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A FLOWER STRUCTURE OF BACKWARD FLOW INVARIANT DOMAINS FOR SEMIGROUPS

Mark Elin, David Shoikhet and Lawrence Zalcman†

ORT Braude College, Department of Mathematics
P.O. Box 78, Karmiel 21982, Israel; mark.elin@gmail.com

ORT Braude College, Department of Mathematics
P.O. Box 78, Karmiel 21982, Israel; davs27@netvision.net.il

Bar-Ilan University, Department of Mathematics
52900 Ramat-Gan, Israel; zalcman@macs.biu.ac.il

Abstract. In this paper, we study conditions which ensure the existence of backward flow invariant domains for semigroups of holomorphic self-mappings of a simply connected domain D. More precisely, the problem is the following. Given a one-parameter semigroup $\mathscr{S}$ on D, find a simply connected subset $\Omega \subset D$ such that each element of $\mathscr{S}$ is an automorphism of Ω, in other words, such that $\mathscr{S}$ forms a one-parameter group on Ω.

On the way to solving this problem, we prove an angle distortion theorem for starlike and spirallike functions with respect to interior and boundary points.

Let D be a simply connected domain in the complex plane $\mathbf{C}$. By $\mathrm{Hol}(D,\Omega)$ we denote the set of all holomorphic functions on D with values in a domain Ω in $\mathbf{C}$. We write $\mathrm{Hol}(D)$ for $\mathrm{Hol}(D,D)$, the set of holomorphic self-mappings of D. This set is a topological semigroup with respect to composition. We denote by $\mathrm{Aut}(D)$ the group of all automorphisms of D; thus $F \in \mathrm{Aut}(D)$ if and only if F is univalent on D and $F(D) = D$.

Definition 1. A family $\mathscr{S} = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(D)$ is said to be a one-parameter continuous semigroup (semiflow) on D if

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$,

(ii) $\lim\limits_{t\to 0^+} F_t(z) = z$ for all $z \in D$.

If, in addition, condition (i) holds for all $t, s \in \mathbf{R}$, then $(F_t)^{-1} = F_{-t}$ for each $t \in \mathbf{R}$; and $\mathscr{S}$ is called a *one-parameter continuous group* (*flow*) on D. In this case, $\mathscr{S} \subset \mathrm{Aut}(D)$.

In this paper, we study the following problem. *Given a one-parameter semigroup* $\mathscr{S} \subset \mathrm{Hol}(D)$, *find a simply connected domain* $\Omega \subset D$ (*if it exists*) *such that* $\mathscr{S} \subset \mathrm{Aut}(\Omega)$.

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It is well-known that condition (ii) and holomorphy, in fact, imply that

$$\lim_{t\to s} F_t(z) = F_s(z)$$

for each $z \in D$ and $s > 0$ ($s \in \mathbf{R}$ in the case when $\mathscr{S} \subset \mathrm{Aut}(D)$); see, for example, [8], [2], [28] and [29]. This explains the name "continuous semigroup" in our terminology.

Furthermore, it follows by a result of Berkson and Porta [8] that each continuous semigroup is differentiable in $t \in \mathbf{R}^+ = [0, \infty)$, (see also [1] and [30]). So, for each continuous semigroup (semiflow) $\mathscr{S} = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(D)$, the limit

$$\lim_{t\to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in D, \tag{1}$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(D, \mathbf{C})$. This mapping f is called the (*infinitesimal*) *generator of* $\mathscr{S} = \{F_t\}_{t\geq 0}$. Moreover, the function $u(= u(t, z))$, $(t, z) \in \mathbf{R}^+ \times D$, defined by $u(t, z) = F_t(z)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t, z)}{\partial t} + f(u(t, z)) = 0, \\ u(0, z) = z, \quad z \in D. \end{cases} \tag{2}$$

Conversely, a mapping $f \in \mathrm{Hol}(D, \mathbf{C})$ is said to be a *semi-complete* (respectively, *complete*) *vector field* on D if the Cauchy problem (2) has a solution $u(= u(t, z)) \in D$ for all $z \in D$ and $t \in \mathbf{R}^+$ (respectively, $t \in \mathbf{R}$). Thus $f \in \mathrm{Hol}(D, \mathbf{C})$ is a semi-complete vector field if and only if it is the generator of a one-parameter continuous semigroup $\mathscr{S}$ (semiflow) on D. It is complete if and only if $\mathscr{S} \subset \mathrm{Aut}(D)$. The set of semi-complete vector fields on D is denoted by $\mathscr{G}(D)$. The set of complete vector fields on D is usually denoted by $\mathrm{aut}(D)$ (see, for example, [23], [35], [32]).

Thus, in these terms, our problem can be rephrased as follows. *Given* $f \in \mathscr{G}(D)$, *find a domain* Ω (*if it exists*) *such that* $f \in \mathrm{aut}(\Omega)$.

Let now $D = \Delta$ be the open unit disk in $\mathbf{C}$. In this case, $\mathscr{G}(\Delta)$ is a real cone in $\mathrm{Hol}(\Delta, \mathbf{C})$, while $\mathrm{aut}(\Delta) \subset \mathscr{G}(\Delta)$ is a real Banach space (see, for example, [30]). Moreover, by the Berkson–Porta representation formula, a function f belongs to $\mathscr{G}(\Delta)$ if and only if there is a point $\tau \in \overline{\Delta}$ and a function $p \in \mathrm{Hol}(\Delta, \mathbf{C})$ with positive real part ($\mathrm{Re}\, p(z) \geq 0$ everywhere) such that

$$f(z) = (z - \tau)(1 - z\overline{\tau})p(z). \tag{3}$$

This representation is unique and is equivalent to

$$f(z) = a - \bar{a}z^2 + zq(z), \quad a \in \mathbf{C},\ \mathrm{Re}\, q(z) \geq 0$$

(see [3]). Moreover, $f \in \mathrm{Hol}(\Delta, \mathbf{C})$ is complete if and only if it admits the representation

$$f(z) = a - \bar{a}z^2 + ibz \tag{4}$$

for some $a \in \mathbf{C}$ and $b \in \mathbf{R}$ (see, [7], [5], [35]).

Note also that if a semigroup $\mathscr{S} = \{F_t\}_{t\ge0}$ generated by $f \in \mathscr{G}(\Delta)$ does not contain an elliptic automorphism of Δ, then the point $\tau \in \overline{\Delta}$ in representation (3) is the unique attractive point for the semigroup $\mathscr{S}$, i.e.,

$$\lim_{t\to\infty} F_t(z) = \tau \tag{5}$$

for all $z \in \Delta$. This point is usually referred as the Denjoy–Wolff point of $\mathscr{S}$. In addition,

- if $\tau \in \Delta$, then $\tau = F_t(\tau)$ is a unique fixed point of $\mathscr{S}$ in Δ;
- if $\tau \in \partial\Delta$, then

$$\tau = \lim_{r\to1^-} F_t(r\tau)$$

is a common boundary fixed point of $\mathscr{S}$ in $\overline{\Delta}$, and no element F_t $(t > 0)$ has an interior fixed point in Δ.

Also, we observe that for $\tau \in \Delta$, formula (3) implies the condition

$$\operatorname{Re} f'(\tau) \ge 0. \tag{6}$$

Comparing this with (3) and (4), we see that $\mathscr{S}$ consists of elliptic automorphisms if and only if

$$\operatorname{Re} f'(\tau) = 0. \tag{7}$$

Consequently, condition (5) is equivalent to

$$\operatorname{Re} f'(\tau) > 0. \tag{8}$$

If τ in (3) belongs to $\partial\Delta$, then if follows by the Riesz–Herglotz representation of the function p in (3) that the angular limits

$$f(\tau) := \angle\lim_{z\to\tau} f(z) = 0 \quad \text{and} \quad f'(\tau) := \angle\lim_{z\to\tau} f'(z) = \beta \tag{9}$$

exist and that β is a nonnegative real number (see also [16]). Moreover, if for some point $\zeta \in \partial\Delta$ there are limits

$$\angle\lim_{z\to\zeta} f(z) = 0$$

and

$$\angle\lim_{z\to\zeta} f'(z) = \gamma$$

with $\gamma \ge 0$, then $\gamma = \beta$ and $\zeta = \tau$ (see [16] and [33]).

In the case where $\beta > 0$, the semigroup $\mathscr{S} = \{F_t\}_{t\ge0}$ consists of mappings $F_t \in \operatorname{Hol}(\Delta)$ of hyperbolic type,

$$\angle\lim_{z\to\tau} \frac{\partial F_t(z)}{\partial z} = e^{-t\beta} < 1;$$

otherwise $(\beta = 0)$, it consists of mappings of parabolic type,

$$\angle\lim_{z\to\tau} \frac{\partial F_t(z)}{\partial z} = 1 \quad \text{for all } t \ge 0.$$

For $\tau \in \overline{\Delta}$, we use the notation $\mathscr{G}^+[\tau]$ for a subcone of $\mathscr{G}(\Delta)$ of functions f defined by (3) for which

$$\operatorname{Re} f'(\tau) > 0. \tag{10}$$

We solve the problem mentioned above for the class $\mathscr{G}^+[\tau]$ of generators.

Definition 2. Let $\mathscr{S} = \{F_t\}_{t\geq 0}$ be a semiflow on Δ. A domain $\Omega \subset \Delta$ is called a (backward) *flow invariant domain* (shortly, FID) for $\mathscr{S}$ if $\mathscr{S} \subset \operatorname{Aut}(\Omega)$.

We need the following notation. We write $f \in \mathscr{G}^+[\tau, \eta]$, where $\tau \in \overline{\Delta}$, $\eta \in \partial\Delta$, $\eta \neq \tau$, if $f \in \mathscr{G}^+[\tau]$, $f(\eta) = \angle\lim\limits_{z\to\eta} f(z) = 0$ and $\gamma = \angle\lim\limits_{z\to\eta} f'(z)$ exists finitely. In fact, in this case γ must be a real negative number (see Lemma 6 below).

Theorem 1. *Let $\mathscr{S} = \{F_t\}_{t\geq 0}$ be a semiflow on Δ generated by $f \in \mathscr{G}^+[\tau]$, for some $\tau \in \overline{\Delta}$ with $f(\tau) = 0$ and $f'(\tau) = \beta$, $\operatorname{Re}\beta > 0$. The following assertions are equivalent.*

(i) *$f \in \mathscr{G}^+[\tau, \eta]$ for some $\eta \in \partial\Delta$.*
(ii) *There is a nonempty (backward) flow invariant domain $\Omega \subset \Delta$, so $\mathscr{S} \subset \operatorname{Aut}(\Omega)$.*
(iii) *For some $\alpha > 0$, the differential equation*

$$\alpha\varphi'(z)(z^2 - 1) = 2f(\varphi(z)) \tag{11}$$

has a locally univalent solution φ with $|\varphi(z)| < 1$ when $z \in \Delta$. Moreover, in this case φ is univalent and is a Riemann mapping of Δ onto a flow invariant domain Ω.

This theorem can be completed by the following result.

Theorem 2. *Let $\mathscr{S} = \{F_t\}_{t\geq 0}$ be a semiflow on Δ generated by $f \in \mathscr{G}^+[\tau]$, for some $\tau \in \overline{\Delta}$ with $f(\tau) = 0$ and $f'(\tau) = \beta$, $\operatorname{Re}\beta > 0$. The following assertions hold.*

(a) *If $f \in \mathscr{G}^+[\tau, \eta]$ for some $\eta \in \partial\Delta$ with $\gamma = \angle\lim\limits_{z\to\eta} f'(z)$, then for each $\alpha \geq -\gamma$, equation (11) has a univalent solution φ such that $\varphi(1) = \tau$, $\varphi(-1) = \eta$ and $\Omega = \varphi(\Delta)$ is a (backward) flow invariant domain for $\mathscr{S}$. In addition, $\tau = \lim\limits_{t\to\infty} F_t(z) \in \partial\Omega$, $z \in \Omega$, and $\lim\limits_{t\to-\infty} F_t(z) = \eta \in \partial\Delta \cap \partial\Omega$ for each $z \in \Omega$.*
(b) *If $\Omega \subset \Delta$ is a nonempty (backward) flow invariant domain, then it is a Jordan domain such that $\tau \in \partial\Omega$, and there is a point $\eta \in \partial\Omega \cap \partial\Delta$ such that $\lim\limits_{t\to-\infty} F_t(z) = \eta$ whenever $z \in \Omega$, $\angle\lim\limits_{z\to\eta} f(z) = 0$ and $\angle\lim\limits_{z\to\eta} f'(z) =: \gamma$ exists with $\gamma < 0$. In addition, there is a conformal mapping φ of Δ onto Ω which satisfies equation (11) with some $\alpha \geq -\gamma$.*
(c) *Conversely, if for some $\alpha > 0$, the differential equation (11) has a locally univalent solution $\varphi \in \operatorname{Hol}(\Delta)$, then it is, in fact, a conformal mapping of Δ onto the FID $\Omega = \varphi(\Delta)$ such that $\varphi(1) = \tau \in \partial\Omega$ and $\varphi(-1) = \eta$ for some $\eta \in \partial\Delta \cap \partial\Omega$.*

In addition, $f(\eta) = 0$ and $f'(\eta) = \gamma$ with $0 > \gamma \geq -\alpha$.

Definition 3. A (backward) flow invariant domain (FID) $\Omega \subset \Delta$ for $\mathscr{S}$ is said to be maximal if there is no $\Omega_1 \supset \Omega,\ \Omega_1 \neq \Omega$, such that $\mathscr{S} \subset \mathrm{Aut}(\Omega_1)$.

Theorem 3. *Let $f \in \mathscr{G}^+[\tau,\eta]$ for some $\tau \in \overline{\Delta},\ \eta \in \partial\Delta$ with $\gamma = f'(\eta)\Big(< 0\Big)$, and let φ be a (univalent) solution of (11) with some $\alpha \geq -\gamma$ normalized by $\varphi(1) = \tau$ and $\varphi(-1) = \eta$. The following assertions are equivalent:*

(i) $\Omega = \varphi(\Delta)$ *is a maximal FID;*
(ii) $\alpha = -\gamma$*;*
(iii) φ *is isogonal at the boundary point $z = -1$ (see Remark 3 below).*

Remark 1. In general, a maximal FID for $\mathscr{S}$ need not be unique. Theorem 1 states that if $\mathscr{S} = \{F_t\}_{t\geq 0}$ is generated by $f \in \mathscr{G}^+[\tau]$, then its *FID is not empty if and only if there is a point $\eta \in \partial\Delta$, such that $f(\eta) = \angle\lim\limits_{z\to\eta} f(z) = 0$ and $f'(\eta) = \angle\lim\limits_{z\to\eta} f'(z)$ exists finitely with $f'(\eta) < 0$.* This point η is a repelling fixed point for $\mathscr{S} = \{F_t\}_{t\geq 0}$ as $t \to \infty$, namely, $F_t(\eta) = \eta$ and $\left.\frac{\partial F_t(z)}{\partial z}\right|_{z=\eta} = e^{-tf'(\eta)} > 1$ (see [16]). Moreover, there is *a one-to-one correspondence between maximal flow invariant domains for $\mathscr{S}$ and such repelling fixed points.*

Theorem 4. *Let $f \in \mathscr{G}^+[\tau,\eta_k]$ for some sequence $\{\eta_k\} \in \partial\Delta$, i.e., $f(\eta_k) = 0$ and $\gamma_k = f'(\eta_k) > -\infty$.*

The following assertions hold.

(i) *There is $\delta > 0$ such that $\gamma_k < -\delta < 0$ for all $k = 1, 2, \ldots$.*
(ii) *For each $a < -\delta < 0$ there is at most a finite number of the points η_k such that $a \leq \gamma_k < -\delta$.*
Consequently equation (11) *has a* (*univalent*) *solution $\varphi \in \mathrm{Hol}(\Delta)$ for each $\alpha \geq -\max\{\gamma_k\} > -\delta$.*
(iii) *If φ_k is a solution of* (11) *normalized by $\varphi_k(1) = \tau,\ \varphi_k(-1) = \eta_k$ with $\alpha = \gamma_k$ and $\Omega_k = \varphi_k(\Delta)$ (i.e., Ω_k are maximal), then for each pair Ω_{k_1} and Ω_{k_2} such that $\eta_{k_1} \neq \eta_{k_2}$ either $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = \{\tau\}$ or $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = l$, where l is a continuous curve joining τ with a point on $\partial\Delta$.*

We illustrate the content of our theorems in the following examples.

Example 1. Consider a generator $f \in \mathscr{G}^+[0]$ defined by

$$f(z) = z(1 - z^n), \quad n \in \mathbf{N}.$$

Solving the Cauchy problem (2), we find

$$F_t(z) = \frac{ze^{-t}}{\sqrt[n]{1 - z^n + z^n e^{-nt}}}\,.$$

In this case, f has n additional null points $\eta_k = e^{\frac{2\pi ik}{n}}$, $k = 1, 2, \ldots, n$, on the unit circle with finite angular derivative $\gamma = f'(\eta_k) = -n$. So the generated semiflow has n repelling fixed points, and there are n maximal flow invariant domains. One can show that the functions

$$\varphi_k(z) = e^{\frac{2\pi ik}{n}} \sqrt[n]{\frac{1-z}{2}}$$

are the solutions of (11) with $\alpha = n$ satisfying $\varphi_k(1) = 0$ and $\varphi_k(-1) = \eta_k$ which map Δ onto n FID's Ω_k (for $n = 2$, these domains form lemniscate) with $\overline{\Omega_i} \cap \overline{\Omega_j} = \{0\}$ when $i \neq j$. The family $\{F_t\}_{t\in\mathbf{R}}$ forms a group of automorphisms of each one of these domains. See Figure 1 for $n = 1, 2, 3$ and 5. For $n = 1$, for instance, it can be seen explicitly that $F_t(\varphi(z))$ is well-defined for all $t \in \mathbf{R}$ and tends to $\eta = 1$ when $t \to -\infty$.

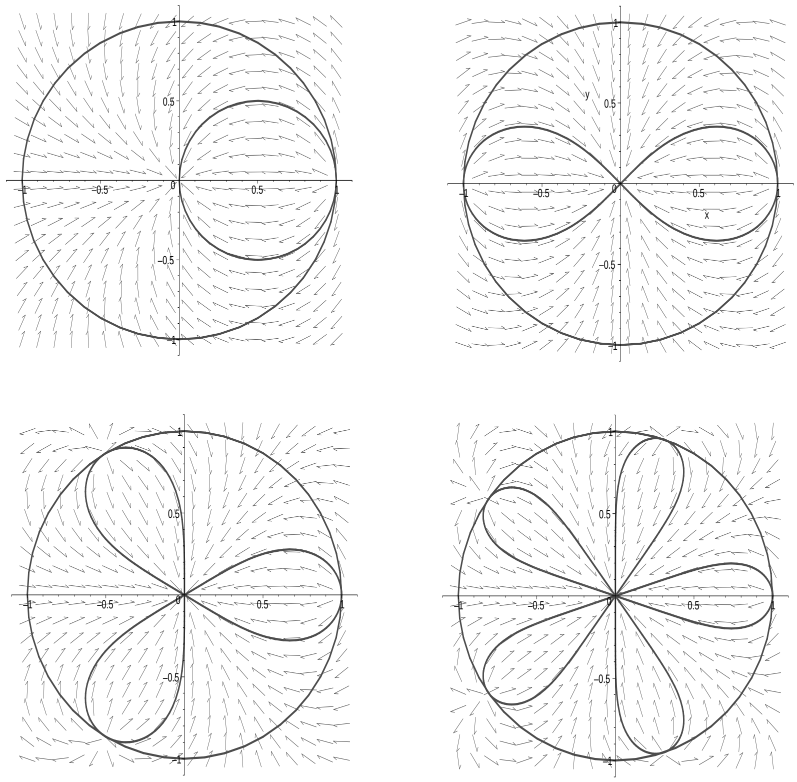

Figure 1. Example 1, $n = 1, 2, 3, 5$.

Example 2. Consider a generator $f \in \mathscr{G}^+[1]$ defined by

$$f(z) = -\frac{(1-z)(1+z^2)}{1+z}.$$

Solving the Cauchy problem (2), we find

$$F_t(z) = \frac{(1+z^2)e^{2t} - (1-z)\sqrt{2(1+z^2)e^{2t} - (1-z)^2}}{(1+z^2)e^{2t} - (1-z)^2}.$$

Since f has the two additional null points $\eta_{1,2} = \pm i \in \partial\Delta$ with finite angular derivative $\gamma = f'(\pm i) = -2$, the generated semiflow has two repelling fixed points. Thus, there are two maximal flow invariant domains Ω_1 and Ω_2. One can show that these domains Ω_j coincide with the upper and the lower half-disks (see Figure 2). So we have $\overline{\Omega_1} \cap \overline{\Omega_2} = \{-1 < x < 1\}$. In each of these two domains, the family $\{F_t\}_{t\in\mathbf{R}}$ is well defined and forms a group of automorphisms.

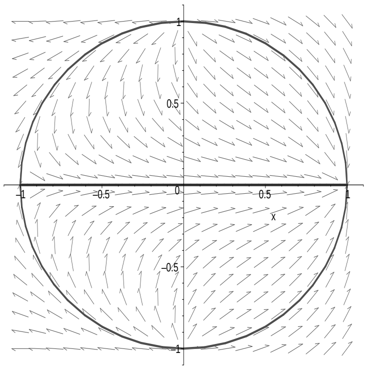

Figure 2. Example 2, the flow generated by $f(z) = -\frac{(1-z)(1+z^2)}{1+z}$ and two flow invariant domains.

The following example shows that a maximal flow invariant domain may be even dense in the open unit disk.

Example 3. Let $f \in \mathscr{G}^+[0]$ be given by

$$f(z) = z\,\frac{1-z}{1+z}\,.$$

In this case, $\tau = 0$ and $\eta = 1$. Also, we have $f'(0) = 1$ and $f'(1) = -\frac{1}{2}$. Solving equation (11) with $\alpha = \frac{1}{2}$, one can write its solution in the form $\varphi(z) = h^{-1}(h_0(z))$, where h is the Koebe function $h(z) = \dfrac{z}{(1-z)^2}$ and $h_0(z) = \left(\dfrac{1-z}{1+z}\right)^2$. We shall see below that each solution of (11) has a similar representation.

Thus φ maps Δ onto the maximal flow invariant domain $\Omega = \varphi(\Delta) = \Delta\backslash\{-1 \le x \le 0\}$; see Figure 3. (All the pictures were obtained by using the vector field drawing tool in Maple 9.)

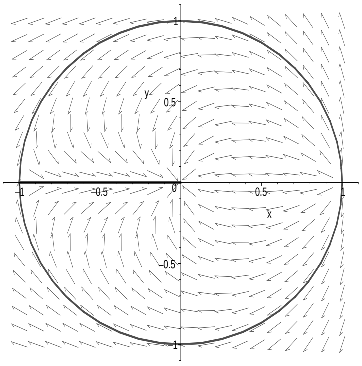

Figure 3. Example 3, the flow generated by $f(z) = z\frac{(1-z)}{1+z}$ and the dense flow invariant domain.

Remark 2. Let $F \in \mathrm{Hol}(\Delta)$ be a single self-mapping of Δ which can be embedded into a continuous semigroup, i.e., there is a semiflow $\mathscr{S} = \{F_t\}_{t\geq 0}$ such that $F = F_1$. In this case, all the fractional iterations F_t of F have the same collection of boundary fixed points for all $t \geq 0$ (see [9]). In turn, our theorem asserts the existence of backward fractional iterations of F defined on a FID Ω whenever F has a repelling boundary fixed point η, i.e.,

$$A = F'(\eta) = \lim_{z\to\eta} F'(z) > 1. \tag{12}$$

As a matter of fact, for a single mapping which is not necessarily embedded into a semiflow (not even necessarily univalent on Δ), the existence of backward integer iterations under condition (12) was proved in [27]. This fact has provided the existence of conjugations near repelling points. More precisely, the main result in [27] asserts that if $\eta = 1,\ a = \dfrac{A-1}{A+1}$ and $G(z) = \dfrac{z-a}{1-az}$, then there is $\varphi \in \mathrm{Hol}(\Delta)$ with $\varphi(1) = 1$ which is a conjugation for F and G, i.e.,

$$\varphi(G(z)) = F(\varphi(z)).$$

However, for the case in which F can be embedded into a continuous semigroup $\mathscr{S} = \{F_t\}$, it is not clear whether φ is a conjugation for the whole semiflow $\mathscr{S}$ and the flow produced by G.

It is natural to expect a more precise result under stronger requirements. A direct consequence of the proof of our Theorem 1 is the following assertion for conjugations.

Corollary 1. *Let $F \subset \mathrm{Hol}(\Delta)$ be embedded into a semiflow $\mathscr{S} = \{F_t\}_{t\geq 0}$ of hyperbolic type and let $\eta \in \partial\Delta$ be a repelling fixed point of F with $A = F'(\eta) > 1$.*

Then for each $B \geq A$ and the automorphism $G(= G_B) \in \mathrm{Aut}(\Delta)$ defined by

$$G(z) = \frac{z+b}{1+zb},$$

where $b = \frac{B-1}{B+1}$, there is a homeomorphism $\varphi(= \varphi_B)$ of $\overline{\Delta}$, $\varphi \in \mathrm{Hol}(\Delta)$, such that $\varphi(\eta) = -1$ and

$$\varphi(G(z)) = F(\varphi(z)), \quad z \in \Delta.$$

Moreover, for all $t \in \mathbf{R}$ and $w \in \varphi(\Delta)$, the flow $\{F_t(w)\}_{t\in\mathbf{R}}$ is well-defined with $F_1 = F$ and

$$F_t(\varphi(z)) = \varphi(G_t(z)), \quad \text{for all } t \in \mathbf{R},$$

where

$$G_t(z) = \frac{z+1+e^{-\alpha t}(z-1)}{z+1-e^{-\alpha t}(z-1)}, \quad t \in \mathbf{R},$$

with $\alpha = \log B$.

In addition, $\varphi_B(\Delta) \subseteq \varphi_A(\Delta)$, with $\varphi_A(\Delta) = \varphi_B(\Delta)$ if and only if $A = B$.

Our approach to construct conjugations is different from that used in [27].

The main tool of the proof of our theorems is a linearization method for semigroups which uses the classes of starlike and spirallike functions on Δ.

Definition 4. A univalent function h is called spirallike (respectively, starlike) on Δ if for some $\mu \in \mathbf{C}$ with $\operatorname{Re}\mu > 0$ (respectively, $\mu \in \mathbf{R}$ with $\mu > 0$) and for each point $z \in \Delta$,

$$\left\{e^{-\mu t}h(z),\ t \geq 0\right\} \subset h(\Delta). \tag{13}$$

In this case, we say that h is μ-spirallike.

Obviously, $0 \in \overline{h(\Delta)}$.

• If $0 \in h(\Delta)$, (i.e., if there is a point $\tau \in \Delta$ such that $h(\tau) = 0$), then h is called *spirallike* (*respectively, starlike*) *with respect to an interior point.*

• If $0 \notin h(\Delta)$ (and hence $0 \in \partial h(\Delta)$), h is called *spirallike* (*respectively, starlike*) *with respect to a boundary point.* In this case, there is a boundary point $\tau \in \partial\Delta$ such that $h(\tau) := \angle\lim\limits_{z\to\tau} h(z) = 0$ (see, for example, [13]).

The class of spirallike (starlike) functions satisfying $h(\tau) = 0,\ \tau \in \overline{\Delta}$, is denoted by $\operatorname{Spiral}[\tau]$ (respectively, $\operatorname{Star}[\tau]$).

It follows from Definition 4 that a family $\mathscr{S} = \{F_t\}_{t\geq 0}$ of holomorphic self-mappings of the open unit disk Δ defined by

$$F_t(z) := h^{-1}\left(e^{-\mu t}h(z)\right)$$

forms a semiflow on Δ. Differentiating this semiflow at $t = 0^+$, one sees that h is a solution of the differential equation

$$\mu h(z) = h'(z)f(z), \tag{14}$$

where $f \in \mathscr{G}^+[\tau]$ is the generator of $\mathscr{S}$. As a matter of fact, the converse assertion also holds [13], [14], [4], [12], [15]. More precisely, we have

Lemma 1. *Let $\mathscr{S} = \{F_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings generated by $f \in \mathscr{G}^+[\tau],\ \tau \in \overline{\Delta}$.*

(i) *If $\tau \in \Delta$, then equation* (14) *has a univalent solution if and only if $\mu = f'(\tau)$.*

(ii) *If $\tau \in \partial\Delta$, then equation* (14) *has a univalent solution h satisfying $h(\tau) = 0$ if and only if $\mu \in \Lambda_\beta := \{w \neq 0 :\ |w - \beta| \leq \beta\}$, where $\beta = f'(\tau)$.*

Moreover, in both cases, this solution h is a spirallike (*starlike*) *function which satisfies Schröder's functional equation*

$$h(F_t(z)) = e^{-\mu t}h(z), \quad t \geq 0,\ z \in \Delta. \tag{15}$$

It is clear that h is λ-spirallike for each λ with $\arg\lambda = \arg\mu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We call this function h the *spirallike* (*starlike*) *function associated with f.*

Since we are interested in generators having additional null points on the boundary, we introduce the following subclasses of $\mathscr{G}^+[\tau]$ and of $\operatorname{Spiral}[\tau]$ ($\operatorname{Star}[\tau]$).

• Given $\tau \in \overline{\Delta}$ and $\eta \in \partial\Delta,\ \eta \neq \tau$, we say that a generator $f \in \mathscr{G}^+[\tau]$ belongs to the subcone $\mathscr{G}^+[\tau,\eta]$ if it vanishes at the point η, i.e., $\angle\lim\limits_{z\to\eta} f(z) = 0$ and the angular derivative at the point η

$$f'(\eta) := \angle\lim_{z\to\eta}\frac{f(z)}{z-\eta}$$

exists finitely.

• We say that a function $h \in \mathrm{Spiral}[\tau]$ ($h \in \mathrm{Star}[\tau]$) belongs to the subclass $\mathrm{Spiral}[\tau,\eta]$ ($\mathrm{Star}[\tau,\eta]$) if the angular limit

$$Q_h(\eta) := \angle\lim_{z\to\eta} \frac{(z-\eta)h'(z)}{h(z)}$$

exists finitely and is different from zero.

Remark 3. We recall that if $\zeta \in \partial\Delta$ and $g \in \mathrm{Hol}(\Delta, \mathbf{C})$ is such that $\angle\lim\limits_{z\to\zeta} g(z) =: g(\zeta)$ exists finitely, the expression

$$Q_g(\zeta, z) := \frac{(z-\zeta)g'(z)}{g(z)-g(\zeta)}$$

is called the *Visser–Ostrowski quotient* of g at ζ (see [26]). If for some $h \in \mathrm{Hol}(\Delta, \mathbf{C})$ we have $\angle\lim\limits_{z\to\zeta} h(z) = \infty$, then the Visser–Ostrowski quotient of h is defined by

$$Q_h(\zeta, z) := Q_{1/h}(\zeta, z).$$

A function g is said to satisfy the *Visser–Ostrowski condition* if

$$Q_g(\zeta) := \angle\lim_{z\to\zeta} Q_g(\zeta, z) = 1.$$

In this context, we recall also that $g \in \mathrm{Hol}(\Delta, C)$ is called *conformal at* $\zeta \in \partial\Delta$ if the angular derivative $g'(\zeta)$ exists and is neither zero nor infinity; g is called *isogonal at* ζ if the limit of $\arg \dfrac{g(z)-g(\zeta)}{z-\zeta}$ as $z \to \zeta$ exists.

It is clear that any function g conformal at a boundary point ζ is isogonal at this point. Also, it is known (see [26]) that any function g isogonal at a boundary point ζ satisfies the Visser–Ostrowski condition at this point, i.e., $Q_g(\zeta) = 1$.

So it is natural to say that g satisfies a *generalized Visser–Ostrowski condition* if $Q_g(\zeta) := \angle\lim\limits_{z\to\zeta} Q_g(\zeta, z)$ exists finitely and is different from zero. Thus each function $h \in \mathrm{Spiral}[\tau,\eta]$ ($h \in \mathrm{Star}[\tau,\eta]$) satisfies a generalized Visser–Ostrowski condition at the boundary point η.

To proceed, we note that the inequality $\eta \neq \tau$ implies that for each $h \in \mathrm{Spiral}[\tau,\eta]$

$$\angle\lim_{z\to\eta} h(z) = \infty.$$

The following fact is an immediate consequence of Lemma 1.

Lemma 2. *Let* $h \in \mathrm{Spiral}[\tau]$ *and* $f \in \mathscr{G}^+[\tau]$ *be connected by* (14). *Then* h *belongs to* $\mathrm{Spiral}[\tau,\eta]$ *if and only if* $f \in \mathscr{G}^+[\tau,\eta]$. *In this case,*

$$Q_h(\eta) = \frac{\mu}{f'(\eta)}\,.$$

We require two representation formulas for the classes of starlike functions $\mathrm{Star}[\tau]$ and $\mathrm{Star}[\tau,\eta]$. For a boundary point w, denote by δ_w the Dirac measure at this point.

Lemma 3. (cf. [19] and [18]) *Let* $\tau \in \overline{\Delta}$ *and* $\eta \in \partial\Delta$, $\eta \neq \tau$. *Let* $h \in \mathrm{Hol}(\Delta, \mathbf{C})$ *satisfy* $h(\tau) = 0$. *Then*

(i) $h \in \mathrm{Star}[\tau]$ *if and only if it has the form*

$$h(z) = C(z-\tau)(1-z\bar{\tau}) \cdot \exp\left[-2 \oint\limits_{\partial\Delta} \log(1-z\bar{\zeta})\, d\widetilde{\sigma}(\zeta)\right], \tag{16}$$

where $d\widetilde{\sigma}$ *is an arbitrary probability measure on the unit circle and* $C \neq 0$.

(ii) *Moreover,* $h \in \mathrm{Star}[\tau, \eta]$ *if and only if it has the form*

$$\begin{aligned} h(z) &= C(z-\tau)(1-z\bar{\tau})(1-z\bar{\eta})^{-2a} \cdot \\ &\cdot \exp\left[-2(1-a) \oint\limits_{\partial\Delta} \log(1-z\bar{\zeta})\, d\sigma(\zeta)\right], \end{aligned} \tag{17}$$

where $d\sigma$ *is a probability measure on the unit circle singular relative to* δ_η, $C \neq 0$ *and* $a \in (0,1]$. *In this case,* $Q_h(\eta) = -2a$.

Remark 4. The constant C can be chosen starting from a normalization of functions under consideration. On the other hand, since a starlike function h is a solution of a linear homogeneous equation (see (14)), C arises in the integration process of this equation.

Proof. First, suppose that $\tau = 0$, and let $h \in \mathrm{Hol}(\Delta, \mathbf{C})$ be normalized by $h(0) = 0$ and $h'(0) = 1$. A well-known criterion of Nevanlinna asserts that $h \in \mathrm{Star}[0]$ if and only if

$$q(z) := \frac{zh'(z)}{h(z)}$$

has positive real part. (Note that the same fact follows by (14), because by the Berkson–Porta representation formula (3), a generator $f \in \mathscr{G}[0]$ has the form $f(z) = zp(z)$ with $\mathrm{Re}\, p(z) > 0$).)

Representing q by the Riesz–Herglotz formula, we write

$$\frac{zh'(z)}{h(z)} = \oint\limits_{\partial\Delta} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\, d\widetilde{\sigma}(\zeta)$$

with some probability measure $d\widetilde{\sigma}$. Integrating this equality, we get

$$h(z) = z \exp\left[-2 \oint_{\partial\Delta} \log(1-z\bar{\zeta})\, d\widetilde{\sigma}(\zeta)\right]. \tag{18}$$

So we have proved (16) for the case $\tau = 0$.

Now let $\tau \in \Delta$ be different from zero, and suppose $h(\tau) = 0$. It was proved by Hummel (see [21], [22] and [32]) that $h \in \mathrm{Star}[\tau]$ if and only if

$$\frac{z}{(z-\tau)(1-z\bar{\tau})}\, h(z) \in \mathrm{Star}[0].$$

Thus, (18) implies (16) for the interior location of τ. The reverse consideration and Hummel's criterion show that if h satisfies (16) with $\tau \in \Delta$, it must be starlike.

Finally, let $\tau \in \partial\Delta$. Following Lyzzaik [25] (see also [11]) one can approximate $h \in \mathrm{Star}[\tau]$ by a sequence $\{h_n\}$ of functions starlike with respect to those interior points τ_n which converge to τ. Also, one can assume that $h_n(0) = h(0)$. Representing each function h_n by (16)

$$h_n(z) = C_n(z-\tau_n)(1-z\bar{\tau}_n)\cdot\exp\left[-2\oint\limits_{\partial\Delta}\log(1-z\bar{\zeta})d\widetilde{\sigma_n}(\zeta)\right],$$

we see that

$$h(0) = h_n(0) = -C_n\tau_n.$$

Thus $C_n \to -\frac{h(0)}{\tau}$. Since the set of all probability measures is compact, $\{d\widetilde{\sigma_n}\}$ has a subsequence converging to some probability measure $d\widetilde{\sigma}$. Therefore, any function $h \in \mathrm{Star}[\tau]$ has the form (16).

To prove the converse assertion, we suppose that h has the form (16) with $\tau \in \partial\Delta$. Note that h is starlike if and only if the function $ah(cz)$, $a \neq 0$, $|c| = 1$, is. Therefore, without loss of generality, one can assume that h is normalized by $h(0) = 1$, i.e.,

$$h(z) = (1-z)^2\cdot\exp\left[-2\oint\limits_{\partial\Delta}\log(1-z\bar{\zeta})d\widetilde{\sigma}(\zeta)\right].$$

Differentiating the latter formula, one sees that h satisfies a modified Robertson inequality (see [34] and [13])

$$\mathrm{Re}\left[\frac{zh'(z)}{h(z)} + \frac{1+z}{1-z}\right] > 0. \tag{19}$$

A main result of [34] and Theorem 7 [13] imply that h is a starlike function with respect to a boundary point with $h(1) = 0$, i.e., $h \in \mathrm{Star}[1]$. The first assertion is proved.

Let

$$d\widetilde{\sigma} = a\delta_\eta + (1-a)d\sigma, \quad 0 \le b \le 1,$$

be the Lebesgue decomposition of $d\widetilde{\sigma}$ relative to the Dirac measure δ_η, where the probability measures $d\sigma$ and δ_η are mutually singular. Using this decomposition, we rewrite (16) in the form (17).

Now we calculate

$$
\begin{aligned}
Q_h(\eta) &= \angle\lim_{z\to\eta} \frac{h'(z)(z-\eta)}{h(z)} \\
&= \angle\lim_{z\to\eta}(z-\eta)\left[\frac{((z-\tau)(1-z\bar{\tau}))'}{(z-\tau)(1-z\bar{\tau})} + \frac{2a\bar{\eta}}{1-z\bar{\eta}}\right. \\
&\quad \left. + 2(1-a)\oint_{\partial\Delta} \frac{\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma(\zeta)\right] \\
&= -2a + 2(1-a)\angle\lim_{z\to -1}\oint_{\partial\Delta} \frac{\bar{\zeta}(z-\eta)}{1-z\bar{\zeta}}\, d\sigma(\zeta)
\end{aligned} \tag{20}
$$

Noting that

$$\left|\frac{\bar{\zeta}(z-\eta)}{1-z\bar{\zeta}}\right| \le \frac{|z-\eta|}{1-|z|},$$

we see that the integrand in the last expression of (20) is bounded on each nontangential approach region $D_{k,\eta} := \{z:\ |z-\eta| < k(1-|z|)\}$, $k \ge 1$, at the point η. Since the measures $d\sigma$ and δ_η are mutually singular, we conclude by the Lebesgue convergence theorem that the last integral in (20) is equal to zero, so

$$Q_h(\eta) = -2a.$$

The proof is complete. □

The following results are angle distortion theorems for starlike and spirallike functions of the classes $\mathrm{Star}[\tau,\eta]$ and $\mathrm{Spiral}[\tau,\eta]$ respectively.

Lemma 4. (cf. [31] and [18]) *Let $h \in \mathrm{Star}[\tau,\eta]$ with $Q_h(\eta) = \nu$. Denote*

$$\theta := \lim_{r\to 1^-} \arg h(r\eta). \tag{21}$$

Then the image $h(\Delta)$ contains the wedge

$$W = \left\{w \in \mathbf{C}:\ |\arg w - \theta| < \frac{|\nu|\pi}{2}\right\} \tag{22}$$

and contains no larger wedge with the same bisector.

Proof. By Lemma 3, the function h has the form (17) with $\nu = -2a$.

First we show that the image $h(\Delta)$ contains the wedge W defined by (22).

Since (as mentioned above) $\angle\lim_{z\to\eta} h(z) = \infty$, for each $\delta \in \left(0, \frac{\pi}{2}\right)$ and each $R > 0$, there exists $r > 0$ such that

$$|h(z)| > R \tag{23}$$

whenever

$$z \in D_{r,\delta} := \{z \in \Delta:\ |1 - z\bar{\eta}| \le r,\ |\arg(1 - z\bar{\eta})| \le \delta\}.$$

Lemma 3 and the Lebesgue bounded convergence theorem imply the existence of

$$\begin{aligned}&\lim_{z\to\eta}\arg\frac{h(z)}{(1-z\bar\eta)^\nu}\\&=\arg\Big(C(\eta-\tau)(1-\eta\bar\tau)\Big)-2(1-a)\lim_{z\to\eta}\oint\limits_{\partial\Delta}\arg(1-z\bar\zeta)\,d\sigma(\zeta).\end{aligned}$$

On the other hand, by formula (17), we have

$$\begin{aligned}\theta&=\lim_{r\to1^-}\arg h(r\eta)\\&=\arg\Big(C(\eta-\tau)(1-\eta\bar\tau)\Big)-2(1-a)\lim_{r\to1^-}\oint_{\partial\Delta}\arg(1-r\eta\bar\zeta)\,d\sigma(\zeta).\end{aligned}$$

Therefore,

$$\lim_{z\to\eta}\arg\frac{h(z)}{(1-z\bar\eta)^\nu}=\theta.$$

Thus, decreasing r (if necessary), we have

$$\theta-\varepsilon<\arg\frac{h(z)}{(1-z\bar\eta)^\nu}<\theta+\varepsilon$$

for all $z\in D_{r,\delta}$. So, for each point z belonging to the arc

$$\Gamma:=\{z\in\Delta:\ |1-z\bar\eta|=r,\ |\arg(1-z\bar\eta)|\le\delta\}\subset D_{r,\delta},$$

i.e., $z=\eta(1-re^{it})$, $|t|\le\delta$, we get

$$\theta-\varepsilon-t|\nu|<\arg h(z)<\theta+\varepsilon-t|\nu|.$$

In particular,

$$\arg h(\eta(1-re^{i\delta}))<\theta+\varepsilon-\delta|\nu| \tag{24}$$

and

$$\arg h(\eta(1-re^{-i\delta}))>\theta-\varepsilon+\delta|\nu|. \tag{25}$$

Thus, the curve $h(\Gamma)$ lies outside the disk $|z|\le R$ and joins two points having arguments less than $\theta+\varepsilon-\delta|\nu|$ and greater than $\theta-\varepsilon+\delta|\nu|$, respectively. Since h is starlike, we see that $h(\Delta)$ contains the sector

$$\{w\in\mathbf{C}:\ |w|<R,\ |\arg w-\theta|<\delta|\nu|-\varepsilon\}.$$

Since R and ε are arbitrary, one concludes

$$\{w\in\mathbf{C}:\ |\arg w-\theta|<\delta|\nu|\}\subset h(\Delta).$$

Letting δ tend to $\frac{\pi}{2}$, we obtain

$$W=\left\{w\in\mathbf{C}:\ |\arg w-\theta|<\frac{|\nu|\pi}{2}\right\}\subset h(\Delta).$$

Further, since h is a starlike function, $\arg h(e^{i\varphi})$ is an increasing function in $\varphi \in (\arg \eta - \pi, \arg \eta + \pi)$. So the limits

$$\lim_{\varphi \to (\arg \eta)^{\pm}} \arg h(e^{i\varphi})$$

exist. Let $\varphi_{n,+} \to (\arg \eta)^{+}$ and $\varphi_{n,-} \to (\arg \eta)^{-}$ be two sequences such that the values $h(e^{i\varphi_{n,\pm}})$ are finite. Then, once again by Lemma 3,

$$\begin{aligned} &\lim_{n\to\infty} \arg h(e^{i\varphi_{n,+}}) - \arg h(e^{i\varphi_{n,-}}) \\ &= \lim_{n\to\infty} \left((\arg(1 - e^{i\varphi_{n,+}}\bar{\eta}))^{\nu} - (\arg(1 - e^{i\varphi_{n,-}}\bar{\eta}))^{\nu} \right) = |\nu|\pi. \end{aligned}$$

Therefore, the image contains no wedge of angle larger than $|\nu|\pi$. Thus, the wedge W defined by (22) is the largest one contained in $h(\Delta)$.

The proof is complete. □

Let $\lambda \in \Lambda = \{w \in \mathbf{C} : |w - 1| \le 1, \ w \ne 0\}$ and $\theta \in [0, 2\pi)$ be given. Define the function $h_{\lambda,\theta} \in \mathrm{Hol}(\Delta)$ by

$$h_{\lambda,\theta}(z) = e^{i\theta} \left(\frac{1-z}{1+z} \right)^{\lambda}. \tag{26}$$

Here and in the sequel, we choose a single-valued branch of the analytic function w^{λ} such that $1^{\lambda} = 1$.

Definition 5. The set $W_{\lambda,\theta} = h_{\lambda,\theta}(\Delta)$ is called a canonical λ-spiral wedge with midline $l_{\theta,\lambda} = \{w \in \mathbf{C} : \ w = e^{i\theta + t\lambda}, \ t \in \mathbf{R}\}$.

To explain this definition, let us observe that $h = h_{\lambda,\theta}$ is a solution of the differential equation

$$\lambda h(z) = h'(z) f(z)$$

normalized by the conditions $h(0) = e^{i\theta}, \ h(1) = 0$, where f is given by

$$f(z) = \frac{1}{2}\,(z^2 - 1).$$

Since $f \in \mathscr{G}^{+}[1]$ with $f'(1) = 1$ and $\lambda \in \Lambda$, it follows by Lemma 1 that h is a λ-spirallike function with respect to the boundary point $h(1) = 0$. Moreover, f is a generator of a one-parameter group (flow) of hyperbolic automorphisms of Δ having two boundary fixed points $z = 1$ and $z = -1$. Hence, for each $w \in W_{\lambda,\theta}$ and $t \in \mathbf{R} = (-\infty, \infty)$, the spiral curve $e^{-t\lambda} w$ belongs to $W_{\lambda,\theta}$ (see (15)).

In [4], the notion of "angle measure" for spirallike domains with respect to a boundary point was introduced. It can be shown that a λ-spiral wedge is of angle measure $\pi\lambda$.

Finally, we see that for real $\lambda \in (0, 2]$, the set $W_{\lambda,\theta}$ is a straight wedge (sector) of angle $\pi\lambda$, whose bisector is $l_{\theta} = \{w \in \mathbf{C} : \ \arg w = \theta\}$.

Lemma 5. *Let $h \in \mathrm{Spiral}[\tau]$ be a μ-spirallike function on Δ. Then the image $h(\Delta)$ contains a canonical λ-spiral wedge with*

$$\arg \lambda = \arg \mu \tag{27}$$

if and only if $h \in \mathrm{Spiral}[\tau, \eta]$ *for some* $\eta \in \partial\Delta$*. Moreover, if* $Q_h(\eta) = \nu$*, then the canonical wedge* $W_{-\nu,\theta} \subset h(\Delta)$ *for some* $\theta \in [0, 2\pi)$*; and it is maximal in the sense that there is no spiral wedge* $W_{\lambda,\theta} \subset h(\Delta)$ *with* λ *satisfying* (27) *which contains* $W_{-\nu,\theta}$ *properly.*

Proof. First, given $h \in \mathrm{Spiral}[\tau, \eta]$ we construct $h_1 \in \mathrm{Spiral}[1, -1]$ which is spirallike with respect to a boundary point whose image eventually coincides with $h(\Delta)$ at ∞. If $\tau \in \partial\Delta$, we just set $h_1 = h(\Phi(z))$, where $\Phi \in \mathrm{Aut}(\Delta)$ is an automorphism of Δ such that $\Phi(1) = \tau$ and $\Phi(-1) = \eta$.

If $\tau \in \Delta$, we take any two points $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ such that $w_1 = h(z_1)$ and $w_2 = h(z_2)$ exist finitely and $\theta_1 \in (\arg\eta - \epsilon, \arg\eta)$, $\theta_2 \in (\arg\eta, \arg\eta - \epsilon)$, so the arc (θ_1, θ_2) on the unit circle contains the point η.

Since h is spirallike with respect to an interior point, it satisfies the equation

$$\beta h(z) = h'(z) f(z), \tag{28}$$

where $f \in \mathscr{G}^+[\tau]$ and $\beta = f'(\tau)$, so $\arg\mu = \arg\beta$. This means that for each $w \in h(\Delta)$ the spiral curve $\{e^{-t\beta}w,\ t \geq 0\}$ belongs to $h(\Delta)$. In turn, the curves $l_1 = \{z = h^{-1}(e^{-t\beta}w_1),\ t \geq 0\}$ and $l_2 = \{z = h^{-1}(e^{-t\beta}w_2),\ t \geq 0\}$ lie in $\overline{\Delta}$ with ends in z_1 and τ and z_2 and τ, respectively.

Since $z_1 \neq z_2$ and the interior points of l_1 and l_2 are semigroup trajectories in Δ, these curves do not intersect except at their common end point $z = \tau$. Consequently, the domain D bounded by l_1, l_2 and the arc (θ_1, θ_2) is simply connected, and there is a conformal mapping Φ of Δ such that $\Phi(\Delta) = D$ and $\Phi(-1) = \eta$, $\Phi(1) = \tau$. Now define $h_1(z) = h(\Phi(z))$. It follows by our construction that $h_1(\Delta) \subset h(\Delta)$ and h_1 is spirallike with respect to a boundary point $h_1(1) = 0$. In addition, since Φ is conformal at the point $z = -1$, it satisfies the Visser–Ostrowski condition and we have

$$\begin{aligned}
\angle \lim_{z\to-1} \frac{(z+1)h_1'(z)}{h_1(z)} &= \angle \lim_{z\to-1} \frac{(z+1)h'(\Phi(z))(\Phi(z)-\eta)}{h(\Phi(z))(\Phi(z)-\eta)} \\
&= \angle \lim_{z\to-1} \frac{(z+1)\Phi'(z)}{\Phi(z)-\Phi(-1)} \cdot \angle \lim_{z\to-1} \frac{(\Phi(z)-\eta)h'(\Phi(z))}{h(\Phi(z))} \\
&= \angle \lim_{z\to-1} \frac{(\Phi(z)+1)h'(\Phi(z))}{h(\Phi(z))}.
\end{aligned} \tag{29}$$

Note also that Φ is a self-mapping of Δ mapping the point $z = -1$ to η and having a finite derivative at this point.

It follows by the Julia–Carathéodory theorem, (see, for example, [32]) that if z converges to -1 nontangentially, then $\Phi(z)$ converges nontangentially to $\eta = \Phi(-1)$. Then (29) implies that

$$Q_{h_1}(-1) = \angle \lim_{z\to-1} \frac{(z+1)h_1'(z)}{h_1(z)} \tag{30}$$

exists finitely if and only if $h \in \mathrm{Spiral}[\tau, \eta]$ and

$$Q_{h_1}(-1) = Q_h(\eta). \tag{31}$$

We claim that this last relation implies that $h_1(\Delta)$ contains a $(-\nu)$-spiral wedge $W_{-\nu,\theta}$ for some $\theta \in [0, 2\pi)$.

To this end, observe that h_1 satisfies the equation

$$\beta h_1(z) = h_1'(z) \cdot f_1(z),$$

where $f_1(z) = \frac{f(\Phi(z))}{\Phi'(z)}$ is a generator of a semigroup of Δ with $f_1(1) = 0$ and $f_1'(1) = \beta_1$ for some $\beta_1 > 0$ such that

$$|\beta - \beta_1| \leq \beta_1.$$

Therefore, h_1 is a complex power of the function $h_2 \in \mathrm{Hol}(\Delta, \mathbf{C})$ defined by the equation

$$\beta_1 h_2(z) = h_2'(z) f_1(z), \quad h_2(1) = 0, \tag{32}$$

i.e.,

$$h_1(z) = h_2^{\mu}(z), \tag{33}$$

where $\mu = \frac{\beta}{\beta_1} \neq 0$, $|\mu - 1| \leq 1$, hence $\arg \mu = \arg \beta$.

On the other hand, if we normalize h_1 by $h_1^{1/\mu}(0) = h_2(0)$, equation (33) has a unique solution which is a starlike function with respect to a boundary point $(h_2(1) = 0)$. Obviously,

$$Q_{h_2}(-1) = \frac{1}{\mu} Q_{h_1}(-1) \left(= \frac{1}{\mu} Q_h(\eta) \right). \tag{34}$$

Note that $\nu_2 := Q_{h_2}(-1)$ is a negative real number, while $\nu_1 := Q_{h_1}(-1) = \nu_2 \mu$ is complex.

Now it follows by Lemma 4 that the starlike set $h_2(\Delta)$ contains a straight wedge (sector) of a nonzero angle $\sigma\pi$ for each $\sigma \in (0, |\nu_2|\pi]$. So the maximal (straight) wedge $W \subset h_2(\Delta)$ is of the form

$$W = W_{-\nu_2,\theta_2} = \left\{ w \in \mathbf{C} : \ w = e^{i\theta_2} \left(\frac{1-z}{1+z} \right)^{-\nu_2} \right\},$$

with

$$\begin{aligned} \theta_2 &= \lim_{r \to 1^-} \arg h_2(-r) = \lim_{r \to 1^-} \arg h_1^{\nu_2/\nu_1}(-r) \\ &= \nu_2 \cdot \lim_{r \to 1^-} \arg h_1^{1/\nu_1}(-r) = \nu_2 \theta_1, \end{aligned}$$

where

$$\theta_1 = \lim_{r \to 1^-} \arg h_1^{1/\nu_1}(-r).$$

Writing W in the form

$$W = \left\{ e^{i\varsigma} e^t, \ t \in \mathbf{R}, \ \varsigma \in \left(\theta_2 + \frac{\pi\nu_2}{2}, \theta_2 - \frac{\pi\nu_2}{2} \right) \right\}$$

and setting $\varsigma_1 = \varsigma/\nu_2,\ s = t/\nu_2$, we see that the set

$$K := W^\mu = \left\{ e^{i\varsigma_1\nu_1} e^{s\nu_1},\ s \in \mathbf{R},\ \varsigma_1 \in \left(\frac{\theta_2}{\nu_2} - \frac{\pi}{2}, \frac{\theta_2}{\nu_2} + \frac{\pi}{2} \right) \right\}$$

is contained in $h_1(\Delta)$; hence in $h(\Delta)$. But $\theta_2/\nu_2 = \theta_1$ and $\nu_1 = \nu\,(= Q_h(-1))$; hence K is of the form

$$\begin{aligned} K &= \left\{ e^{i\varsigma_1\nu} e^{s\nu},\ s \in \mathbf{R},\ \varsigma_1 \in \left(\theta_1 - \frac{\pi}{2}, \theta_1 + \frac{\pi}{2} \right) \right\} \\ &= \left\{ e^{i\theta_1\nu} e^{i\varsigma_1\nu} e^{s\nu},\ s \in \mathbf{R},\ \varsigma_1 \in \left(-\frac{\pi}{2}, +\frac{\pi}{2} \right) \right\}. \end{aligned}$$

Setting $\theta = \frac{|\nu|^2\theta_1}{\operatorname{Re}\nu} \in \mathbf{R}$, we get

$$i\theta_1\nu + s\nu = i\theta + \nu\left(\frac{\theta \operatorname{Re}\nu}{|\nu|^2} + s - \frac{i\theta}{\nu} \right) = i\theta + \nu\left(s - \frac{\theta \operatorname{Im}\nu}{|\nu|^2} \right).$$

Since s takes all real values, so does $t = s - \frac{\theta \operatorname{Im}\nu}{|\nu|^2}$. Therefore, the set K has the form

$$K = e^{i\theta} \left\{ e^{i\varsigma_1\nu} e^{t\nu},\ t \in \mathbf{R},\ \varsigma_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right\},$$

i.e., coincides with $W_{-\nu,\theta}$. Finally, it follows by (34) that $\lambda := -\nu = |\nu_2|\mu$. This implies (27).

Conversely, let h be a μ-spirallike function on Δ such that $h(\Delta)$ contains a canonical λ-spiral wedge $W_{\lambda,\theta}$ for some λ satisfying (27) and $\theta \in [0, 2\pi)$. Then for each $w_0 \in W_{\lambda,\theta}$, the curve $l := \left\{ w \in \mathbf{C} :\ w = e^{-t\lambda} w_0,\ t \in \mathbf{R} \right\}$ belongs to $h(\Delta)$. Hence the curve $h^{-1}(l) \subset \Delta$ joints the point $\tau \in \overline{\Delta}$ with a point $\eta \in \partial\Delta$. Again, as in the first step of the proof, one can find a conformal mapping $\Phi \in \operatorname{Hol}(\Delta)$ with $\Phi(1) = \tau,\ \Phi(-1) = \eta$ such that $h_1 = h \circ \Phi$ is a μ-spirallike function with respect to a boundary point $h_1(1) = 0$ and

$$W_{\lambda,\theta} \subset h_1(\Delta) \subset h(\Delta). \tag{35}$$

Again the function $h_2 = h_1^{1/\mu}$ is starlike with respect to a boundary point, and $h_2(\Delta)$ contains the set

$$K = \left\{ w \in \mathbf{C} :\ w = e^{i\frac{\theta}{\mu}} \left(\frac{1-z}{1+z} \right)^{\frac{\lambda}{\mu}} \right\}$$

because of (35).

Setting $\dfrac{\lambda}{\mu} = \kappa$ and $\theta_1 = \theta \dfrac{\operatorname{Re}\mu}{|\mu|^2}$, we see by (27) that κ is real and K can be written as

$$K = \left\{ w \in \mathbf{C} :\ w = Re^{i\theta_1} \left(\frac{1-z}{1+z} \right)^{\kappa} \right\},$$

with $R = \exp\left[\frac{\theta_1 \operatorname{Im}\mu}{\operatorname{Re}\mu} \right]$ real and positive.

Hence, $h_2(\Delta)$ contains a straight canonical wedge

$$W_{\kappa,\theta_1} = \left\{ w \in \mathbf{C} : \ w = e^{i\theta_1} \left(\frac{1-z}{1+z} \right)^{\kappa} \right\}$$

with $0 < \kappa|\nu_2|$, where $\nu_2 = Q_{h_2}(-1)$ exists finitely and $W_{|\nu_2|,\theta_1}$ is the maximal wedge contained in $h_2(\Delta)$. But, as before, we have

$$\nu = Q_h(\eta) = \mu Q_{h_2}(-1) = \mu\nu_2.$$

The latter relations show that ν is finite and λ must satisfy the conditions $\arg\lambda = \arg\mu = \arg(-\nu)$ and $0 < |\lambda| \le |\nu|$. So the wedge $W_{-\nu,\theta}$ is a maximal wedge contained in $h(\Delta)$ satisfying condition (27). The lemma is proved. □

Remark 5. By using Lemma 4 and the proof of Lemma 5, one can show that the number θ in the formulation of Lemma 5 is defined by the formula

$$\theta = \frac{|\nu|^2}{\operatorname{Re}\nu} \lim_{r\to 1^-} \arg h^{1/\nu}(-r).$$

For real r, this formula coincides with (21). Hence, in fact, Lemma 5 contains Lemma 4.

Lemma 6. *Let $f \in \mathscr{G}^+[\tau,\eta]$ for some $\tau \in \overline{\Delta}$ (which is the Denjoy–Wolff point for the semiflow $\mathscr{S}$ generated by f) with $\beta = f'(\tau) > 0$ and some $\eta \in \partial\Delta$, such that $f(\eta) := \angle\lim_{z\to\eta} f(z) = 0$ and $\gamma = f'(\eta) = \angle\lim_{z\to\eta} f'(z)$ exists finitely.*

The following assertions hold.

(i) *If $\tau \in \Delta$, then $\gamma < -\frac{1}{2}\operatorname{Re}\beta$.*

(ii) *If $\tau \in \partial\Delta$, then $\gamma \le -\beta < 0$ and the equality $\gamma = -\beta$ holds if and only if $f \subset \operatorname{aut}(\Delta)$ or, what is the same, $\mathscr{S} \subset \operatorname{Aut}(\Delta)$ consists of hyperbolic automorphisms of Δ.*

Proof. (i) Let $\tau \in \Delta$. Then $f \in \mathscr{G}^+[\tau]$ admits the representation

$$f(z) = (z-\tau)(1-z\bar{\tau})p(z)$$

with $\operatorname{Re} p(z) > 0,\ z \in \Delta$ and

$$\beta\,(= f'(\tau)) = (1-|\tau|^2)p(\tau).$$

Assume that for some $\eta \in \partial\Delta$

$$f(\eta) := \angle\lim_{z\to\eta} f(z) = 0$$

and

$$\gamma = \angle\lim_{z\to\eta} \frac{f(z)}{z-\eta}$$

exists finitely. Then $\angle\lim_{z\to\eta} p(z) = 0$, and

$$\gamma = \eta|\eta-\tau|^2 \cdot p'(\eta),$$

where

$$p'(\eta) = \angle\lim_{z\to\eta} \frac{p(z)}{z-\eta}.$$

To find an estimate for $p'(\eta)$, we introduce a function p_1 of positive real part by the formula

$$p_1(z) = (1 - |\tau|^2)p(m(z)),$$

where

$$m(z) = \frac{\tau - z}{1 - z\bar{\tau}}$$

is the Möbius transformation (involution) taking τ to 0 and 0 to τ. Thus

$$p_1(0) = (1 - |\tau|^2)p(\tau) = \beta;$$

and, setting $\eta_1 = m(\eta)$, we have

$$p_1'(\eta_1) = (1 - |\tau|^2)p'(\eta) \cdot m'(\eta_1) = \frac{1 - |\tau|^2}{m'(\eta)} \cdot p'(\eta) = -(1 - \eta\bar{\tau})^2 p'(\eta).$$

On the other hand, using the Riesz–Herglotz formula for the function $q = 1/p$, we obtain

$$\begin{aligned}\angle \lim_{z\to\eta_1} (z - \eta_1)q(z) &= \angle \lim_{z\to\eta_1} \int_{\partial\Delta} \frac{(z - \eta_1)(1 + z\bar{\zeta})}{1 - z\bar{\zeta}}\, d\mu_q(\zeta) \\ &= -\eta_1 \cdot \angle \lim_{z\to\eta_1} \int_{\partial\Delta} \frac{(1 - z\bar{\eta}_1)(1 + z\bar{\zeta})}{1 - z\bar{\zeta}}\, d\mu_q(\zeta) \\ &= -\eta_1 2\mu_q(\eta_1),\end{aligned}$$

where μ_q is a positive measure on $\partial\Delta$ such that $\int_{\partial\Delta} d\mu_q(\zeta) = \operatorname{Re} q(0)$. Consequently,

$$p_1'(\eta_1) = \angle \lim_{z\to\eta_1} \frac{p_1(z)}{z - \eta_1} = \angle \lim_{z\to\eta_1} \frac{1}{(z - \eta_1)q(z)} = \frac{-\overline{\eta_1}}{2\mu_q(\eta_1)} = -(1 - \eta\bar{\tau})^2 p'(\eta).$$

Hence

$$p'(\eta) = \frac{\overline{\eta_1}}{(1 - \eta\bar{\tau})^2 2\mu_q(\eta_1)}$$

and

$$\gamma = \frac{1}{2}\frac{\eta|\eta - \tau|^2\overline{\eta_1}}{(1 - \eta\bar{\tau})^2} \cdot \frac{1}{\mu_q(\eta_1)}.$$

Since $\mu_q(\eta_1) \le \operatorname{Re} q(0) \le \dfrac{1}{\operatorname{Re} p_1(0)} = \dfrac{1}{\operatorname{Re}\beta}$, we have

$$|\gamma| \ge \frac{1}{2}\operatorname{Re}\beta.$$

Note that equality is impossible since otherwise q (and hence p_1 and p) are constant. But $\angle \lim\limits_{z\to\eta_1} p(z) = 0$, which means that $p(z) \equiv 0$.

This proves assertion (i).

(ii) Let now $\tau \in \partial\Delta$. In this case, we know already that

$$\beta = f'(\tau) = \angle \lim_{z\to\tau} f'(z) > 0.$$

Without loss of generality, let us assume that $\tau = 1$ and $\eta = -1$. In other words, we assume that $f \in \mathscr{G}^+[1,-1]$. We have to show that in that case $\gamma = \angle \lim\limits_{z \to -1} f'(z) \le -\beta$, and equality holds if and only if f is a complete vector field.

Indeed, suppose to the contrary that $\gamma \in (-\beta, 0)$. Then the function $g \in \mathrm{Hol}(\Delta, \mathbf{C})$ defined by

$$g(z) = f(z) + \frac{\gamma}{2}(z^2 - 1)$$

belongs to the class $\mathscr{G}^+[1,-1]$, because this class is a real cone. In addition,

$$g'(1) = \beta + \gamma \ge 0,$$

while

$$g'(-1) = \gamma - \gamma = 0.$$

Then either $g(z) \equiv 0$, or $g \ne 0$ and both points 1 and -1 are sink points of the semigroup generated by f, which is impossible. This contradiction shows that g must be identically zero, hence $\gamma = -\beta$ and

$$f(z) = -\frac{\gamma}{2}(z^2 - 1).$$

Thus f belongs to $\mathrm{aut}(\Delta)$, and the flow $\mathscr{S} = \{F_t\}_{t \in \mathbf{R}}$ consists of hyperbolic automorphisms of Δ. The lemma is proved. □

Now we are ready to prove our theorems. Since Theorem 2 is a compliment of Theorem 1, we give their proofs simultaneously.

Proof of Theorems 1 and 2. We prove implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) of Theorem 1 successively, while assertions (a), (b) and (c) of Theorem 2 will be obtained in the process. Let $\mathscr{S} = \{F_t\}_{t \ge 0}$ be a semiflow on Δ generated by $f \in \mathscr{G}^+[\tau]$ with $\beta = f'(\tau)$, $\operatorname{Re}\beta > 0$. Let $h \in \mathrm{Hol}(\Delta, \mathbf{C})$ be the associated spirallike (starlike) function on Δ defined by equation (14) with $\mu = \beta$. Then by Lemma 1, h satisfies Schröder's equation (15)

$$h(F_1(z)) = e^{-t\beta} h(z) \tag{36}$$

for all $t \ge 0$ and $z \in \Delta$.

Step 1 ((i) $\Longrightarrow$ (ii)). If now $f \in \mathscr{G}^+[\tau, \eta]$ for some $\eta \in \partial\Delta$, that is $f(\eta)(= \angle \lim_{z \to \eta} f(z)) = 0$ and $\gamma = f'(\eta)$ $(= \angle \lim_{z \to \eta} f'(z))$ exists finitely, then by Lemma 2 the function h belongs to the class $\mathrm{Spiral}[\tau, \eta]$ with

$$Q_h(\eta) = \angle \lim_{z \to \eta} \frac{(z - \eta) h'(z)}{h(z)} = \frac{\beta}{\gamma}\,.$$

Since $\gamma \ne 0$ (actually, $\gamma < 0$), $Q_h(\eta)$ is finite.

In turn, Lemma 5 implies that there is a non-empty (spiral) wedge $W \subset h(\Delta)$ with vertex at the origin such that for each $w \in W$ the spiral curve $\{e^{-t\beta} w\}$ belongs to W, for all $t \in \mathbf{R}$.

Define the simply connected domain $\Omega \subset \Delta$ by

$$\Omega = h^{-1}(W).$$

Then the family $\widetilde{F}_t\colon \Omega \mapsto \Omega$

$$\widetilde{F}_t(z) = h^{-1}\left(e^{-t\beta}h(z)\right), \quad z \in \Omega,\ t \in \mathbf{R},$$

forms a flow (one-parameter group) of holomorphic self-mappings of Ω. Comparing the latter formula with (36), we see that for $t \geq 0$, $\widetilde{F}_t(z) = F_t(z)$ whenever $z \in \Omega$ and $\left(F_t|_\Omega\right)^{-1} = \widetilde{F_{-t}}$. Thus $\mathscr{S} \subset \mathrm{Aut}(\Omega)$.

Step 2 ((ii) $\Longrightarrow$ (iii)). Let again $\mathscr{S} = \{F_t\}_{t\geq 0}$ be a semiflow generated by $f \in \mathscr{G}^+[\tau]$ so that

$$\lim_{t\to\infty} F_t(z) = \tau \in \partial\Delta \quad \text{and} \quad \operatorname{Re}\beta > 0, \quad \text{where } \beta = f'(\tau), \tag{37}$$

and let $\Omega \subset \Delta$ be a simply connected domain such that $\mathscr{S} \subset \mathrm{Aut}(\Omega)$. Let $\psi\colon \Delta \mapsto \Omega$ be any Riemann conformal mapping of Δ onto Ω. Consider the flow $\{G_t\}_{t\in\mathbf{R}} \subset \mathrm{Aut}(\Delta)$ defined by

$$G_t(z) = \psi^{-1}(F_t(\psi(z))),\ t \in \mathbf{R}. \tag{38}$$

In this case, ψ is a conjugation for G_t and F_t for each $t \in \mathbf{R}$, i.e.,

$$\psi(G_t(z)) = F_t(\psi(z)), \quad z \in \Delta,\ t \in \mathbf{R}. \tag{39}$$

Denote by $g \in \mathrm{aut}(\Delta)$ the generator of $\{G_t\}_{t\in\mathbf{R}}$:

$$g(z) = \lim_{t\to 0} \frac{z - G_t(z)}{t}.$$

Then by (39), ψ satisfies the differential equation

$$\psi'(z) \cdot g(z) = f(\psi(z)). \tag{40}$$

First we show that the family $\{G_t\}_{t\in\mathbf{R}} \subset \mathrm{Aut}(\Delta)$ consists of hyperbolic automorphisms or, what is the same, that it does not contains neither elliptic nor parabolic automorphisms.

Indeed, suppose $\{G_t\}_{t\in\mathbf{R}}$ contains an elliptic automorphism. Then there is a point $a \in \Delta$ such that $G_t(a) = a$ for all $t \in \mathbf{R}$; hence $g(a) = 0$ and $\operatorname{Re} g'(a) = 0$. By (40), $f(\psi(a)) = 0$; and thus $\psi(a) = \tau$. On the other hand, differentiating (40) with respect to z and setting $z = a$, we get $g'(a) = f'(\tau)$. Hence $\operatorname{Re} f'(\tau) = 0$, which contradicts (37).

Thus $\{G_t\}$ has no interior fixed point in Δ; hence there are boundary points ζ_1 and ζ_2 such that

$$\lim_{t\to\infty} G_t(z) = \zeta_1 \in \partial\Delta, \quad z \in \Delta, \tag{41}$$

and

$$\lim_{t\to-\infty} G_t(z) = \zeta_2 \in \partial\Delta, \quad z \in \Delta. \tag{42}$$

To show that the family $\{G_t\}_{t\in\mathbf{R}}$ does not contain a parabolic automorphism it is sufficient to prove that $\zeta_1 \neq \zeta_2$.

To this end, we again consider the associated spirallike (starlike) function h defined by equation (14) with $\mu = \beta$ and normalized by the conditions $h(\tau) =$

$0,\ h'(\tau)=1$ if $\tau\in\Delta$ or by $h(\tau)=0$ and $h(0)=1$ if $\tau\in\partial\Delta$ (see Lemma 1). Define $h_0\in \mathrm{Hol}(\Delta,\mathbf{C})$ by

$$h_0(z)=h(\psi(z)). \tag{43}$$

Since h satisfies Schröder's equation (36), it follows from (39) that for all $t\geq 0$,

$$h_0(G_t(z))=h(\psi(G_t(z))=h(F_t(\psi(z))=e^{-t\beta}h(\psi(z))=e^{-t\beta}h_0(z).$$

Since the mapping $G_t\in\mathrm{Hol}(\Delta)$ is an automorphism of Δ for each $t\in\mathbf{R}^+$, we have, in fact,

$$h_0(G_t(z))=e^{-t\beta}h_0(z) \tag{44}$$

for all $t\in\mathbf{R}$.

From (44) we conclude that h is a univalent spirallike (starlike) function on Δ. Moreover, (44) and Corollary 2.17 of [26] imply that

$$\angle\lim_{z\in\zeta_1}h_0(z)=0,$$

while

$$\angle\lim_{z\in\zeta_2}h_0(z)=\infty.$$

Thus $\zeta_1\neq\zeta_2$, and it follows that $\{G_t\}_{t\in\mathbf{R}}$ consists of hyperbolic automorphisms.

Now observe that $W=h_0(\Delta)$ is a spirallike (starlike) wedge with vertex at the origin belonging to $h(\Delta)$. Since all the points of $\partial h(\Delta)$ are admissible, $\psi=h^{-1}\circ h_0$ is a homeomorphism of $\overline{\Delta}$ onto $\overline{\Omega}$; hence $\partial\Omega$ is a Jordan curve. Now (39) implies that

$$\lim_{t\to\infty}\psi(G_t(z))=\lim_{t\to\infty}F_t(\psi(z))=\tau \tag{45}$$

and

$$\lim_{t\to-\infty}\psi(G_t(z))=\lim_{t\to-\infty}F_t(\psi(z))=\eta \tag{46}$$

for some $\eta\in\overline{\Delta}$. Applying again Corollary 2.17 in [26], we obtain

$$\psi(\zeta_1):=\lim_{z\to\zeta_1}\psi(z)=\tau \tag{47}$$

and

$$\psi(\zeta_2):=\angle\lim_{z\to\zeta_2}\psi(z)=\eta. \tag{48}$$

Thus $\eta\neq\tau$ and, moreover, $\eta\in\partial\Delta$. Indeed, if η is an interior point of Δ, then

$$\eta=\psi(\zeta_2)=\psi(G_t(\zeta_2))=F_t(\psi(\zeta_2))=F_t(\eta),\quad t\geq 0,$$

i.e., it must be an interior fixed point for all $F_t\in\mathscr{S},\ t\geq 0$, which is impossible.

So $\tau\in\partial\Omega$ by (47), and $\eta\in\partial\Delta\cap\partial\Omega$ by (48).

To show that equation (11) has a locally univalent (even univalent) solution $\varphi\in\mathrm{Hol}(\Delta)$ for some $\alpha>0$, we use a Möbius transformation $m\in\mathrm{Aut}(\Delta)$ such that

$m(1) = \zeta_1$ and $m(-1) = \zeta_2$. Then $\psi_1 = \psi \circ m$ is a conformal mapping of Δ onto Ω with normalization

$$\psi_1(1) = \tau, \quad \psi_1(-1) = \eta.$$

For $s \in (-1, 1)$, define another conformal mapping φ_s of Δ onto Ω by

$$\varphi_s(z) := \psi_1\left(\frac{z-s}{1-zs}\right), \quad -1 < s < 1.$$

Clearly $\varphi_s(1) = \psi_1(1) = \tau$ and $\varphi_s(-1) = \psi_1(-1) = \eta$. Note also that $l = \{z = \varphi_s(0)\,(= \psi_1(-s)),\ s \in [-1, 1])\}$ is a continuous curve joining the points $z = 1$ and $z = -1$, and so $l_1 = \{z = h(\varphi_s(0))(= h(\psi_1(-s)))\}$ is a continuous curve joining $h(\tau) = 0$ and $h(\eta) = \infty$. Hence, there exists $s \in (-1, 1)$ such that $|h(\varphi_s(0))| = 1$.

Thus there exists a homeomorphism $\varphi(= \varphi_s)$ of $\overline{\Delta}$ onto $\overline{\Omega}$ holomorphic in Δ such that $\varphi(1) = \tau$, $\varphi(-1) = \eta$ and $h(\varphi(0)) = e^{i\theta}$ for some $\theta \in \mathbf{R}$.

Since the mapping ψ in our previous consideration was arbitrary, we can replace it by φ. In this case, the "new" flow $\{G_t\}_{t\in 0}$ defined by

$$G_t(z) = \varphi^{-1}(F_t(\varphi(z)))$$

is a one-parameter group of hyperbolic automorphisms of Δ having the fixed points $z = 1$ and $z = -1$ on $\partial\Delta$. In turn, its generator $g \in \mathrm{Hol}(\Delta, \mathbf{C})$ must have the form

$$g(z) = \frac{\alpha}{2}(z^2 - 1)\,, \tag{49}$$

where $\alpha = g'(1) > 0$.

Hence, equation (40) (with φ in place of ψ) becomes (11)

$$\alpha\varphi'(z)(z^2 - 1) = 2f(\varphi(z)). \tag{50}$$

Combining this with (14), we show that α must be greater than or equal to $-\gamma > 0$. Namely, defining $h_0 \in \mathrm{Spiral}[1]$ as in (43) by

$$h_0(z) = h(\varphi(z)), \tag{51}$$

we have from (50) and (14) that

$$\beta h_0(z) = \frac{\alpha}{2}(z^2 - 1)h_0'(z) \tag{52}$$

with $h_0(0) = h(\varphi(0)) = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Solving this equation, we obtain

$$h_0(z) = e^{i\theta}\left(\frac{1-z}{1+z}\right)^{\beta/\alpha} \tag{53}$$

with $\alpha = g'(1)$.

On the other hand, by Lemma 5, the maximal (spiral) wedge contained in $h(\Delta)$ is of the form $W_{-\nu,\theta} = \left\{w \in \mathbf{C} :\ w = e^{i\theta}\left(\frac{1-z}{1+z}\right)^{-\nu}\right\}$, where

$$\nu = \angle \lim_{z\to\eta} \frac{(z-\eta)h'(z)}{h(z)} = \angle \lim_{z\to\eta} \frac{z-\eta}{f(z)} = \frac{\beta}{\gamma}\,. \tag{54}$$

Thus γ is finite and $\varphi = h^{-1} \circ h_0$ is a well-defined self-mapping of Δ if and only if $\alpha \geq -\gamma$. This completes the proof of the implication (ii) $\Longrightarrow$ (iii) of Theorem 1, as well as assertions (a) and (b) of Theorem 2. Note in passing that we have also proved the implication (ii) $\Longrightarrow$ (i) of Theorem 1.

Step 3 ((iii) $\Longrightarrow$ (i)). Suppose now that $\varphi \in \mathrm{Hol}(\Delta)$ is locally univalent and satisfies (11) for some $\alpha \in \mathbf{R}^+$. Solving this differential equation explicitly, we get

$$\alpha \int_{\varphi(0)}^{\varphi(z)} \frac{dw}{f(w)} = \int_0^z \frac{2dz}{z^2 - 1} = \log\left(\frac{1-z}{1+z}\right). \tag{55}$$

Since $\varphi'(z) \neq 0,\ z \in \Delta$, we have by (11) that there is no $z \in \Delta$ such that $\varphi(z) = \tau$. So if l is a curve joining 0 and z, the curve $\varphi(l)$ joining $\varphi(0)$ and $\varphi(z)$ does not contain τ.

Consider now the differential equation (14) with initial data $h(\varphi(0)) = 1$. Separating variables in this equation, we see that

$$\beta \int_{\varphi(0)}^{\varphi(z)} \frac{dw}{f(w)} = \int_1^{h(\varphi(z))} \frac{dh}{h} = \log(h(\varphi(z)). \tag{56}$$

Comparing (55) with (56), we have

$$\log\left(h(\varphi(z)\right) = \frac{\beta}{\alpha} \log\left(\frac{1-z}{1+z}\right),$$

or

$$h(\varphi(z)) = \left(\frac{1-z}{1+z}\right)^{\frac{\beta}{\alpha}}.$$

This equality implies that the set $\left\{\left(\frac{1-z}{1+z}\right)^{\frac{\beta}{\alpha}} \, : \, z \in \Delta\right\}$ is a subset of $h(\Delta)$, so this set is different from $\mathbf{C} \setminus \{0\}$. It follows by [4] that in this case $\left|\frac{\beta}{\alpha} - 1\right| \leq 1$, the function $h_0 := \left(\frac{1-z}{1+z}\right)^{\beta/\alpha}$ is univalent on Δ, and its image $W = h_0(\Delta)$ is a spiral wedge with vertex at the origin. So, by Lemma 5, there is a point $\eta \in \partial\Delta$ such that $h(\eta) = \infty$ and $Q_h(\eta)$ exists finitely with $\arg Q_h(\eta) = \arg \beta$ and $\left|\frac{\beta}{\alpha}\right| \leq |Q_h(\eta)|$.

Finally, we note that $\varphi(z) = h^{-1}(h_0(z))$ is, in fact, a univalent function on Δ. Now, applying Lemma 2 with $\mu = \beta$, we complete the proof of the implication (iii) $\Longrightarrow$ (i) of Theorem 1, as well as assertion (c) of Theorem 2.

Theorems 1 and 2 are proved. □

Proof of Theorem 3. We already know by (51) and (53) that $\varphi = h^{-1} \circ h_0$, where h is the spirallike (starlike) function associated to f and $h_0(z)) = e^{i\theta}\left(\frac{1-z}{1+z}\right)^{\beta/\alpha}$ with $\beta = f'(\tau),\ \mathrm{Re}\,\beta > 0$ and $\alpha \geq -\gamma$. So, by Definition 5,

$$h_0(\Delta) = W_{\frac{\beta}{\alpha},\theta} \subset h((\Delta).$$

Thus, $\Omega = \varphi(\Delta) = h^{-1}\left(W_{\frac{\beta}{\alpha},\theta}\right)$ is maximal if and only if the spiral wedge $W_{\frac{\beta}{\alpha},\theta}$ is maximal. In turn, by Lemma 5, this wedge $W_{\frac{\beta}{\alpha},\theta}$ is maximal if and only if $\frac{\beta}{\alpha} = -\nu$. Comparing this fact with (54), we obtain the equivalence of assertions (i) and (ii) of the theorem.

We prove the equivalence of assertions (ii) and (iii) for the case where $\tau = 1$. Namely, let $f_1 \in \mathscr{G}^+[1,\eta]$ with $f_1'(1) = \beta_1 > 0$ and $f_1'(\eta) = \gamma_1 < 0$. Let ψ be a univalent solution of equation (11), i.e.,

$$\alpha\psi'(z)(z^2-1) = 2f(\psi(z)) \tag{57}$$

for some $\alpha \geq -\gamma_1$, normalized by $\psi(1) = 1,\ \psi(-1) = \eta$.

Substituting in formula (51) the explicit form of h_0 (see (53)) and the integral representation (17) with $\tau = 1$ for the spirallike function h and taking into account that $Q_h(\eta) = \nu = \frac{\beta_1}{\gamma_1}$ (cf. (54)), we get

$$\begin{aligned}&(\psi(z)-1)(1-\psi(z)\bar{\eta})^{\beta_1/\gamma_1}\cdot\exp\left[-(2+\beta_1/\gamma_1)\int_{\partial\Delta}\log(1-\psi(z)\bar{\zeta})\,d\sigma(\zeta)\right]\\ &= C_1\left(\frac{1-z}{1+z}\right)^{\beta_1/\alpha}\end{aligned}$$

or

$$\begin{aligned}&\frac{\psi(z)-\eta}{z+1}\\ &= (z+1)^{-1-\gamma_1/\alpha}\frac{C_1(1-z)^{\gamma_1/\alpha}}{(1-\psi(z))^{\gamma_1/\beta_1}}\cdot\exp\left[\frac{2\gamma_1+\beta_1}{\beta_1}\int_{\partial\Delta}\log(1-\psi(z)\bar{\zeta})\,d\sigma(\zeta)\right].\end{aligned}$$

Note that one can choose an analytic branch of the multivalued function

$$C_1\,\frac{(1-z)^{\gamma_1/\alpha}}{(1-\psi(z))^{\gamma_1/\beta_1}}.$$

We denote this branch by $\chi(z)$. It is a continuous function which does not vanish at the point $z = -1$. Hence its argument is a well-defined continuous function at this point. Thus

$$\begin{aligned}&\arg\frac{\psi(z)-\eta}{z+1}\\ &= \arg\left((z+1)^{-1-\gamma_1/\alpha}\right) + \arg\chi(z) + \frac{2\gamma_1+\beta_1}{\beta_1}\int_{\partial\Delta}\arg(1-\psi(z)\bar{\zeta})\,d\sigma(\zeta).\end{aligned}$$

Exactly as in the proof of Lemma 4, we conclude that the limit of the last summand exists finitely. Therefore, the function ψ is isogonal if and only if the limit

$$\lim_{z\to-1}\arg\left((z+1)^{-1-\gamma_1/\alpha}\right)$$

exists. Obviously, this happens if and only if the exponent vanishes, i.e., $\alpha = -\gamma_1$.

Now let $\tau \in \overline{\Delta}$ be arbitrary, and let $f \in \mathcal{G}^+[\tau,\eta]$ with $f'(\tau)=\beta$, $\operatorname{Re}\beta>0$, and $f'(\eta)=\gamma<0$. Let φ be a univalent solution of equation (11) for some $\alpha \geq -\gamma$, normalized by $\varphi(1)=\tau$, $\varphi(-1)=\eta$. Denote by h the spirallike function associated to f, that is, h satisfies equation (14) with $\mu=\beta$. As above, let h_0 be the function which maps the disk Δ onto a spiral wedge, namely, $h_0(z)=e^{i\theta}\left(\frac{1-z}{1+z}\right)^{\beta/\alpha}$, such that $\varphi = h^{-1}\circ h_0$.

Repeating the constructions in the proof of Lemma 5, we find a conformal mapping Φ of Δ such that $\Phi(1)=\tau$, $\Phi(-1)=\eta$, and $h_1 = h\circ\Phi$ is a spirallike function with respect to a boundary point. Note here that the domain $D=\Phi(\Delta)$ has a corner of opening π at the point η because Φ maps a circular arc containing $z=-1$ onto a circular arc which contains $z=\eta$. By Theorem 3.7 of [26], the limit $\lim\limits_{z\to-1}\arg\frac{\Phi(z)-\eta}{z+1}$ exists. Hence Φ is isogonal at the point -1. Moreover, by Proposition 4.11 of [26], the function Φ satisfies the Visser–Ostrowski condition

$$\angle\lim_{z\to-1}\frac{\Phi(z)-\eta}{z+1}=1. \tag{58}$$

Now write

$$\varphi = h^{-1}\circ h_0 = \Phi\circ\left(h_1^{-1}\circ h_0\right) = \Phi\circ\psi, \tag{59}$$

where $\psi = h_1^{-1}\circ h_0$. One sees that

$$\psi(-1) := \lim_{s\to-1^+}\psi(s) = \lim_{s\to-1^+} h_1^{-1}\left(h_0(s)\right) = -1$$

and

$$\psi(1) := \lim_{s\to1^-}\psi(s) = \lim_{s\to1^-} h_1^{-1}\left(h_0(s)\right) = 1.$$

Using this notation, we have

$$\arg\frac{\varphi(z)-\eta}{z+1} = \arg\frac{\Phi(\psi(z))-\eta}{\psi(z)+1} + \arg\frac{\psi(z)+1}{z+1}. \tag{60}$$

Thus (58) and (60) imply that φ is isogonal at the point η if and only if ψ is isogonal at the point $z=-1$.

Now we check that function ψ satisfies equation (57). We have seen already in the proof of Lemma 5 that $\beta h_1(z) = h_1'(z)f_1(z)$, where $f_1\in\mathcal{G}[1,-1]$ is defined by $f_1(z)=\frac{f(\Phi(z))}{\Phi'(z)}$. Using (58), we get

$$f_1'(-1) = \angle\lim_{z\to-1}\frac{f_1(z)}{z+1} = \angle\lim_{z\to-1}\frac{f(\Phi(z))}{\Phi(z)-\eta}\cdot\frac{\Phi(z)-\eta}{(z+1)\Phi'(z)} = \gamma.$$

Furthermore,

$$\begin{aligned} h_0(z) = h_1(\psi(z)) &= \frac{1}{\beta}\,h_1'(\psi(z))f_1(\psi(z)) \\ &= \frac{1}{\beta}\,\frac{(h_1(\psi(z)))'}{\psi'(z)}\,f_1(\psi(z)) = \frac{h_0'(z)}{\beta\psi'(z)}\,f_1(\psi(z)). \end{aligned}$$

Substituting $h_0(z) = e^{i\theta}\left(\frac{1-z}{1+z}\right)^{\beta/\alpha}$ in the last equality and differentiating, we see that equation (57) holds. But we have already shown that in that case ψ (hence, φ) is isogonal if and only if $\alpha = -f_1'(-1) = -\gamma$. This completes the proof. □

Proof of Theorem 4. Assertions (i) and (ii) of the theorem are direct consequences of Lemma 6. To prove assertion (iii), we first note that the inclusion $\tau \in \cap_k \partial\Omega_k$ follows by assertion (a) of Theorem 2.

Also observe that for each pair k_1 and k_2 such that $\eta_{k_1} \neq \eta_{k_2}$, the set $\Omega_{k_1,k_2} = \Omega_{k_1} \cap \Omega_{k_2}$ is empty. Indeed, otherwise Ω_{k_1,k_2} is a FID for $\mathscr{S}$. Hence, it must contain a point $\eta \in \partial\Omega_{k_1,k_2} \cap \partial\Delta$ such that

$$\eta = \angle \lim_{t\to-\infty} F_t(z)$$

whenever $z \in \Omega_{k_1,k_2}$. Hence we should have a contradiction $\eta = \eta_{k_1} = \eta_{k_2}$.

Let us suppose now that for a pair k_1 and k_2 there is a point $z_0 \neq \tau$, $z_0 \in \Delta$, such that $z_0 \in \partial\Omega_{k_1} \cap \partial\Omega_{k_2}$. Then the whole curve

$$l = \{z \in \Delta :\ z = F_t(z_0),\ t \geq 0\}$$

ending at τ must belong to both $\overline{\Omega}_{k_1}$ and $\overline{\Omega}_{k_2}$, hence to $l \subset \partial\Omega_{k_1} \cap \partial\Omega_{k_2}$, since $\Omega_{k_1} \cap \Omega_{k_2} = \emptyset$.

Finally, we have that $f \in \mathrm{Hol}(\Delta, \mathbf{C})$ is locally Lipshitzian. Therefore, if $\zeta \in \Delta$ is an interior end point of l, $\zeta \neq \tau$, then there is $\delta > 0$ such that the Cauchy problem (2) has a solution $u(t,\zeta)$ $(= F_t(\zeta))$ for all $t \in [-\delta, \infty)$; and the curve $l_1 = \{z \in \Delta :$ $z = u(t,\zeta),\ t \in [-\delta,\infty)\}$ also belongs to $\partial\Omega_{k_1} \cap \partial\Omega_{k_2}$. But l_1 properly contains l, which is impossible. So ζ must belong to $\partial\Delta$. The corollary is proved. □

Remark 6. The complete solution to the problem of finding FID's requires the treating the case in which $\tau \in \partial\Delta$ and $f'(\tau) = 0$. In this case, the semiflow $\mathscr{S} = \{F_t\}_{t\geq 0}$ generated by f consists of self-mappings of Δ of parabolic type. This delicate question is equivalent to the following problem. Associate with f a univalent function $h \in \mathrm{Hol}(\Delta, \mathbf{C})$ which is a solution of Abel's functional equation

$$h(F_t(z)) = h(z) + Kt, \quad t \geq 0, \tag{61}$$

for some $K \in \mathbf{C}$ which does not depend on $t \geq 0$. Under what conditions does the image $h(\Delta)$ contain a strip W such that equation (61) holds for all $t \in \mathbf{R}$, whenever $z \in \Omega = h^{-1}(W)$? We hope to consider this problem elsewhere.

Appendix. Quoting Harris [20], we note that "a classical problem of analysis is a problem that has interested mathematicians since the time of Abel: how to define the n-th iterate of a function when n is not an integer."

In other words, the question is given a function F, to find a family of functions $\{F_t\}_{t\geq 0}$, with $F_1 = F$ satisfying the semigroup (group) property for all $t \geq 0$ (respectively, $t \in \mathbf{R}$). This problem is called the embedding problem into a continuous semiflow (respectively, flow).

The possibility of such an embedding is important, in particular, in problems of conformal mapping and in the study of Markov branching processes with continuous time (whose first general formulation appears to have been given by Kolmogorov (1947)).

When F is a holomorphic function, Kœnigs (1884) showed how the problem may be solved locally near a fixed point z_0 such that $0 < |F'(z_0)| < 1$.

The limit

$$\lim_{n\to\infty} \frac{F^n(z_0) - z_0}{(F'(z_0))^n} = h(z)$$

can be shown to exist for z near z_0 and to satisfy Schröder's functional equation

$$h(F(z)) = F'(z_0)h(z), \tag{62}$$

whence

$$F(z) = h^{-1}[F'(z_0)h(z)].$$

The latter expression then serves as a definition of F_t when t is not necessarily an integer:

$$F_t = h^{-1}[(F'(z_0))^t h(z)].$$

Consequently, if $F \in \mathrm{Hol}(\Delta)$ is a self-mapping of the unit disk Δ and $z_0 \in \Delta$, then $\mathscr{S} = \{F_t\}_{t\geq 0}$ is *globally* well-defined on Δ if and only if h is a μ-spirallike function on Δ with $\arg\mu = \arg(-\log F'(z_0))$.

Following the work of Baker [6], Karlin and McGregor [24] considered the local embedding problem of holomorphic functions with two fixed points into a continuous group. In particular, they studied a class $\mathscr{L}$ of functions holomorphic in the extended complex plane $\overline{\mathbf{C}}$ except for an at most countable closed set in $\overline{\mathbf{C}}$ and proved the following result.

Let F be a function of class $\mathscr{L}$ with two fixed points z_0 and z_1, such that the segment $[z_0, z_1]$ is in the domain of regularity of F and is mapped onto itself. Assume that $0 < |F'(z_0)| < 1 < |F'(z_1)|$ and that for z in the open segment (z_0, z_1), $F(z) \neq z$, $F'(z) \neq 0$. Then there is a continuous one-parameter group $\{F_t\}_{t\in\mathbf{R}}$ of functions with common fixed points z_0 and z_1 and invariant segment $[z_0, z_1]$ such that $F_1(z) = F(z)$ if and only if $F(z)$ is a linear fractional transformation on $\mathbf{C}$.

First we note that the condition that F map $[z_0, z_1]$ into itself implies that $F'(z)$ is real on this segment.

Suppose now that F is linear fractional, $F(z) \not\equiv z$, and let z_0 and z_1 be its finite fixed points, $z_0 \neq z_1$. The following simple assertion can be obtained by using the linear model of mappings having two fixed points 0 and ∞ and applying the Julia–Carathéodory theorem.

Lemma 7. *The following are equivalent.*

(i) *There is an open disk D such that either $z_0 \in \partial D$ and $z_1 \notin \overline{D}$, or $z_0 \in D$ and $z_1 \in \partial D$, which is F-invariant.*

(ii) *Each open disk D such that $z_0 \in \overline{D}$ and $z_1 \notin D$ is F-invariant.*

(iii) *The segment $[z_0, z_1]$ is F-invariant and $|F'(z_0)| \leq 1$.*

(iv) *If $a = F'(z_0)$ then $0 < a < 1$.*

Since Schröder's equation (62) with linear-fractional F has a linear-fractional solution h, we have that h is starlike; hence F can be embedded into a one-parameter semigroup $\{F_t\}_{t\in\mathbf{R}}$ on each disk D containing z_0 and such that $z_1 \notin D$. This disk is F_t-invariant for all $t \geq 0$.

In turn, for the embedding property into a continuous group, we obtain the following assertion by using our Theorems 1 and 2 and Theorem 1 in [24].

Corollary 2. *Let F be a function of class $\mathscr{L}$ with two different fixed points z_0 and z_1. Assume that $0 < |F'(z_0)| < 1 < |F'(z_1)|$, and that for z in the open segment (z_0, z_1), $F(z) \neq z$, $F'(z) \neq 0$. The following assertions are equivalent.*

(i) *For each open disk D such that $z_0 \in \overline{D}$ and $z_1 \notin D$, there is a semiflow $\mathscr{S} = \{F_t\}_{t\geq 0}$ with $F_1 = F$ such that $\mathscr{S} \subset \mathrm{Hol}(D)$.*

(ii) *For each domain Ω bounded by two circles passing through z_0 and z_1, there is a one-parameter flow $\mathscr{S} = \{F_t\}_{t\in\mathbf{R}}$ such that $\mathscr{S} \subset \mathrm{Aut}(\Omega)$ and $F = F_1$.*

(iii) *The function F is linear fractional with $0 < F'(z_0) < 1$.*

Consequently, in this case, for any disk D such that $z_0 \in \overline{D}$ and $z_1 \in \partial D$, the maximal (backward) flow invariant domain is the disk $\Omega \subset D$ whose boundary passes through z_0 and is internally tangent to ∂D at z_1.

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