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A NOTE ON A THEOREM OF CHUAQUI AND GEVIRTZ

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Abstract. For a subdomain Ω of the right half-plane $\mathbf{H}$, Chuaqui and Gevirtz showed the following theorem: the image $f(\mathbf{D})$ of the unit disk $\mathbf{D}$ under an analytic function f on $\mathbf{D}$ is a quasidisk whenever $f'(\mathbf{D}) \subset \Omega$ if and only if there exists a compact subset K of $\mathbf{H}$ such that $sK \cap (\mathbf{H} \setminus \Omega) \neq \emptyset$ for any positive number s. We show that this condition is equivalent to the inequality $W(\Omega) < 2$, where $W(\Omega)$ stands for the circular width of the domain Ω.

1. Introduction

Let f be an analytic function on a convex domain D in the complex plane $\mathbf{C}$. The Noshiro–Warschawski theorem asserts that if the derivative f' maps D into the right half-plane $\mathbf{H} = \{w \in \mathbf{C} : \operatorname{Re} w > 0\}$, then f must be univalent on D. The second author observed in [7] that furthermore if D is mapped by f' into the disk $|(w - f'(0))/(w + \overline{f'(0)})| < k(< 1)$ then f extends to a k-quasiconformal mapping of the Riemann sphere. Here, a homeomorphism g of the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is called k-quasiconformal if g has locally square integrable partial derivatives on $\mathbf{C} \setminus \{g^{-1}(\infty)\}$ with $|g_{\bar{z}}/g_z| \leq k$ a.e. A homeomorphism of the Riemann sphere is called quasiconformal if it is k-quasiconformal for some constant $0 \leq k < 1$.

In the case when D is the unit disk $\mathbf{D}$, Chuaqui and Gevirtz [1] obtained a more refined result. To state their result, we introduce terminology due to them.

Definition 1. A closed subset X of the right half-plane $\mathbf{H}$ is said to have *property M* if there exists a compact subset K of $\mathbf{H}$ for which $sK \cap X \neq \emptyset$ for every $s > 0$.

In the above, sK means the set $\{w : w/s \in K\}$. We are now ready to state the theorem of Chuaqui and Gevirtz.

Theorem A. (Chuaqui–Gevirtz [1]) *Let Ω be a subdomain of the right half-plane $\mathbf{H}$. Every analytic function f on the unit disk $\mathbf{D}$ with $f'(\mathbf{D}) \subset \Omega$ extends to a quasiconformal mapping of the Riemann sphere if and only if $\mathbf{H} \setminus \Omega$ has property M.*

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A subdomain of the Riemann sphere is called a quasidisk if it is the image of the unit disk under a quasiconformal mapping of the Riemann sphere. Note that for a univalent analytic function f on $\mathbf{D}$, it extends to a quasiconformal mapping of $\widehat{\mathbf{C}}$ if and only if $f(\mathbf{D})$ is a quasidisk (see [5]).

Though the above theorem seems to be very useful, the property M is not necessarily easy to handle. The main objective of this note is to provide more convenient quantities characterizing the property M. To this end, we employ the hyperbolic geometry of the domains involved.

We denote by $\lambda_\Omega(z)|dz|$ the hyperbolic metric with curvature -4 of a hyperbolic subdomain Ω of $\mathbf{C}$. Note that the hyperbolic metric has the monotonicity property: $\lambda_\Omega(w) \le \lambda_{\Omega_0}(w)$ for $\Omega_0 \subset \Omega$. Let $d_\Omega(w_1, w_2)$ denote the hyperbolic distance induced by λ_Ω. For instance, the right half-plane has the hyperbolic metric $\lambda_{\mathbf{H}}(w) = 1/(2\mathrm{Re}\, w)$ and

$$d_{\mathbf{H}}(w_1, w_2) = \operatorname{arctanh}\left|\frac{w_1 - w_2}{w_1 + \overline{w_2}}\right|.$$

We also denote by $d_\Omega(w, A)$ the hyperbolic distance from a point $w \in \Omega$ to a subset A of the closure $\overline{\Omega}$ of Ω, namely, $d_\Omega(w, A) = \inf_{a\in A} d_\Omega(w, a)$. Here, we define $d_\Omega(w, a)$ to be $+\infty$ when $a \in \partial\Omega$.

The authors introduced in [4] the notion of circular width of a proper subdomain of the punctured plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. We now recall the definition of the circular width. If $0 \notin \Omega$, then the circular width $W(\Omega)$ of Ω (about the origin) is defined by

$$W(\Omega) = \left(\inf_{w\in\Omega} |w|\lambda_\Omega(w)\right)^{-1}.$$

Various properties of circular width were given in [4]. Among them, the monotonicity property is most important here: $W(\Omega_0) \le W(\Omega)$ if $\Omega_0 \subset \Omega \subset \mathbf{C}^*$. This is an immediate consequence of the monotonicity of the hyperbolic metric. Since $W(\mathbf{H}) = 2$, we see that the inequality $W(\Omega) \le 2$ holds for any subdomain Ω of $\mathbf{H}$. Now those subdomains of $\mathbf{H}$ whose complements have property M can be characterized by the following.

Theorem 1. *Let Ω be a subdomain of the right half-plane $\mathbf{H}$. Then the following three conditions are equivalent.*

(i) *$\mathbf{H} \setminus \Omega$ has property M.*
(ii) *The quantity $\delta(\Omega) = \sup_{a\in\Omega\cap\mathbf{R}} d_{\mathbf{H}}(a, \partial\Omega)$ is finite.*
(iii) *The circular width $W(\Omega)$ of Ω is less than 2.*

Here, we define $\delta(\Omega)$ to be 0 if $\Omega \cap \mathbf{R} = \emptyset$. The proof of this theorem will be given in a more quantitative form in the following sections.

In [4], the authors made attempts to give a sufficient or a necessary condition for a subdomain Ω of $\mathbf{H}$ to satisfy $W(\Omega) < 2$. Theorem 1 also gives a complete solution to this problem.

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2. Proof of the theorem

Let us start with the easier part, namely, the proof of the equivalence (i) $\Leftrightarrow$ (ii). Here and in what follows, we denote by $\Delta(a,\rho)$ the (open) hyperbolic disk within $\mathbf{H}$ centered at a with radius ρ, or more concretely,

$$\Delta(a,\rho) = \{w \in \mathbf{H} : d_{\mathbf{H}}(w,a) < \rho\} = \{w : \left|\tfrac{w-a}{w+\bar{a}}\right| < \tanh\rho\}. \tag{2.1}$$

Note that the disk $\Delta(a,\rho),\ a>0$, is described as the Euclidean disk $\mathbf{D}(c,r) = \{w : |w-c| < r\}$, where

$$c = a\frac{1+m^2}{1-m^2} = a\cosh(2\rho), \quad r = a\frac{2m}{1-m^2} = a\sinh(2\rho), \quad m = \tanh\rho. \tag{2.2}$$

We denote by $\bar{\Delta}(a,\rho)$ the closure of $\Delta(a,\rho)$. Since any compact subset of $\mathbf{H}$ is contained in the closed hyperbolic disk $\bar{\Delta}(a,\rho)$ for a suitable choice of $a>0$ and $\rho>0$, we may replace K in Definition 1 by a closed disk of this form.

We now prove that (i) implies (ii). We assume that for some $a_0>0$ and ρ, $K=\bar{\Delta}(a_0,\rho)$ satisfies $sK\cap(\mathbf{H}\setminus\Omega)\neq\emptyset$ for all $s>0$. Let $a\in\Omega\cap\mathbf{R}$ and choose $s>0$ so that $a=sa_0$. Then, we can take a point b in $sK\cap\partial\Omega$. In view of (2.1), we get $sK=\bar{\Delta}(sa_0,\rho)=\bar{\Delta}(a,\rho)$. Thus $d_{\mathbf{H}}(a,b)\le\rho$, which implies $d_{\mathbf{H}}(a,\partial\Omega)\le\rho$. Since $a\in\Omega\cap\mathbf{R}$ is arbitrary, we obtain $\delta(\Omega)\le\rho$.

Conversely, we assume that $\rho:=\delta(\Omega)$ is finite. Let $K=\bar{\Delta}(1,\rho)$. If $sK\cap(\mathbf{H}\setminus\Omega)=\emptyset$, then $sK=\bar{\Delta}(s,\rho)\subset\Omega$. Since sK is compact, $\Delta(s,\rho')\subset\Omega$ for some $\rho'>\rho$. Thus $d_{\mathbf{H}}(s,\partial\Omega)\ge\rho'>\rho$, which contradicts the assumption that $\delta(\Omega)=\rho$.

Thus the proof of (i) $\Leftrightarrow$ (ii) is complete. □

We next show the equivalence of (ii) and (iii) with concrete estimates. To this end, we estimate the hyperbolic metric of the punctured half-plane $\mathbf{H}_b := \mathbf{H}\setminus\{b\}$.

Lemma 2. *For $b\in\mathbf{H}$ and $w\in\mathbf{H}_b$, the inequality*

$$2|w|\lambda_{\mathbf{H}_b}(w) \ge \frac{\sinh t}{t}, \quad t = \log\left|\frac{w+\bar{b}}{w-b}\right|,$$

holds.

Note that $G(w)=\log\left|\frac{w+\bar{b}}{w-b}\right|$ is nothing but Green's function on $\mathbf{H}$ with pole at b.

Proof of Lemma 2. Let $g(w)=(w-b)/(w+\bar{b})$. Then g maps $\mathbf{H}$ conformally onto the unit disk $\mathbf{D}$ and $\mathbf{H}_b$ onto the punctured disk $\mathbf{D}^*=\mathbf{D}\setminus\{0\}$. Recall that $\lambda_{\mathbf{D}}(z)=1/(1-|z|^2)$, $\lambda_{\mathbf{D}^*}(z)=1/(-2|z|\log|z|)$ and thus

$$\frac{\lambda_{\mathbf{D}^*}(z)}{\lambda_{\mathbf{D}}(z)} = \frac{1-|z|^2}{-2|z|\log|z|} = \frac{\sinh t}{t}, \quad t = -\log|z|.$$

Since $\lambda_{\mathbf{H}_b}(w)/\lambda_{\mathbf{H}}(w) = \lambda_{\mathbf{D}^*}(g(w))/\lambda_{\mathbf{D}}(g(w))$, we obtain

$$2|w|\lambda_{\mathbf{H}_b}(w) = \frac{|w|}{\operatorname{Re} w} \cdot \frac{\sinh t}{t} \geq \frac{\sinh t}{t}, \tag{2.3}$$

where $t = -\log|g(w)|$. □

We are now ready to prove that (ii) implies (iii). Let $\rho = \delta(\Omega) < \infty$. If $\rho = 0$, then Ω is contained either in the sector $\mathbf{H}_+ = \{w \in \mathbf{H} : \operatorname{Im} w > 0\}$ or in $\mathbf{H}_- = \{w \in \mathbf{H} : \operatorname{Im} w < 0\}$. Since $W(\mathbf{H}_+) = W(\mathbf{H}_-) = 1$ (see [4, Example 5.1]), we have $W(\Omega) \leq W(\mathbf{H}_\pm) = 1 < 2$ in this case.

From now on, we suppose that $\rho > 0$, in other words, $\Omega \cap \mathbf{R} \neq \emptyset$. Let $m = \tanh\rho$ and take $\theta \in (0, \pi/2)$ so that $\sin\theta = 2m/(1+m^2) = \tanh(2\rho)$. Note that $m = \tan(\theta/2)$. In view of (2.2), one can see that a ray emanating from the origin which is tangent to the circle $\partial\Delta(a,\rho)$, $a > 0$, forms an angle θ or $-\theta$ with the positive real axis.

Let $w \in \Omega$. Note that with respect to the hyperbolic distance $d_{\mathbf{H}}$, $|w|$ is the nearest point to w among the positive real axis and

$$d = d_{\mathbf{H}}(w, |w|) = \operatorname{arctanh}\left(\tan\frac{\psi}{2}\right), \quad \psi = |\arg w|. \tag{2.4}$$

Set $a = |w|$. If $a \in \Omega$, then there exists a point b in $\bar{\Delta}(a,\rho) \cap \partial\Omega$ because $d_{\mathbf{H}}(a, \partial\Omega) \leq \delta(\Omega) = \rho$. Therefore, $d_{\mathbf{H}}(w,b) \leq d_{\mathbf{H}}(w,a) + d_{\mathbf{H}}(a,b) \leq d + \rho$. If $a \notin \Omega$, then there exists a point $b \in \partial\Omega$ on the closed hyperbolic segment joining w and a in $\mathbf{H}$. Thus $d_{\mathbf{H}}(w,b) \leq d_{\mathbf{H}}(w,a) = d$. In either case, we therefore have a point $b \in \partial\Omega$ such that $d_{\mathbf{H}}(w,b) \leq d + \rho$.

We now assume that $\psi \leq \theta$. Then, we see that $d \leq \operatorname{arctanh}(\tan(\theta/2)) = \rho$ and hence, $d_{\mathbf{H}}(w,b) \leq 2\rho$. Lemma 2 now yields

$$2|w|\lambda_\Omega(w) \geq 2|w|\lambda_{\mathbf{H}_b}(w) \geq \frac{\sinh t}{t}.$$

Here,

$$e^t = \left|\frac{w + \bar{b}}{w - b}\right| = \frac{1}{\tanh d_{\mathbf{H}}(w,b)} \geq \frac{1}{\tanh(2\rho)} = \coth(2\rho).$$

Since the function $\sinh x/x$ is increasing in $x > 0$, we have

$$\frac{\sinh t}{t} \geq \frac{\sinh(\log(\coth(2\rho)))}{\log(\coth(2\rho))} = \frac{1}{\sinh(4\rho)\log(\coth(2\rho))}.$$

Let us memorize the fact that the right-hand side is greater than 1 since $\sinh x/x > 1$ for $x > 0$.

When $w \in \Omega$ satisfies $|\arg w| > \theta$, we encounter a difficulty with the above method (see computations in the final section). However, we have a much simpler but crude estimate:

$$2|w|\lambda_\Omega(w) \geq 2|w|\lambda_{\mathbf{H}}(w) = \frac{1}{\cos\arg w} \geq \frac{1}{\cos\theta} = \frac{1+m^2}{1-m^2} = \cosh(2\rho).$$

Therefore, for an arbitrary $w \in \Omega$, we obtain the inequality

$$2|w|\lambda_\Omega(w) \geq \min\left\{\cosh(2\rho), \frac{1}{\sinh(4\rho)\log(\coth(2\rho))}\right\}.$$

Since the right-hand side in the above depends only on ρ and is greater than 1, we conclude that $W(\Omega) < 2$.

We next show that (iii) implies (ii). We will need to compute the circular width of $\Delta(a,\rho)$.

Lemma 3. $W(\Delta(a,\rho)) = 2\tanh\rho$ *for* $a > 0$ *and* $\rho > 0$.

Proof. First we recall that the Euclidean disk $\mathbf{D}(c,r) = \{w : |w-c| < r\}$ with $0 < r \leq c$ has circular width

$$W(\mathbf{D}(c,r)) = \frac{2r/c}{1+\sqrt{1-(r/c)^2}}, \tag{2.5}$$

see [4, Example 5.4]. Set $m = \tanh\rho$. Then, in view of (2.2), we have

$$W(\Delta(a,\rho)) = \frac{4m/(1+m^2)}{1+\sqrt{1-(2m/(1+m^2))^2}} = 2m,$$

and thus the proof is complete. □

Let $a \in \Omega \cap \mathbf{R}$ and let $\rho \leq d_{\mathbf{H}}(a, \partial\Omega)$. Then $\Delta(a,\rho) \subset \Omega$. The monotonicity of the circular width together with the last lemma implies $W(\Omega) \geq W(\Delta(a,\rho)) = 2\tanh\rho$, which implies $\rho \leq \operatorname{arctanh}(W(\Omega)/2)$. Thus we have proved the inequality $\delta(\Omega) \leq \operatorname{arctanh}(W(\Omega)/2)$ and we conclude that (iii) implies (ii).

3. Concluding remarks

It is well known that a bounded simply connected domain in $\mathbf{C}$ is a quasidisk precisely if it is a linearly connected John disk. Chuaqui and Gevirtz define a John disk to be a bounded simply connected domain with certain property, and in fact, they prove that f is bounded if $f'(\mathbf{D}) \subset \Omega$ for a subdomain Ω of $\mathbf{H}$ for which $\mathbf{H}\setminus\Omega$ has property M. We can now give another proof for that with a concrete bound by combining Theorem 1 with the following result (see [4, Theorem 6.1] and its proof).

Theorem B. *Let* Ω *be a proper subdomain of the punctured plane* $\mathbf{C}^*$ *with* $W(\Omega) < 2$. *Suppose that an analytic function* f *on the unit disk* $\mathbf{D}$ *with* $f(0) = f'(0) - 1 = 0$ *is given. If* $f'(\mathbf{D}) \subset \Omega$, *then the pre-Schwarzian derivative* $T_f = f''/f'$ *of* f *satisfies the inequality*

$$\|T_f\| := \sup_{z\in\mathbf{D}}(1-|z|^2)|T_f(z)| \leq W(\Omega)$$

and the image $f(\mathbf{D})$ *is contained in the disk* $|w| < 2^{W(\Omega)/2}/(1-W(\Omega)/2)$.

We remark that the bound $2^{W(\Omega)/2}/(1-W(\Omega)/2)$ can be replaced by the sharp one $\int_0^1[(1+x)/(1-x)]^{W(\Omega)/2}\mathrm{d}x$. For more information about the bound, see also [3, §2].

One might raise a similar problem: What is a characterizing property of subdomains Ω of $\mathbf{H}$ for which $f'(\mathbf{D}) \subset \Omega$ implies boundedness of the function f? As we observed above, the condition $W(\Omega) < 2$ is sufficient. But, it is not necessary. The simplest example is $\Omega = \mathbf{D}(1,1) = \{w : |w-1| < 1\}$. Obviously $f'(\mathbf{D}) \subset \mathbf{D}(1,1)$ implies $|f(z) - f(0)| < 2$ but $W(\mathbf{D}(1,1)) = 2$ by (2.5). In this problem, the shape of the domain Ω near the point at infinity is dominating the boundedness property. This sort of problem was also considered by MacGregor and Rønning [6].

We also remark that, keeping Theorem 1 in mind, Theorem B gives another way to prove a part of the theorem of Chuaqui and Gevirtz. Indeed, if $f'(\mathbf{D}) \subset \Omega$ and $W(\Omega) < 2$, then Theorem B implies $\|T_f\| < 2$. Now we recall a theorem of Kari and Per Hag [2, Theorem 4.3]: if a univalent function f on $\mathbf{D}$ satisfies $\|T_f\| < 2$, then $f(\mathbf{D})$ is a John disk. In our case, we know that f is univalent by the Noshiro–Warschawski theorem when $\operatorname{Re} f' > 0$. Thus, the theorem of Hag implies that $f(\mathbf{D})$ is a John disk if $f'(\mathbf{D}) \subset \Omega$ and if $W(\Omega) < 2$.

As a by-product of the proof of Theorem 1 in Section 2, we obtain the following quantitative result:

$$2\tanh\delta(\Omega) \le W(\Omega) \le 2\max\left\{\frac{1}{\cosh(2\delta(\Omega))}, \sinh(4\delta(\Omega))\log(\coth(2\delta(\Omega)))\right\},$$

whenever $\delta(\Omega) > 0$. This estimate is unfortunately not good when $\delta(\Omega)$ is small. We supply a better but more complicated estimate, which might be of future use.

In the first part of the proof of (ii) $\Leftrightarrow$ (iii) in Theorem 1, we took a point $b \in \partial\Omega$ such that $d_{\mathbf{H}}(w,b) \le d + \rho$, where d is given by (2.4). By using the relation in (2.3), we now have

$$2|w|\lambda_\Omega(w) \ge 2|w|\lambda_{\mathbf{H}_b}(w) = \frac{1}{\cos\psi}\cdot\frac{\sinh t}{t}, \tag{3.1}$$

where $\psi = |\arg w|$ and $t = \log|(w+\bar{b})/(w-b)|$. Since $\mathrm{e}^t = \coth(d_{\mathbf{H}}(w,b)) \ge \coth(d+\rho)$ and $1/\cos\psi = (1+\tan^2(\psi/2))/(1-\tan^2(\psi/2)) = (1+\tanh^2 d)/(1-\tanh^2 d) = \cosh(2d)$, we obtain

$$2|w|\lambda_\Omega(w) \ge \frac{\cosh(2d)\sinh(\log(\coth(d+\rho)))}{\log(\coth(d+\rho))} = \frac{\cosh(2d)}{\sinh(2d+2\rho)\log(\coth(d+\rho))}.$$

Set

$$\begin{aligned} F(x,\rho) &= \frac{\sinh(2x+2\rho)\log(\coth(x+\rho))}{\cosh(2x)} \\ &= (\sinh(2\rho) + \cosh(2\rho)\tanh(2x))\log(\coth(x+\rho)). \end{aligned}$$

Then

$$\partial_x F(x,\rho) = 2\frac{\cosh(2\rho)\log(\coth(x+\rho)) - \cosh(2x)}{\cosh^2(2x)} = \frac{2G(x,\rho)}{\cosh^2(2x)}.$$

Since $G(x,\rho)$ is decreasing in $x > 0$, $G(0,\rho) = \cosh(2\rho)\log(\coth\rho) - 1 > 0$ and $G(x,\rho) \to -\infty$ as $x \to +\infty$, we see that there is a unique root $x = \xi(\rho)$ of the equation $G(x,\rho)=0$ in $x>0$ for a fixed $\rho\geq 0$. The function $F(x,\rho)$ takes its maximum at the point $x=\xi(\rho)$, and thus the inequalities

$$\begin{aligned}\frac{1}{2|w|\lambda_\Omega(w)}&\leq F(d,\rho)\leq F(\xi(\rho),\rho)=\frac{\sinh(2\xi(\rho)+2\rho)}{\cosh(2\rho)}\\&=\sinh(2\xi(\rho))+\cosh(2\xi(\rho))\tanh(2\rho)=:\mu(\rho)\end{aligned}$$

hold. The partial derivative $\partial_\rho F(x,\rho)=2(\cosh(2x+2\rho)\log(\coth(x+\rho))-1)/\cosh(2x)$ is positive and thus $F(x,\rho)$ is increasing in ρ. Therefore, for $0\leq\rho<\rho'$, we have

$$\mu(\rho)=F(\xi(\rho),\rho)<F(\xi(\rho),\rho')\leq F(\xi(\rho'),\rho')=\mu(\rho'),$$

which means that $\mu(\rho)$ is increasing in ρ. Note that $\mu(\rho)<1$ since $F(x,\rho)<1$. (We can also show that $\xi(\rho)$ is decreasing in ρ.) We summarize the above observation in the following form.

Proposition 4. *Let Ω be a subdomain of the right half-plane $\mathbf{H}$ with $\delta(\Omega)>0$. Then*

$$W(\Omega)\leq 2\mu(\delta(\Omega)).$$

Here, the function μ is given by

$$\mu(\rho)=\sinh(2\xi(\rho))+\cosh(2\xi(\rho))\tanh(2\rho),\quad \rho\geq 0,$$

where $x=\xi(\rho)$ is the unique root of the equation

$$\cosh(2\rho)\log(\coth(x+\rho))=\cosh(2x)$$

in $x>0$. The function μ is strictly increasing and less than 1 on $[0,\infty)$.

For instance, $\xi(0)\approx 0.3109$ and $\mu(0)\approx 0.6627$. On the other hand, as we saw before, when $\delta(\Omega)=0$, we have $W(\Omega)/2\leq 1/2$. Thus, the above estimate is not asymptotically sharp. The main reason is probably that we used only one point b of $\partial\Omega$ in the estimate (3.1) of $\lambda_\Omega(w)$ in spite of the fact that there would be many other boundary points near the positive real axis when $\delta(\Omega)$ is very small.

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