u_i + Vu_i = \lambda_i\rho u_i \quad \text{in } M, \quad u_i|_{\partial M} = 0, \tag{2.2}$$

$$\int_M \rho u_i u_j = \delta_{ij}, \quad \text{for any } i,j = 1,2,\ldots. \tag{2.3}$$

Then for any function $h \in C^3(M) \cap C^2(\partial M)$ and any integer k, we have

$$\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 ||u_i\nabla h||^2 \leq \sum_{i=1}^k (\lambda_{k+1}-\lambda_i) \left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h, \nabla u_i\rangle + u_i\Delta h)\right|\right|^2, \tag{2.4}$$

where

$$||f||^2 = \int_M f^2.$$

Remark 2.2. Theorem 2.1 generalizes the general inequality for eigenvalues of the Laplacian in [CY2]. It is easy to see that when $\partial M \neq \emptyset$, the first eigenvalue of the problem (2.1) is always positive. One can also check that when $\partial M = \emptyset$, the first eigenvalue of the problem (2.1) satisfies $\lambda_1 \geq 0$ with equality holding if and only if $V \equiv 0$. In both cases, we use the same notations $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ to represent the set of the eigenvalues of the problem (2.1).

Proof of Theorem 2.1. Set $S = -\Delta + V$ and consider the inner product given by $\langle\langle f, g\rangle\rangle = \int_M \rho f g$. If a nontrivial function ϕ on M satisfying $\phi|_{\partial M} = 0$ is orthogonal to $u_1, u_2, \ldots, u_k$ with respect to the above inner product, then the Rayleigh–Ritz inequality says that

$$\lambda_{k+1} \leq \frac{\int_M \phi(S\phi)}{\int_M \rho\phi^2}. \tag{2.5}$$

For each $i = 1, \ldots, k$, following Payne, Pólya and Weinberger, we consider the functions $\phi_i \colon M \to \mathbf{R}$, given by

$$\phi_i = hu_i - \sum_{j=1}^k a_{ij} u_j, \tag{2.6}$$

where

$$a_{ij} = \int_\Omega \rho h u_i u_j = a_{ji}. \tag{2.7}$$

Since

$$\phi_i|_{\partial M} = 0 \tag{2.8}$$

and

$$\int_M \rho u_j \phi_i = 0, \quad \forall\ i, j = 1, \ldots, k, \tag{2.9}$$

it follows from the Rayleigh–Ritz inequality that

$$\lambda_{k+1} \leq \frac{\int_M \phi_i(S\phi_i)}{\int_M \rho\phi_i^2}. \tag{2.10}$$

We have

$$\int_M \rho\phi_i^2 = \int_M \rho h u_i \phi_i = \int_M \rho h^2 u_i^2 - \sum_{j=1}^k a_{ij}^2, \tag{2.11}$$

$$
\begin{aligned}
S\phi_i &= (-\Delta + v)\phi_i \\
&= -u_i\Delta h + \lambda_i \rho h u_i - 2\langle \nabla h, \nabla u_i\rangle - \sum_{j=1}^{k} \lambda_j a_{ij}\rho u_j
\end{aligned} \tag{2.12}
$$

Multiplying (2.12) by ϕ_i and integrating over M, we get

$$
\int_M \phi_i S\phi_i = \lambda_i \int_M \rho\phi_i^2 - 2\int_M \phi_i \langle \nabla h, \nabla u_i\rangle - \int_M u_i\phi_i\Delta h \tag{2.13}
$$

Introducing (2.13) into (2.10), one arrives at

$$
(\lambda_{k+1} - \lambda_i)\int_M \rho\phi_i^2 \le -\int_M \phi_i \left(2\langle \nabla h, \nabla u_i\rangle + u_i\Delta h\right) \equiv w_i. \tag{2.14}
$$

Setting

$$
b_{ij} = \int_M u_j \left(\langle \nabla u_i, \nabla h\rangle + \frac{1}{2}u_i\Delta h\right),
$$

one gets from integration by parts that

$$
\begin{aligned}
b_{ij} + b_{ji} &= \int_M \langle \nabla h, u_j\nabla u_i + u_i \nabla u_j\rangle + \int_M u_i u_j \Delta h \\
&= \int_M \langle \nabla h, \nabla(u_i u_j)\rangle + \int_M u_i u_j\Delta h = 0
\end{aligned} \tag{2.15}
$$

and

$$
\begin{aligned}
b_{ij} &= \int_M \langle \nabla u_i, u_j\nabla h\rangle + \frac{1}{2}\int_M u_i u_j \Delta h \\
&= -\int_M u_i \operatorname{div}(u_j\nabla h) + \frac{1}{2}\int_M u_i u_j\Delta h \\
&= -\int_M \langle u_i\nabla u_j, \nabla h\rangle - \frac{1}{2}\int_M u_i u_j \Delta h \\
&= \int_M h(\langle \nabla u_i, \nabla u_j\rangle + u_i\Delta u_j) - \frac{1}{2}\int_M u_i u_j\Delta h \\
&= \int_M h\langle \nabla u_i, \nabla u_j\rangle + \int_M h u_i (V u_j - \lambda_j \rho u_j) - \frac{1}{2}\int_M u_i u_j \Delta h \\
&= -\lambda_j a_{ij} + \int_M h\langle \nabla u_i, \nabla u_j\rangle + \int_M V h u_i u_j - \frac{1}{2}\int_M u_i u_j\Delta h,
\end{aligned} \tag{2.16}
$$

where $\operatorname{div}(Z)$ denotes the divergence of Z. Hence, we have

$$
b_{ij} - b_{ji} = (\lambda_i - \lambda_j)a_{ij} \tag{2.17}
$$

which, combining with (2.15), gives

$$
2b_{ij} = (\lambda_i - \lambda_j)a_{ij}. \tag{2.18}
$$

Since

$$\int_M hu_i(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h) = \frac{1}{2}\int_M\langle\nabla h^2,\nabla u_i^2\rangle + \int_M hu_i^2\Delta h$$
$$= -\frac{1}{2}\int_M u_i^2\Delta h^2 + \int_M hu_i^2\Delta h = -||u_i\nabla h||^2,$$

we have

$$\begin{aligned} w_i &= -\int_M\left(hu_i - \sum_{j=1}^k a_{ij}u_j\right)(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h) \\ &= -\int_M hu_i(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h) + 2\sum_{j=1}^k a_{ij}b_{ij} \qquad (2.19) \\ &= ||u_i\nabla h||^2 + \sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2. \end{aligned}$$

By Schwarz inequality and (2.14), we infer

$$\begin{aligned} &(\lambda_{k+1}-\lambda_i)w_i^2 \\ &= (\lambda_{k+1}-\lambda_i)\left\{\int_M\sqrt{\rho}\phi_i\left(\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h) - 2\sum_{j=1}^k b_{ij}\sqrt{\rho}u_j\right)\right\}^2 \\ &\le (\lambda_{k+1}-\lambda_i)||\sqrt{\rho}\phi_i||^2\cdot\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h) - 2\sum_{j=1}^k b_{ij}\sqrt{\rho}u_j\right|\right|^2 \\ &= (\lambda_{k+1}-\lambda_i)||\sqrt{\rho}\phi_i||^2\left(\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h)\right|\right|^2 - 4\sum_{j=1}^k b_{ij}^2\right) \\ &\le w_i\left(\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h)\right|\right|^2 - \sum_{j=1}^k(\lambda_i-\lambda_j)^2a_{ij}^2\right). \end{aligned}$$

Hence

$$(\lambda_{k+1}-\lambda_i)w_i \le \left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h)\right|\right|^2 - \sum_{j=1}^k(\lambda_i-\lambda_j)^2a_{ij}^2 \qquad (2.20)$$

Multiplying (2.20) by $(\lambda_{k+1}-\lambda_i)$ and taking sum on i from 1 to k, we get

$$\begin{aligned} \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2w_i \le &-\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2 \\ &+\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla h,\nabla u_i\rangle + u_i\Delta h)\right|\right|^2. \end{aligned} \qquad (2.21)$$

On the other hand, since $a_{ij} = a_{ji}$, it follows from (2.19) that

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 w_i &= \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla h\|^2 + \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)a_{ij}^2\\
&= \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla h\|^2 - \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2. \qquad (2.22)\end{aligned}$$

Introducing (2.22) into (2.21), one gets (2.4). This completes the proof of Theorem 2.1. □

3. Eigenvalues of Schrödinger operator on compact domains in $S^n(1)$, $CP^n(4)$ and minimal submanifolds in $\mathbf{R}^m$

In this section, we will prove universal inequalities for eigenvalues of Schrödinger operator on compact connected domains in a unit sphere, a complex projective space and a minimal submanifold of a Euclidean space by using Theorem 2.1.

Theorem 3.1. *Let $(\overline{\Omega}, \langle,\rangle)$ be a compact connected Riemannian manifold with smooth boundary $\partial\Omega$. Let V be a nonnegative continuous function on $\overline{\Omega}$ and ρ a positive continuous function on $\overline{\Omega}$. Set $V_0 = \min_{x\in\overline{\Omega}} V(x)$, $P = \max_{x\in\overline{\Omega}}\rho(x)$ and $Q = \min_{x\in\Omega}\rho(x)$. Denote by Δ the Laplacian of Ω and let λ_i be the i-th eigenvalue of the eigenvalue problem*

$$-\Delta u + Vu = \lambda\rho u \ \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \qquad (3.1)$$

i) *If Ω is a domain in a unit n-sphere, then*

$$\begin{aligned}\lambda_{k+1} \le{}& \frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}\right) + \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i\\
&+\left\{\left(\frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2\right.\\
&\left.-\left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}. \qquad (3.2)\end{aligned}$$

ii) *If Ω is a domain in a complex projetive space $CP^n(4)$ of complex dimension n and of holomorphic sectional curvature 4, then*

$$\begin{aligned}\lambda_{k+1} \le{}& \frac{P}{nQ}\left(\frac{2n(n+1)}{Q}-\frac{V_0}{P}\right) + \left(1+\frac{P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i\\
&+\left\{\left(\frac{P}{nQ}\left(\frac{2n(n+1)}{Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2\right.\\
&\left.-\left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}. \qquad (3.3)\end{aligned}$$

iii) *If* Ω *is a domian in an* n*-dimensional complete minimal submanifold of* $\mathbf{R}^m$, *then*

$$\begin{aligned}\lambda_{k+1} \leq & -\frac{2V_0}{nQ} + \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^k \lambda_i + \left\{\left(\frac{2P}{nQ}\left(-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^k \lambda_i\right)\right)^2\right. \\ & \left. - \left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^k\left(\lambda_j - \frac{1}{k}\sum_{i=1}^k \lambda_i\right)^2\right\}^{1/2}.\end{aligned} \tag{3.4}$$

Remark 3.2 The items i), ii) and iii) in Theorem 3.1 generalize Theorem 1 in [CY1], Theorem 1 in [CY2] and Theorem 4.1 in [A2], respectively.

Proof of Theorem 3.1. Let ∇ be the gradient operator on Ω and let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i, $i = 1, 2, \ldots,$ that is, u_i satisfies

$$-\Delta u_i + V u_i = \lambda_i \rho u_i \quad \text{in } \Omega, \quad u_i|_{\partial\Omega} = 0, \tag{3.5}$$

$$\int_\Omega \rho u_i u_j = \delta_{ij}, \quad \forall\ i, j. \tag{3.6}$$

Multiplying (3.5) by u_i and integrating on Ω, one has

$$\int_\Omega |\nabla u_i|^2 = \lambda_i \int_\Omega \rho u_i^2 - \int_\Omega V u_i^2 \leq \lambda_i - \frac{V_0}{P}. \tag{3.7}$$

We shall use (2.4) to prove the inequalities (3.2), (3.3) and (3.4), respectively.

i) Assume that Ω is a domain in a unit n-sphere $S^n(1)$. Denote by $x_1, x_2, \ldots,$ x_{n+1} the standard coordinate functons of the Euclidean space $\mathbf{R}^{n+1}$; then

$$S^n(1) = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}; \sum_{\alpha=1}^{n+1} x_\alpha^2 = 1\}.$$

It is well known that

$$\Delta x_\alpha = -n x_\alpha, \quad \alpha = 1, \ldots, n+1. \tag{3.8}$$

It follows by taking $h = x_\alpha$ in (2.4) that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 ||u_i \nabla x_\alpha||^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left|\left|\frac{1}{\sqrt{\rho}}(2\langle \nabla x_\alpha, \nabla u_i\rangle + u_i \Delta x_\alpha)\right|\right|^2.$$

Summing over α, we get

$$\begin{aligned}\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{n+1} ||u_i \nabla x_\alpha||^2 &\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \sum_{\alpha=1}^{n+1} \left|\left|\frac{1}{\sqrt{\rho}}(2\langle \nabla x_\alpha, \nabla u_i\rangle + u_i \Delta x_\alpha)\right|\right|^2 \\ &= \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \sum_{\alpha=1}^{n+1} \left|\left|\frac{1}{\sqrt{\rho}}(2\langle \nabla x_\alpha, \nabla u_i\rangle - n u_i x_\alpha)\right|\right|^2.\end{aligned} \tag{3.9}$$

From $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$ and (3.8), we have

$$\sum_{\alpha=1}^{n+1} |\nabla x_\alpha|^2 = n \tag{3.10}$$

and so

$$\sum_{\alpha=1}^{n+1} ||u_i \nabla x_\alpha||^2 = n \int_\Omega u_i^2 \geq n \int_\Omega \frac{\rho u_i^2}{P} = \frac{n}{P}. \tag{3.11}$$

Since

$$\sum_{\alpha=1}^{n+1} \langle \nabla x_\alpha, \nabla u_i \rangle^2 = \sum_{\alpha=1}^{n+1} (\nabla u_i(x_\alpha))^2 = |\nabla u_i|^2, \tag{3.12}$$

one gets from (3.8) that

$$\sum_{\alpha=1}^{n+1} \int_\Omega \langle \nabla x_\alpha, \nabla u_i \rangle^2 \leq \lambda_i - \frac{V_0}{P}. \tag{3.13}$$

Thus

$$\begin{aligned} &\sum_{\alpha=1}^{n+1} \left\| \frac{1}{\sqrt{\rho}} (2\langle \nabla x_\alpha, \nabla u_i \rangle - n u_i x_\alpha) \right\|^2 = \sum_{\alpha=1}^{n+1} \int_\Omega \rho^{-1} \left(2\langle \nabla x_\alpha, \nabla u_i \rangle - n x_\alpha u_i \right)^2 \\ &= \int_\Omega \rho^{-1} \left(4 \sum_{\alpha=1}^{n+1} \langle \nabla x_\alpha, \nabla u_i \rangle^2 - n \left\langle \nabla \left(\sum_{\alpha=1}^{n+1} x_\alpha^2 \right), \ \nabla u_i^2 \right\rangle \right. \\ &\quad \left. + n^2 \left(\sum_{\alpha=1}^{n+1} x_\alpha^2 \right) u_i^2 \right) = 4 \int_\Omega \rho^{-1} |\nabla u_i|^2 + n^2 \int_\Omega \rho^{-1} u_i^2 \\ &\leq \frac{4}{Q} \int_\Omega |\nabla u_i|^2 + \frac{n^2}{Q^2} \int_\Omega \rho u_i^2 \leq \frac{4}{Q} \left(\lambda_i - \frac{V_0}{P} \right) + \frac{n^2}{Q^2}. \end{aligned} \tag{3.14}$$

By introducing (3.11) and (3.14) into (3.9), we infer

$$\frac{n}{P} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\frac{4}{Q} \left(\lambda_i - \frac{V_0}{P} \right) + \frac{n^2}{Q^2} \right), \tag{3.15}$$

that is

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4P}{nQ} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i - \frac{V_0}{P} + \frac{n^2}{4Q} \right). \tag{3.16}$$

Setting

$$\nu_i = \lambda_i - \frac{V_0}{P} + \frac{n^2}{4Q},$$

then (3.16) gives

$$\sum_{i=1}^{k}(\nu_{k+1}-\nu_i)^2 \le \frac{4P}{nQ}\sum_{i=1}^{k}(\nu_{k+1}-\nu_i)\nu_i. \tag{3.17}$$

Solving this quadratic polynomial, we get

$$\begin{aligned}\nu_{k+1} \le{}& \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\nu_i \\ &+\left\{\left(\frac{2P}{nQ}\frac{1}{k}\sum_{i=1}^{k}\nu_i\right)^2-\left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\nu_j-\frac{1}{k}\sum_{i=1}^{k}\nu_i\right)^2\right\}^{1/2},\end{aligned}$$

that is

$$\begin{aligned}\lambda_{k+1} \le{}& \frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}\right)+\left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i+\left\{\left(\frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2\right. \\ &\left.-\left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}.\end{aligned}$$

This proves (3.2).

ii) Assume that Ω is a domain in $CP^n(4)$, the complex projective space of complex dimension n and of holomorphic sectional curvature 4. Let $z=(z_0,z_1,\cdots,z_n)$ be a homogeneous coordinate system of $CP^n(4)$, $(z_i \in \mathbf{C})$ and consider the functions

$$h_{p\bar{q}}=\frac{z_p\overline{z_q}}{\sum_{r=0}^{n} z_r\overline{z_r}}, \quad p,q=0,1,\ldots,n. \tag{3.18}$$

Setting $g_{p\bar{q}}=\mathrm{Re}(h_{p\bar{q}})$ and $f_{p\bar{q}}=\mathrm{Im}(h_{p\bar{q}})$, $p,q=0,1,\ldots,n$, we have (cf. [CY2])

$$\sum_{p,q=0}^{n}(g_{p\bar{q}}^2+f_{p\bar{q}}^2)=1, \tag{3.19}$$

$$\sum_{p,q=0}^{n}(g_{p\bar{q}}\nabla g_{p\bar{q}}+f_{p\bar{q}}\nabla f_{p\bar{q}})=0, \tag{3.20}$$

$$\sum_{p,q=0}^{n}(\langle\nabla g_{p\bar{q}},\nabla g_{p\bar{q}}\rangle+\langle\nabla f_{p\bar{q}},\nabla f_{p\bar{q}}\rangle)=-\sum_{p,q=0}^{n}(g_{p\bar{q}}\Delta g_{p\bar{q}}+f_{p\bar{q}}\Delta f_{p\bar{q}})=4n, \tag{3.21}$$

$$\sum_{p,q=0}^{n}(\Delta g_{p\bar{q}}\nabla g_{p\bar{q}}+\Delta f_{p\bar{q}}\nabla f_{p\bar{q}})=0, \tag{3.22}$$

$$\sum_{p,q=0}^{n}(\Delta g_{p\bar{q}}\Delta g_{p\bar{q}}+\Delta f_{p\bar{q}}\Delta f_{p\bar{q}})=16n(n+1), \tag{3.23}$$

$$\sum_{p,q=0}^{n}(\langle\nabla g_{p\bar{q}},\nabla u_i\rangle^2+\langle\nabla f_{p\bar{q}},\nabla u_i\rangle^2)=2|\nabla u_i|^2. \tag{3.24}$$

Applying Theorem 2.1 to the functions $g_{p\bar{q}}$ and $f_{p\bar{q}}$ and summing over p and q, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega\sum_{p,q=0}^{n}(||u_i\nabla g_{p\bar{q}}||^2+||u_i\nabla f_{p\bar{q}}||^2)\\&\le\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{p,q=0}^{n}\left(\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla g_{p\bar{q}},\nabla u_i\rangle+u_i\Delta g_{p\bar{q}})\right|\right|^2\right.\\&\quad\left.+\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla f_{p\bar{q}},\nabla u_i\rangle+u_i\Delta f_{p\bar{q}})\right|\right|^2\right).\end{aligned} \tag{3.25}$$

From (3.21), we have

$$\sum_{p,q=0}^{n}(||u_i\nabla g_{p\bar{q}}||^2+||u_i\nabla f_{p\bar{q}}||^2)=4n\int_\Omega u_i^2\ge\frac{4n}{P}. \tag{3.26}$$

It follows from (3.22)–(3.24) that

$$\begin{aligned}&\sum_{p,q=0}^{n}\left(\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla g_{p\bar{q}},\nabla u_i\rangle+u_i\Delta g_{p\bar{q}})\right|\right|^2+\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla f_{p\bar{q}},\nabla u_i\rangle+u_i\Delta f_{p\bar{q}})\right|\right|^2\right)\\&=\sum_{p,q=0}^{n}\int_\Omega\frac{4}{\rho}\left(\langle\nabla g_{p\bar{q}},\nabla u_i\rangle^2+\langle\nabla f_{p\bar{q}},\nabla u_i\rangle^2\right)\\&\quad+\sum_{p,q=0}^{n}\int_\Omega\frac{4}{\rho}\langle\Delta g_{p\bar{q}}\nabla g_{p\bar{q}}+\Delta f_{p\bar{q}}\nabla f_{p\bar{q}},u_i\nabla u_i\rangle\\&\quad+\sum_{p,q=0}^{n}\int_\Omega\frac{1}{\rho}\left((\Delta g_{p\bar{q}})^2+(\Delta f_{p\bar{q}})^2\right)u_i^2\\&=\int_\Omega\frac{8}{\rho}|\nabla u_i|^2+16n(n+1)\int_\Omega\frac{u_i^2}{\rho}\le\frac{8}{Q}\left(\lambda_i-\frac{V_0}{P}\right)+\frac{16n(n+1)}{Q^2}.\end{aligned} \tag{3.27}$$

Substituting (3.26)–(3.27) into (3.25), we infer

$$\frac{4n}{P}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\le\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{8}{Q}\left(\lambda_i-\frac{V_0}{P}\right)+\frac{16n(n+1)}{Q^2}\right).$$

Therefore,

$$\begin{aligned}\lambda_{k+1} \leq \frac{P}{nQ}\left(\frac{2n(n+1)}{Q} - \frac{V_0}{P}\right) + \left(1 + \frac{P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \\ + \left\{ \left(\frac{P}{nQ}\left(\frac{2n(n+1)}{Q} - \frac{V_0}{P} + \frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2 \right. \\ \left. - \left(1 + \frac{2P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j - \frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2 \right\}^{1/2}.\end{aligned}$$

Thus (2.4) holds.

iii) Finally, assume that Ω is a domain in an n-dimensional minimal submainfold M of $\mathbf{R}^m$. Let $x_1, x_2, \ldots, x_m$ be the standard coordinate functons of $\mathbf{R}^m$. Since M is a minimal submanifold in $\mathbf{R}^m$, we have

$$\Delta x_\alpha = 0, \quad \alpha = 1, \cdots, m. \tag{3.28}$$

Taking $h = x_\alpha$ in (2.4) and summing over α, we get

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{m} ||u_i \nabla x_\alpha||^2 \leq 4\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\sum_{\alpha=1}^{m}\left|\left|\frac{1}{\sqrt{\rho}}\langle \nabla x_\alpha, \nabla u_i\rangle\right|\right|^2. \tag{3.29}$$

Since

$$\sum_{\alpha=1}^{n}\langle \nabla x_\alpha, \nabla u_i\rangle^2 = \sum_{\alpha=1}^{n}(\nabla u_i(x_\alpha)^2 = |\nabla u_i|^2 \tag{3.30}$$

and (cf. [D])

$$\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2 = n, \tag{3.31}$$

we have

$$\sum_{\alpha=1}^{m}||u_i \nabla x_\alpha|| = n\int_\Omega u_i^2 \geq \frac{n}{P} \tag{3.32}$$

and

$$\sum_{\alpha=1}^{m}\left|\left|\frac{1}{\sqrt{\rho}}\langle \nabla x_\alpha, \nabla u_i\rangle\right|\right|^2 \leq \frac{1}{Q}\left(\lambda_i - \frac{V_0}{P}\right). \tag{3.33}$$

Substituting (3.32)–(3.33) into (3.29), one gets

$$\frac{n}{P}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{Q}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left(\lambda_i - \frac{V_0}{P}\right). \tag{3.34}$$

Consequently, we have

$$\lambda_{k+1} \leq -\frac{2V_0}{nQ} + \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \left\{\left(\frac{2P}{nQ}\left(-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2 - \left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j - \frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2},$$

which proves (2.5). This completes the proof of Theorem 3.1. □

4. Eigenvalues of Schrödinger operator on closed minimal submanifolds in $S^m(1)$, complex hypersurfaces in $CP^{n+1}(4)$ and homogeneous spaces

In this section, we shall use the similar methods as in the last section to prove the following result:

Theorem 4.1. *Let $(M, \langle , \rangle)$ be a closed connected Riemannian manifold. Let V be a nonnegative continuous function and ρ a positive continuous function on M. Set $V_0 = \min_{x\in M} V(x)$, $P = \max_{x\in M}\rho(x)$ and $Q = \min_{x\in M}\rho(x)$. Denote by Δ the Laplacian of M and let λ_i be the i-th eigenvalue of the eigenvalue problem*

$$-\Delta u + Vu = \lambda\rho u \quad \text{on } M. \tag{4.1}$$

i) *If M is an n-dimensional minimal submanifold in $S^m(1)$, then*

$$\begin{aligned}\lambda_{k+1} \leq{}& \frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}\right) + \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \\ &+ \left\{\left(\frac{2P}{nQ}\left(\frac{n^2}{4Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2 \right. \\ &\left. - \left(1+\frac{4P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}.\end{aligned} \tag{4.2}$$

ii) *If M is a complex hypersurface in $CP^{n+1}(4)$, then*

$$\begin{aligned}\lambda_{k+1} \leq{}& \frac{P}{nQ}\left(\frac{2n(n+1)}{Q}-\frac{V_0}{P}\right) + \left(1+\frac{P}{nQ}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \\ &+ \left\{\left(\frac{P}{nQ}\left(\frac{2n(n+1)}{Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2 \right. \\ &\left. - \left(1+\frac{2P}{nQ}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}.\end{aligned} \tag{4.3}$$

iii) *If M is an n-dimensional homogeneous space, then*

$$
\begin{aligned}
\lambda_{k+1} \le\ & \frac{2P}{Q}\left(\frac{\nu_1}{4Q}-\frac{V_0}{P}\right)+\left(1+\frac{2P}{Q}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \\
& +\left\{\left(\frac{2P}{Q}\left(\frac{\nu_1}{4Q}-\frac{V_0}{P}+\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)\right)^2\right. \\
& \left.-\left(1+\frac{4P}{Q}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2},
\end{aligned}
$$

where ν_1 is the first nonzero eigenvalue of the Laplacian of M.

Remark 4.2. The items i), ii) and iii) in Theorem 4.1 generalize Theorem 3 in [CY1], Theorem 2 in [CY2] and Theorem 2 in [CY1], respectively.

Proof of Theorem 4.1. Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i, $i = 1, 2, \ldots$, that is, we have

$$
-\Delta u_i + V u_i = \lambda_i \rho u_i, \quad \int_\Omega \rho u_i u_j = \delta_{ij}, \quad \forall\ i, j. \tag{4.4}
$$

Multiplying (4.5) by u_i and integrating on Ω, one has

$$
\int_\Omega |\nabla u_i|^2 = \lambda_i \int_\Omega \rho u_i^2 - \int_\Omega V u_i^2 \le \lambda_i - \frac{V_0}{P}. \tag{4.5}
$$

The proof of (4.2) and (4.3) is similar to that of (3.2) and (3.3), respectively and we will only give the outlines.

i) Assume that M is an n-dimensional minimal submanifold in $S^m(1)$. Denote by $x_1, x_2, \ldots, x_{m+1}$ the standard coordinate functons of the Euclidean space $\mathbf{R}^{m+1}$; then

$$
S^m(1) = \{(x_1, \ldots, x_{m+1}) \in \mathbf{R}^{m+1}; \sum_{\alpha=1}^{m+1} x_\alpha^2 = 1\}.
$$

Since M is a minimal submanifold in $S^m(1)$, we have

$$
\Delta x_\alpha = -n x_\alpha, \quad \alpha = 1, \ldots, m+1. \tag{4.6}
$$

It follows by taking $h = x_\alpha$ in (2.4) and summing over α that

$$
\begin{aligned}
& \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \sum_{\alpha=1}^{m+1} ||u_i \nabla x_\alpha||^2 \\
& \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) \sum_{\alpha=1}^{m+1} \left|\left|\frac{1}{\sqrt{\rho}}(2\langle \nabla x_\alpha, \nabla u_i\rangle - n u_i x_\alpha)\right|\right|^2.
\end{aligned} \tag{4.7}
$$

Since $\sum_{\alpha=1}^{m+1} x_\alpha^2 = 1$ and (4.7) hold, we have

$$\sum_{\alpha=1}^{m+1} |\nabla x_\alpha|^2 = n \tag{4.8}$$

which gives

$$\sum_{\alpha=1}^{m+1} ||u_i \nabla x_\alpha||^2 = n \int_M u_i^2 \geq n \int_\Omega \frac{\rho u_i^2}{P} = \frac{n}{P}. \tag{4.9}$$

It follows from (4.4) and

$$\sum_{\alpha=1}^{m+1} \langle \nabla x_\alpha, \nabla u_i \rangle^2 = \sum_{\alpha=1}^{m+1} (\nabla u_i(x_\alpha))^2 = |\nabla u_i|^2, \tag{4.10}$$

that

$$\sum_{\alpha=1}^{m+1} \int_M \langle \nabla x_\alpha, \nabla u_i \rangle^2 \leq \lambda_i - \frac{V_0}{P}. \tag{4.11}$$

Using the same arguments as in the proof of i) in Theorem 3.1, one knows that (4.2) is true.

ii) Now consider the case that M is a complex hypersurface in $CP^{n+1}(4)$. Let $z = (z_0, z_1, \ldots, z_{n+1})$ be a homogeneous coordinate system of $CP^{n+1}(4)$, $(z_i \in \mathbf{C})$ and consider the functions

$$h_{p\overline{q}} = \frac{z_p \overline{z_q}}{\sum_{r=0}^{n+1} z_r \overline{z_r}}, \quad p, q = 0, 1, \cdots, n+1. \tag{4.12}$$

Setting $g_{p\overline{q}} = \mathrm{Re}(h_{p\overline{q}})$ and $f_{p\overline{q}} = \mathrm{Im}(h_{p\overline{q}})$, $\quad p, q = 0, 1, \ldots, n$, we have (cf. [CY2])

$$\sum_{p,q=0}^{n+1} (g_{p\overline{q}}^2 + f_{p\overline{q}}^2) = 1, \tag{4.13}$$

$$\sum_{p,q=0}^{n+1} (g_{p\overline{q}} \nabla g_{p\overline{q}} + f_{p\overline{q}} \nabla f_{p\overline{q}}) = 0, \tag{4.14}$$

$$\sum_{p,q=0}^{n+1} (\langle \nabla g_{p\overline{q}}, \nabla g_{p\overline{q}} \rangle + \langle \nabla f_{p\overline{q}}, \nabla f_{p\overline{q}} \rangle) = -\sum_{p,q=0}^{n+1} (g_{p\overline{q}} \Delta g_{p\overline{q}} + f_{p\overline{q}} \Delta f_{p\overline{q}}) = 4n, \tag{4.15}$$

$$\sum_{p,q=0}^{n+1} (\Delta g_{p\overline{q}} \nabla g_{p\overline{q}} + \Delta f_{p\overline{q}} \nabla f_{p\overline{q}}) = 0, \tag{4.16}$$

$$\sum_{p,q=0}^{n+1} (\Delta g_{p\overline{q}} \Delta g_{p\overline{q}} + \Delta f_{p\overline{q}} \Delta f_{p\overline{q}}) = 16n(n+1), \tag{4.17}$$

$$\sum_{p,q=0}^{n+1}(\langle\nabla g_{p\bar{q}},\nabla u_i\rangle^2+\langle\nabla f_{p\bar{q}},\nabla u_i\rangle^2)=2|\nabla u_i|^2. \tag{4.18}$$

Applying Theorem 2.1 to the functions $g_{p\bar{q}}$ and $f_{p\bar{q}}$ and summing over p and q, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega\sum_{p,q=0}^{n+1}(||u_i\nabla g_{p\bar{q}}||^2+||u_i\nabla f_{p\bar{q}}||^2)\\ &\leq\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{p,q=0}^{n+1}\left(\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla g_{p\bar{q}},\nabla u_i\rangle+u_i\Delta g_{p\bar{q}})\right|\right|^2\right.\\ &\quad\left.+\left|\left|\frac{1}{\sqrt{\rho}}(2\langle\nabla f_{p\bar{q}},\nabla u_i\rangle+u_i\Delta f_{p\bar{q}})\right|\right|^2\right).\end{aligned} \tag{4.19}$$

Since (4.14)–(4.20) hold, one can use the same discussions as in the proof of ii) in Theorem 3.1 to conclude that (4.3) is true.

iii) Let $\{f_\alpha\}_{\alpha=1}^{l}$ be an orthonormal basis of the first eigenspace E_{ν_1} corresponding to the first nonzero eigenvalue ν_1 of the eigenvalue problema

$$\Delta f=-\nu f\quad\text{on } M,$$

that is, we have

$$\Delta f_\alpha=-\nu_1 f_\alpha,\quad \alpha=1,\ldots,l. \tag{4.20}$$

It is known that (cf. [Li])

$$\sum_{\alpha=1}^{l}f_\alpha^2=C^2=\text{Const}. \tag{4.21}$$

Applying Theorem 2.1 to the functions f_α and summing over α, we get

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M\left(\sum_{\alpha=1}^{l}|u_i\nabla f_\alpha|^2\right)\\ &\leq\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_M\left\{\frac{1}{\rho}\sum_{\alpha=1}^{l}(2\langle\nabla f_\alpha,\nabla u_i\rangle-\nu_1u_if_\alpha)^2\right\}\end{aligned} \tag{4.22}$$

From (4.20)–(4.21), we have

$$\sum_{\alpha=1}^{l}f_\alpha\nabla f_\alpha=0,\quad \sum_{\alpha=1}^{l}|\nabla f_\alpha|^2=\nu_1C^2. \tag{4.23}$$

Also, we have from Schwarz inequality that

$$\langle\nabla f_\alpha,\nabla u_i\rangle^2\leq\sum_{\alpha=1}^{l}|\nabla f_\alpha|^2|\nabla u_i|^2=\nu_1C^2|\nabla u_i|^2 \tag{4.24}$$

Substituting (4.23)–(4.24) into (4.22), we infer

$$\nu_1 C^2 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_M u_i^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_M \frac{1}{\rho} \left(4\nu_1 C^2 \left| \nabla u_i \right|^2 + \nu_1^2 u_i^2 C^2 \right)$$

That is

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_M u_i^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_M \frac{1}{\rho} \left(4 \left| \nabla u_i \right|^2 + \nu_1 u_i^2 \right) \tag{4.25}$$

Since

$$\int_M u_i^2 \geq \frac{1}{P}, \quad \int_M \frac{u_i^2}{\rho} \leq \frac{1}{Q^2}, \quad \int_M \frac{\left| \nabla u_i \right|^2}{\rho} \leq \frac{1}{Q} \left(\lambda_i - \frac{V_0}{P} \right), \tag{4.26}$$

we infer from (4.25) that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4P}{Q} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i - \frac{V_0}{P} + \frac{\nu_1}{4Q} \right). \tag{4.27}$$

Consequently, we have

$$\begin{aligned} \lambda_{k+1} \leq{} & \frac{2P}{Q} \left(\frac{\nu_1}{4Q} - \frac{V_0}{P} \right) + \left(1 + \frac{2P}{Q} \right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i \\ & + \left\{ \left(\frac{2P}{Q} \left(\frac{\nu_1}{4Q} - \frac{V_0}{P} + \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right) \right)^2 \right. \\ & \left. - \left(1 + \frac{4P}{Q} \right) \frac{1}{k} \sum_{j=1}^{k} \left(\lambda_j - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 \right\}^{1/2}. \end{aligned}$$

This shows that (4.4) holds and completes the proof of Theorem 4.1. □

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