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ON A FUNCTION-THEORETIC RUIN PROBLEM

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Abstract. Let $\varphi\colon \mathbf{D}\to\mathbf{D}$ be analytic, $m\in\mathbf{N}$ and $g_0\in\mathscr{H}^2$. We define the $g_n\in\mathscr{H}^2$ by

$$(*)\qquad g_{n+1}(z)=\big(g_n(z)\varphi(z)z^{-m}\big)^+\quad\text{for } n\in\mathbf{N}_0$$

where $(\ldots)^+$ denotes the analytic part of the Laurent series. We derive explicit formulas for the coefficients $b_{n,k}$ of the g_n.

The recursion $(*)$ comes from the study of the random variables

$$S_{n+1}=S_0+X_1+\ldots+X_n-mn,$$

where the X_ν are i.i.d. with generating function φ. Ruin occurs when S_n becomes negative. We have $b_{n,k}=\mathbf{P}(S_n=k,\text{not yet ruined})$.

1. Introduction

1.1. In the language of classical probability theory, we consider a gambler that, at every turn, pays a fee m and wins a random amount X. In business language, we consider a firm that each month has the fixed cost (such as rents and interest payments) of m and a variable net income of $X\geq 0$. The gambler or firm is ruined when the capital becomes negative. We give a precise formulation in Section 2 where we also address possible interpretations in insurance.

We restrict ourselves to integer values and to discrete time. The capital S_n at time n is a random variable with values in $\mathbf{Z}$ and ruin occurs at the time n when S_n becomes negative for the first time. The main purpose of the paper is to calculate the probability that S_n has a given value k provided that ruin has not yet occurred, that is, to calculate

$$b_{n,k}=\mathbf{P}(S_n=k,\ S_\nu\geq 0 \text{ for } \nu<n)\quad\text{for } n,k\in\mathbf{N}_0$$

using function-theoretic methods.

Let $m\in\mathbf{N}$ and let X be a random variable with values in $\mathbf{N}_0$. Then the generating function of X is a power series

$$\varphi(z)=\sum_{k=0}^{\infty}a_kz^k$$

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analytic in the unit disk $\mathbf{D}$, where $a_k = \mathbf{P}(X = k) \geq 0$ and $|\varphi(z)| \leq \varphi(1) = 1$. In Section 2 we formulate the ruin problem as a function-theoretic problem.

Then we solve this function-theoretic problem assuming only that $\varphi(\mathbf{D}) \subset \mathbf{D}$ but not that the coefficients a_k are non-negative. The solution in Section 3 is in the form of a double generating function and is still rather implicit. In Section 4 we give explicit algorithms to compute the coefficients $b_{n,k}$ of this double generating function in terms of the a_k, using only finitely many additions and multiplications.

In Section 5, we return to the original ruin problem and compute the above probabilities $b_{n,k}$ and the probability that ruin occurs at time n.

In Section 6, we study two classical quantities, in particular the probability that ruin never occurs. Compare e.g. Chapters III and IV of [Asm00] where the compound Poisson model in continuous time is used.

It is a substantial restriction of generality to consider only integer-valued random variables. But this allows the use of function-theoretic methods and furthermore reduces very much the computation time and storage requirements needed to obtain explicit numerical results.

It turns out that the computational effort increases considerably with m and that the case $m = 1$ is much simpler than $m > 1$. In applications, the value of m depends on the approximation of the distribution of income as integer multiples of the whole or of one half or of one third etc. of the fixed cost. After a suitable scaling this leads to the cases $m = 1$ or $m = 2$ or $m = 3$ etc.

Our risk processes are generalized random walks. In the language of the dividend model explained in Section 2.1, no special assumptions about the distribution of claim numbers and claim amounts are made. Basically the explicit formulas are, of course, convolutions, but due to the repeated application of the Laurent separation operator these convolutions are incomplete. This operation makes unnecessary the computation of quantities that are unessential for the barrier problem under consideration.

No knowledge of risk theory is required; see the book of Asmussen [Asm00] for a modern exposition. Only the basic concepts of probability theory are needed.

In a later paper we shall apply the present results to the study of down & out barrier options in financial mathematics. See e.g. [Sch05] for an application of function-theoretic methods to barrier option pricing.

1.2. For a Laurent series $h(z) = \sum_{k\in\mathbf{Z}} c_k z^k$ convergent in $r < |z| < 1$, we define the Laurent separation operators by

$$h(z)^+ = \sum_{k=0}^{\infty} c_k z^k, \quad h(z)^- = \sum_{k=1}^{\infty} c_{-k} z^{-k}. \tag{1.1}$$

If h contains further variables, we agree to use z as the separation variable. We define a norm $\|\cdot\|$ by

$$\|h\|^2 = \sum_{k\in\mathbf{Z}} |c_k|^2. \tag{1.2}$$

It follows from (1.1) that

$$\|h\|^2 = \|h^+\|^2 + \|h^-\|^2. \tag{1.3}$$

Since $\|h\|^2 = \lim_{\rho\to 1-0} \frac{1}{2\pi}\int_0^{2\pi} |h(\rho e^{i\theta})|^2\,d\theta$, it follows for ψ analytic in $r < |z| < 1$ that

$$\|h\psi\|^2 = \lim_{\rho\to 1-0} \frac{1}{2\pi}\int_0^{2\pi} |h(\rho e^{i\theta})|^2 |\psi(\rho e^{i\theta})|^2\,d\theta \le \|h\|^2 \cdot \sup_{r<|z|<1} |\psi(z)|^2. \tag{1.4}$$

2. The probabilistic ruin problem

2.1. We consider a probability space that is so large that all the following (discrete) random variables are defined. So we take the same elementary point of view as in the first volume of Feller [Fel68].

Let X be a random variable with values in $\mathbf{N}_0$ and let

$$\varphi(z) = \sum_{k=0}^{\infty} \mathbf{P}(X = k) z^k = \sum_{k=0}^{\infty} a_k z^k \tag{2.1}$$

be its generating function. Thus $a_j \ge 0$ for all j and $\varphi(1) = \sum_{k=0}^{\infty} a_k = 1$. We assume that $a_0 > 0$. The expectation of X is $\mathbf{E}(X) = \varphi'(1) \le \infty$. Furthermore let $m \in \mathbf{N}$ be given.

Let S_0 be another random variable with values in $\mathbf{N}_0$; we will often assume that S_0 is constant. We inductively define

$$S_n = S_{n-1} + X_n - m \quad \text{for } n \in \mathbf{N} \tag{2.2}$$

where the X_ν are independent random variables distributed like X. Hence we have

$$S_n = S_0 + X_1 + \ldots + X_n - mn, \tag{2.3}$$

so that the S_n are random variables with values in $\mathbf{Z}$ which also may assume negative values because $a_0 > 0$.

This ruin problem has connections with issues arising in insurance business. In some branches of insurance, such as life annuity insurance, the company is steadily paying out sums (life annuities) to the policyholders, while certain discrete sums (annuity reserves) become available to the company as the effect of random events (death of policyholders), see [Cra55, p. 6]. Ruin occurs when the initial fund is used up for the payment of annuities and not enough reserves have been released.

In this case, ruin occurred when the steady cash flow outweighed the random flow. However, more often than not, the situation will be reversed. So our model cannot be applied to non-life insurance or term life insurance, where the company receives risk premiums from the policyholders at a uniform rate and has to settle random claims in return. Here ruin occurs when the random flow outweighs the steady flow, so it is not only the sign of the cash flows which makes this situation different from the one considered by us.

On the other hand, if ruin can be disregarded (for example because its probability is extremely small, or because the portfolio under consideration is only a part of the company's whole business), then our model can be applied to the occurrence of a surplus. A common task in life insurance is the assessment of dividend policies (see [Büh70, Sec. 6.4]). Let $\tilde{S}_n$ be the capital of a company with a monthly premium income m and random losses X_n that intends to pay dividends to the policy holder as soon as the surplus of the portfolio under consideration exceeds the level l; for the sake of simplicity we assume that $\tilde{S}_0 = 0$. Dividends are not paid while $\tilde{S}_n = mn - X_1 - \ldots - X_n \leq l$ or

$$S_n = l + X_1 + \ldots + X_n - mn \geq 0\,. \tag{2.4}$$

This relation is analogous to (2.3), but the interpretation is different; ruin in the original setting corresponds to $\tilde{S}_n$ crossing the barrier l from below in the new setting. In addition, this time the barrier is not absorbing; instead of stopping when $\tilde{S}_n$ exceeds l, we now assume that $\tilde{S}_n$ is distributed and the process continues immediately by starting again with the initial capital 0. See Section 5.4 for further details.

Situations of this type were considered by Gerber et al. ([Ger90], [PL94], [WP02], [Fro05]). Of course, $m > \mathbf{E}(X)$ in this case, because the premium must contain the administration cost and a safety loading. If the probability of claims is small, the insurance premium will be small compared to the amount of the possible individual claims, and this leads to a small m.

A broad account of various types of ruin problems is found in [Cra55], where also function-theoretic methods are used extensively. Predominantly risk theory [Asm00] studies a quantity somewhat different from (2.3), namely (in our notation)

$$S_n = S_0 + mn - (X_1 + \ldots + X_{N_n})\,,$$

where m is the constant premium received by the insurance company, the X_k are the random sizes of the claims and N_n the number of claims to be paid up to time n, often assumed to be Poisson distributed.

2.2. We define new random variables E_n for $n \in \mathbf{N}_0$ by

$$E_n = \begin{cases} 1 & \text{if } S_\nu \geq 0 \text{ for all } \nu \leq n, \\ 0 & \text{if } S_\nu < 0 \text{ for some } \nu \leq n. \end{cases} \tag{2.5}$$

Since $S_0 \geq 0$ we have $E_0 = 1$, and $E_n = 1$ continues to hold until, for some random time R, we have $E_n = 0$ for $n \geq R$. We set $R = \infty$ if $E_n = 1$ for all n. Thus we define

$$R = \begin{cases} n & \text{if } S_\nu \geq 0 \text{ for } 0 \leq \nu < n \text{ but } S_n < 0, \\ \infty & \text{if } S_\nu \geq 0 \text{ for all } \nu. \end{cases} \tag{2.6}$$

Hence R is the time when ruin occurs, and E_n drops to 0 on the event of ruin.

The probability of the event that $S_n = k \geq 0$ and $S_\nu \geq 0$ for $\nu < n$ has the generating function

$$(2.7)\qquad g_n(z) = \sum_{k=0}^{\infty} \mathbf{P}(S_n = k, E_n = 1) z^k = \sum_{k=0}^{\infty} \mathbf{P}(S_n = k, R > n) z^k \quad (n \in \mathbf{N}_0).$$

In particular, the initial random variable S_0 has the generating function

$$(2.8)\qquad g_0(z) = \sum_{k=0}^{\infty} \mathbf{P}(S_0 = k) z^k.$$

Proposition 2.1. *Using the Laurent separation operator (1.1) we have*

$$(2.9)\qquad g_{n+1}(z) = \bigl(g_n(z)\varphi(z)z^{-m}\bigr)^+ \quad \textit{for } n \in \mathbf{N}_0.$$

Proof. We see from (2.1), (2.5) and (2.7) that

$$g_n(z)\varphi(z)z^{-m} = \sum_{j=0}^{\infty} \mathbf{P}\bigl(S_n = j,\ S_\nu \geq 0\ \ (\nu \leq n)\bigr) z^j \sum_{k=0}^{\infty} \mathbf{P}(X_{n+1} = k) z^{k-m},$$

where X_{n+1} is independent of the X_ν for $\nu \leq n$. With $l = j + k - m$ this expression becomes equal to

$$\begin{aligned}
&\sum_{l=-m}^{\infty} \Bigl(\sum_{j=0}^{l+m} \mathbf{P}\bigl(S_n = j,\ S_\nu \geq 0\ \ (\nu \leq n)\bigr) \mathbf{P}(X_{n+1} = l + m - j) \Bigr) z^l \\
&= \sum_{l=-m}^{\infty} \Bigl(\sum_{j=0}^{l+m} \mathbf{P}\bigl(S_{n+1} = l,\ S_n = j,\ S_\nu \geq 0\ \ (\nu \leq n)\bigr) \Bigr) z^l \\
&= \sum_{l=-m}^{\infty} \mathbf{P}\bigl(S_{n+1} = l,\ S_\nu \geq 0\ \ (\nu \leq n)\bigr) z^l
\end{aligned}$$

because of (2.2). The coefficient of z^l with $l \geq 0$ can be written as $\mathbf{P}\bigl(S_{n+1} = l,\ S_\nu \geq 0\ \ (\nu \leq n+1)\bigr)$. Hence (2.9) follows from the definition (1.1) of the Laurent separation. □

3. The function-theoretic problem

3.1. Throughout this section we consider an analytic function

$$(3.1)\qquad \varphi(z) = \sum_{k=0}^{\infty} a_k z^k \quad (z \in \mathbf{D})$$

with complex coefficients that satisfies

$$(3.2)\qquad \varphi(\mathbf{D}) \subset \mathbf{D}, \quad \varphi(0) = a_0 \neq 0.$$

Let $m \in \mathbf{N}$ and let g_0 be analytic in $\mathbf{D}$ with $\|g_0\| < \infty$; see (1.2). Thus g_0 belongs to the Hardy space $\mathscr{H}^2$. In accordance with Proposition 2.1 we define the functions g_n recursively by

$$g_{n+1}(z) = \big(g_n(z)\,\varphi(z)\,z^{-m}\big)^+ \quad \text{for } n \in \mathbf{N}_0; \tag{3.3}$$

the Laurent separation operators were introduced in (1.1). The generating function of this sequence is, by definition,

$$g(z,s) = \sum_{n=0}^{\infty} g_n(z) s^n. \tag{3.4}$$

Theorem 3.1. *The functions g_n are analytic in $\mathbf{D}$ and satisfy $\|g_n\| \le \|g_0\|$ for all n. The generating function $g(z,s)$ is analytic in $(z,s) \in \mathbf{D} \times \mathbf{D}$ and satisfies*

$$g(z,s) = \frac{z^m g_0(z) - h(z,s)}{z^m - \varphi(z)s}, \tag{3.5}$$

where $h(z,s)$ is a polynomial of degree $< m$ in z.

Proof. Since φ and g_0 are analytic in $\mathbf{D}$ it follows from (3.3) and (1.1) that the functions g_n are analytic in $\mathbf{D}$. Furthermore we see from (1.3) that, for $n \in \mathbf{N}_0$,

$$\|g_{n+1}(z)\| \le \|g_n(z)\,\varphi(z)\,z^{-m}\| = \|g_n(z)\,\varphi(z)\| \le \|g_n(z)\|;$$

the last inequality follows from (1.4) because $\varphi(\mathbf{D}) \subset \mathbf{D}$. Hence we conclude that $\|g_n\| \le \|g_0\| < \infty$ so that $g(z,s)$ is analytic in $(z,s) \in \mathbf{D} \times \mathbf{D}$ in view of (3.4).

We obtain from (3.4), (3.3) and (1.1) that

$$\begin{aligned} g(z,s) &= g_0(z) + \sum_{n=0}^{\infty} g_{n+1}(z) s^{n+1} \\ &= g_0(z) + \sum_{n=0}^{\infty} \Big(g_n(z)\,\varphi(z)\,z^{-m} - \big(g_n(z)\,\varphi(z)\,z^{-m}\big)^- \Big) s^{n+1} \\ &= g_0(z) + s\,\varphi(z)\,z^{-m}\,g(z,s) - z^{-m}\,h(z,s), \end{aligned}$$

where we have put

$$h(z,s) = z^m \sum_{n=0}^{\infty} \big(g_n(z)\,\varphi(z)\,z^{-m}\big)^- s^{n+1}. \tag{3.6}$$

Now (3.5) follows by rearranging. Since $g_n(z)\,\varphi(z)\,z^{-m}$ contains only powers z^k with $-m \le k < \infty$, it follows from (1.1) and (3.6) that $h(z,s)$ is a polynomial of degree $< m$ in z. □

3.2. Next we determine the polynomial $h(z,s)$ of Theorem 3.1. We use the fact that $g(z,s)$ is analytic in $z \in \mathbf{D}$ so that, in the quotient (3.5), the zeros of the denominator that lie in $\mathbf{D}$ must cancel with the zeros of the numerator.

The special case $m = 1$ leads to the fixed point function f of φ, the unique inverse function $z = f(s)$ of $s = z/\varphi(z)$ for $s \in \mathbf{D}$. This function was introduced by Mejía and the second author [MP05], and Solynin [Sol07] has shown that f maps $\mathbf{D}$ conformally onto a hyperbolically convex subdomain of $\mathbf{D}$.

Corollary 3.2. *Let $m = 1$. If f denotes the fixed point function of φ then*

$$g(z,s) = \frac{zg_0(z) - f(s)\,g_0(f(s))}{z - \varphi(z)s}. \tag{3.7}$$

Proof. It follows from Theorem 3.1 for $m = 1$ that $h(z,s)$ is a polynomial of degree < 1 so that $h(z,s) \equiv h(s)$ is independent of z. If we put $z = f(s)$ in (3.5) then the denominator $\varphi(z)\big(z/\varphi(z) - s\big)$ is zero and therefore also the numerator. Hence we have

$$h(z,s) = h(s) = zg_0(z) = f(s)\,g_0(f(s)). \qquad \square$$

Now we turn to the general case $m \geq 1$ which is more complicated than the special case $m = 1$ because the inverse function of $s = z^m/\varphi(z)$ is in general multi-valued. A similar phenomenon occurs in the context of composition operators [CGG].

Let $s \in \mathbf{D}$ and $|s|^{1/m} < r < 1$. Since $|\varphi(z)| < 1$ we have

$$|z|^m = r^m > |s| > |\varphi(z)s| \quad \text{for } |z| = r$$

so that $z^m - \varphi(z)s$ has precisely m zeros in $\{|z| < r\}$ by Rouché's theorem.

We conclude that $z^m - \varphi(z)s$ has precisely m zeros $z_\mu(s)$ in $\mathbf{D}$ for each $s \in \mathbf{D}$. We consider the polynomial

$$p(z,s) = \prod_{\mu=1}^{m} \big(z - z_\mu(s)\big) \tag{3.8}$$

and we see that, for every $s \in \mathbf{D}$,

$$\frac{z^m - \varphi(z)s}{p(z,s)} \ \text{ is analytic and nonzero in } z \in \mathbf{D}. \tag{3.9}$$

We shall determine $p(z,s)$ in Section 4. All the following considerations are based on this polynomial.

Theorem 3.3. *For $m \geq 1$ we have, in the notations of (3.4) and (3.5),*

$$g(z,s) = \frac{p(z,s)}{z^m - \varphi(z)s}\left(\frac{z^m g_0(z)}{p(z,s)}\right)^{+}, \tag{3.10}$$

$$h(z,s) = p(z,s)\left(\frac{z^m g_0(z)}{p(z,s)}\right)^{-}. \tag{3.11}$$

Proof. Let $s \in \mathbf{D}$ be fixed. We obtain from (3.5) that

$$\frac{z^m g_0(z)}{p(z,s)} = \frac{z^m - \varphi(z)s}{p(z,s)}\, g(z,s) + \frac{h(z,s)}{p(z,s)}. \tag{3.12}$$

The first term on the right-hand side is analytic in $\mathbf{D}$ by (3.9) and Theorem 3.1. Since h is a polynomial of degree $< m$ and p is a polynomial of exact degree m, the Laurent expansion in $\{|z| \geq 1\}$ of the last term of (3.12) contains only negative powers of z. Hence (3.12) is a Laurent separation as in (1.1), and this implies (3.10) and (3.11). □

4. The determination of the coefficients

4.1. We again consider the function φ given by (3.1) and (3.2). Its power coefficients $a_{n,k}$ are defined by

$$\varphi(z)^n = \sum_{k=0}^{\infty} a_{n,k} z^k \qquad \text{for } n \in \mathbf{Z}; \tag{4.1}$$

we set $a_{n,k} = 0$ for $k < 0$. Then $a_{1,k} = a_k$ and

$$a_{n,0} = a_0^n, \quad a_{0,k} = 0 \quad (k > 0). \tag{4.2}$$

From (4.1) we obtain the recursion formula

$$a_{n,k} = \sum_{j=0}^{k} a_{k-j}\, a_{n-1,j} \qquad \text{for } n \in \mathbf{N},\ k \in \mathbf{N}_0. \tag{4.3}$$

If the $a_{n,k}$ ($k \in \mathbf{N}_0$) are needed for only a single n, then it is quicker to use recursions based on the binary expansion of n.

Proposition 4.1. *The polynomial p defined by (3.8) satisfies*

$$\log \frac{p(z,s)}{z^m} = -\sum_{k=1}^{\infty} \Bigg(\sum_{n \ge k/m} \frac{1}{n}\, a_{n,mn-k} s^n \Bigg) z^{-k} \tag{4.4}$$

for $|s| < 1$ and $|s|^{1/m} < |z| < \infty$, where the summation indices are in $\mathbf{N}$.

Proof. Let $s \in \mathbf{D}$ and $r = r(s) = \max_\mu |z_\mu(s)|$ so that $r < 1$. We write

$$\log\Big(1 - \frac{\varphi(z)s}{z^m}\Big) = \log \frac{z^m - \varphi(z)s}{p(z,s)} + \log \frac{p(z,s)}{z^m}; \tag{4.5}$$

the additive constant $2\pi i k$ is determined such that equality holds for $z = (1+r)/2$ and thus for $r < |z| < 1$. By (3.9) the first term on the right-hand side is analytic in $z \in \mathbf{D}$ and, by (3.8), the second term vanishes at ∞ and is analytic in $\{|z| > r\}$. Hence (4.5) is a Laurent separation; see (1.1).

Now let $|s|^{1/m} < |z| < 1$. Then $|z^{-m}\varphi(z)s| < 1$ and thus, by (4.1),

$$\log\Big(1 - \frac{\varphi(z)s}{z^m}\Big) = -\sum_{n=1}^{\infty} \frac{1}{n} \big(z^{-m}\varphi(z)\big)^n s^n = -\sum_{n=1}^{\infty} \sum_{j=0}^{\infty} a_{n,j} z^{j-mn} s^n.$$

Since (4.5) is a Laurent separation, we obtain (4.4) by collecting the terms with z^{-k} for $k = mn - j \ge 1$. □

Motivated by (4.4) we set

$$f_k(s) = \sum_{n \ge k/m} \frac{1}{n}\, a_{n,mn-k}\, s^n \quad \text{for } k \in \mathbf{N} \tag{4.6}$$

and define $p_k(s)$ $(0 \le k \le m)$ and $q_k(s)$ $(k \in \mathbf{N}_0)$ by

$$p(z,s) = \sum_{k=0}^{m} p_k(s) z^{m-k}, \quad \frac{z^m}{p(z,s)} = \sum_{k=0}^{\infty} q_k(s) z^{-k}; \tag{4.7}$$

see (3.8). It follows from (4.4), (4.6) and the first formula (4.7) that

$$\log\left(\sum_{k=0}^{m} p_k(s) z^{-k}\right) = \log \frac{p(z,s)}{z^m} = -\sum_{k=0}^{\infty} f_k(s) z^{-k},$$

and differentiation with respect to z leads to

$$\sum_{k=1}^{m} k p_k(s) z^{-k} = -\sum_{k=1}^{\infty} k f_k(s) z^{-k} \sum_{k=0}^{m} p_k(s) z^{-k}.$$

This yields the recursion formula

$$p_k(s) = -\frac{1}{k} \sum_{j=1}^{k} j\, f_j(s) p_{k-j}(s) \quad \text{for } k = 1, \dots, m. \tag{4.8}$$

In a similar way the second formula (4.7) leads to the recursion

$$q_k(s) = \frac{1}{k} \sum_{j=1}^{k} j\, f_j(s) q_{k-j}(s) \quad \text{for } k \in \mathbf{N}. \tag{4.9}$$

We have $p_0 = q_0 = 1$ by (4.7) and (3.8), and (4.8) and (4.9) show for instance that

$$p_1 = -f_1, \quad q_1 = f_1, \quad p_2 = f_2 - \frac{1}{2} f_1^2, \quad q_2 = f_2 + \frac{1}{2} f_1^2. \tag{4.10}$$

Proposition 4.2. *The functions f_k, p_k and q_k are analytic in $\mathbf{D}$ and the polynomial p is analytic in $\mathbf{C} \times \mathbf{D}$.*

Proof. Since φ^n is bounded by 1 it follows that $|a_{n,j}| \le 1$. Hence f_k is analytic in $\mathbf{D}$ because of (4.6). Since $q_0(s) = 1$ it follows from (4.9) by induction that $q_k(s)$ is analytic in $\mathbf{D}$ for $k \in \mathbf{N}$. Since $p_0(s) = 1$ it follows from (4.8) by induction that $p_k(s)$ is analytic in $\mathbf{D}$ for $0 < k \le m$, and we deduce from (4.7) that $p(z,s)$ is analytic in $z \in \mathbf{C}$ and $s \in \mathbf{D}$. □

Now we expand the functions in (4.7) into the power series

$$p_k(s) = \sum_{n=0}^{\infty} p_{n,k}\, s^n, \quad q_k(s) = \sum_{n=0}^{\infty} q_{n,k}\, s^n. \tag{4.11}$$

Putting the definition (4.6) of $f_k(s)$ into (4.8) and (4.9), we obtain

$$p_{n,k} = -\sum_{j=1}^{k}\sum_{\nu=1}^{n} \frac{j}{k\nu} a_{\nu,m\nu-j} p_{n-\nu,k-j} \quad \text{for } 1 \le k \le m, \tag{4.12}$$

$$q_{n,k} = \sum_{j=1}^{k}\sum_{j/m\le\nu\le n} \frac{j}{k\nu} a_{\nu,m\nu-j} q_{n-\nu,k-j} \quad \text{for } k \in \mathbf{N}. \tag{4.13}$$

These formulas give recursions in n; for $n = 0$ we have $p_{0,0} = q_{0,0} = 1$ and $p_{0,k} = q_{0,k} = 0$ for $k \ge 1$. We see from (4.3), (4.12) and (4.13) that the $p_{n,k}$ and $q_{n,k}$ are polynomials with rational coefficients in the a_j given by (3.1).

4.2. It follows from Theorem 3.3 that g is linear in g_0. Hence it is no essential restriction to assume that $g_0(z) = z^l$ with $l \in \mathbf{N}_0$.

Proposition 4.3. *Let $g_0(z) = z^l$ with $l \in \mathbf{N}_0$. Then, see* (4.7),

$$g(z,s) = \frac{p(z,s)}{z^m - \varphi(z)s} \sum_{k=0}^{l} q_{l-k}(s) z^k, \tag{4.14}$$

and the polynomial $h(z,s)$ of Theorem 3.1 satisfies

$$h(z,s) = p(z,s)\sum_{k=1}^{\infty} q_{l+k}(s) z^{-k} = \sum_{\mu=1}^{m}\left(\sum_{j=1}^{\mu-1} p_j(s) q_{\mu+l-j}(s)\right) z^{m-\mu}. \tag{4.15}$$

Proof. The second formula (4.7) shows that

$$\left(\frac{z^m g_0(z)}{p(z,s)}\right)^+ = \left(\sum_{j=0}^{\infty} q_j(s) z^{l-j}\right)^+ = \sum_{k=0}^{l} q_{l-k}(s) z^k$$

so that (4.14) follows from (3.10) in Theorem 3.3. Furthermore

$$\left(\frac{z^m g_0(z)}{p(z,s)}\right)^- = \left(\sum_{j=0}^{\infty} q_j(s) z^{l-j}\right)^- = \sum_{k=1}^{\infty} q_{l+k}(s) z^{-k},$$

and the first equation (4.15) follows from (3.11). Replacing $p(z,s)$ by the first sum in (4.11) we obtain

$$h(z,s) = \sum_{j=0}^{m} p_j(s) z^{m-j} \sum_{k=1}^{\infty} q_{l+k}(s) z^{-k},$$

which yields the second equation (4.15) if we put $\mu = j + k$; note that the sums with $\mu > m$ have to vanish because $h(z,s)$ is analytic in $z \in \mathbf{D}$. □

We define $c_{n,\mu}$ $(1 \le \mu \le m)$ by

$$h(z,s) = \sum_{\mu=1}^{m}\sum_{n=0}^{\infty} c_{n,\mu} s^n z^{m-\mu}. \tag{4.16}$$

Then it follows from (4.15) and (4.11) that

$$c_{n,\mu} = \sum_{\nu=0}^{n} \sum_{j=0}^{\mu-1} p_{n-\nu,j}\, q_{\nu,\mu+l-j}. \tag{4.17}$$

Theorem 4.4. *Let $g_0(z) = z^l$ with $l \in \mathbf{N}_0$. We write*

$$g(z,s) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{n,k} z^k s^n. \tag{4.18}$$

With the notation in (4.1) and (4.7), we have

$$b_{n,k} = a_{n,mn+k-l} - \sum_{\nu=0}^{n} \sum_{\mu=1}^{m} a_{\nu,m\nu+k+\mu} c_{n-\nu,\mu}. \tag{4.19}$$

Proof. By (3.5) with $g_0(z) = z^l$, we have

$$\begin{aligned} g(z,s) &= \frac{z^l - z^{-m}h(z,s)}{1 - z^m \varphi(z)s} = \left(z^l - \frac{h(z,s)}{z^m}\right) \sum_{\nu=0}^{\infty} \frac{\varphi(z)^\nu}{z^{m\nu}} s^\nu \\ &= \left(z^l - \sum_{N=0}^{\infty} \sum_{\mu=1}^{m} c_{N,\mu} z^{-\mu} s^N\right) \sum_{\nu=0}^{\infty} \sum_{j=0}^{\infty} a_{\nu,j} z^{j-m\nu} s^\nu \end{aligned}$$

by (4.16) and (4.1).

Now we multiply out. In the contribution from z^l, we set $n = \nu$ and $k = l+j-m\nu$ so that $j = mn+k-l$. In the contribution from the first double sum, we set $n = N+\nu$ and $k = -\mu + j - m\nu$ so that $j = m\nu + k + \mu$. Thus we obtain (4.19) in view of (4.18). $\square$

The expression (4.19) for the $b_{n,k}$ is a finite but complicated formula. In Section 5 we shall study the computational effort necessary to obtain numerical values for the $b_{n,k}$.

There is an alternative formula for the $b_{n,k}$ which is simpler for small l. We restrict ourselves to the case $l = 0$.

Proposition 4.5. *If $g_0(z) = 1$ then*

$$b_{n,k} = a_{n,mn+k} + \sum_{\nu=0}^{n-1} \sum_{j=1}^{m} p_{n-\nu,j}\, a_{\nu,m\nu+k+j}. \tag{4.20}$$

Proof. Since $q_0(s) = 1$ it follows from (4.14) with $l = 0$ that

$$g(z,s) = \frac{p(z,s)}{z^m - \varphi(z)s} = \frac{p(z,s)}{z^m} \left(1 - \frac{\varphi(z)}{z^m}\, s\right)^{-1}.$$

Hence we obtain from (4.7) and (4.11) that

$$g(z,s) = \sum_{N=0}^{\infty} \sum_{j=0}^{m} p_{N,j}\, z^{-j} s^N \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} a_{\nu,\mu} z^{\mu-mn} s^\nu.$$

We multiply out and set $n = N + \nu$ and $k = -j + \mu - mn$. Then we obtain (4.20) because $p_{0,0} = 1$, $p_{0,j} = 0$ for $j \geq 1$ and $p_{n-\nu,0} = 0$ for $\nu < n$. $\square$

4.3. The special case $m = 1$ is again simpler and the resulting formula is much more explicit.

Corollary 4.6. *If* $m = 1$ *and* $g_0(z) = z^l$ *then*

$$\begin{aligned} b_{n,k} &= a_{n,n+k-l} - \sum_{\nu=l+1}^{n} \frac{l+1}{\nu}\, a_{\nu,\nu-l-1}\, a_{n-\nu,n-\nu+k+1} \\ &= \sum_{j=1}^{k+1} \frac{l+1}{n+j}\, a_{n+j,n-j-l-1}\, a_{-j,k+1-j}\,. \end{aligned} \tag{4.21}$$

Note that the second representation contains the coefficients of negative powers of φ. An advantage is that the number of terms in the second sum is independent of n.

Proof. The Bürmann–Lagrange formula shows [MP05, (5.6)] that

$$f(s)^{l+1} = \sum_{\nu=l+1}^{\infty} \frac{l+1}{\nu} a_{\nu,\nu-l-1} s^{\nu} \quad \text{for } l \in \mathbf{N}_0. \tag{4.22}$$

Comparing Corollary 3.2 with Theorem 3.1 for $m = 1$, we obtain that

$$f(s)^{l+1} = h(z, s) = \sum_{\nu=0}^{\infty} c_{\nu,1} s^{\nu};$$

see (4.16). Hence the first equation (4.21) follows from (4.19) with $n - \nu$ instead of ν and from (4.22).

Furthermore we have [MP05, (5.5)]

$$sf'(s)f(s)^{\mu-1} = \sum_{n=\mu}^{\infty} a_{n,n-\mu} s^n \quad \text{for } \mu \in \mathbf{Z}. \tag{4.23}$$

We write $sf'(s)f(s)^{l-k-1} = sf'(s)f(s)^{-k-2}f(s)^{l+1}$. Now we apply (4.23) with $\mu = l - k$ and then again with $\mu = -k - 1$. Using also (4.22) we obtain

$$a_{n,n+k-l} = \sum_{\nu=l+1}^{n+k+1} \frac{l+1}{\nu} a_{\nu,\nu-l-1} a_{n-\nu,n-\nu+k+1}.$$

Hence the terms in the middle part of (4.21) cancel except for those with $n + 1 \leq \nu \leq n + k + 1$, and the substitution $\nu = n + j$ now gives the second line of (4.21). $\square$

5. Application to the ruin problem

5.1. Now we return to the original probabilistic ruin problem described in Section 2. Thus φ is the generating function of a random variable X with values in

$\mathbf{N}_0$ and $\mathbf{P}(X=0)>0$. The coefficients $a_{n,k}$ of the powers φ^n satisfy

$$a_{n,k} = \mathbf{P}(X_1 + \ldots + X_n = k) \quad \text{for } n \in \mathbf{N},\ k \in \mathbf{N}_0 \tag{5.1}$$

where the X_ν are independent and distributed like X; see (2.3). The functions f_k introduced in (4.6) can be written as

$$f_k(s) = \sum_{n \geq k/m} \frac{1}{n}\, \mathbf{P}(S_n = mn - k)\, s^n \quad \text{for } s \in \mathbf{D},$$

whereas the functions p_k and q_k defined in (4.7) have no obvious probabilistic interpretation.

Now we can calculate the probability of the event that $S_n = k$ provided that ruin has not occurred up to time n, see (2.5) and (2.6).

Corollary 5.1. *If $S_0 = l \in \mathbf{N}_0$ and $m \in \mathbf{N}$ then*

$$\mathbf{P}(S_n = k, E_n = 1) = \mathbf{P}(S_n = k, R > n) = b_{n,k}, \tag{5.2}$$

where the $b_{n,k}$ are given by (4.19) if $m \geq 1$ and by (4.21) if $m = 1$.

Proof. It follows from (2.7) and (3.4) that

$$g(z,s) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{P}(S_n = k, E_n = 1)\, z^k s^n. \tag{5.3}$$

The function-theoretic problem of Section 3 is a generalization of the probabilistic problem of Section 2 by Proposition 2.1 and by (3.3). Hence formulas (4.18) and (5.2) agree, and we can apply Theorem 4.4 and Corollary 4.6. □

All formulas in this section are finite expressions which however are so complicated that a computer is needed for their evaluation. As we see from (4.12), (4.13) and (4.17), only the power coefficients $a_{\nu,m\nu}$ for $\nu \leq n$ are involved in calculating the $c_{\nu,k}$ for $\nu \leq n$ and any k. The recursion formula (4.3) shows that $O(nk^2)$ arithmetic operations are required to calculate the $a_{\nu,j}$ for $\nu \leq n$ and $j \leq k$. Hence $O(m^2n^3)$ operations are required for the $a_{\nu,m\nu}$ for $\nu \leq n$. The evaluations in (4.12), (4.13) and (4.17) can be done with $O\big((\mu + m + l)^2 n^2\big)$ further operations.

In our probabilistic case, we have $a_k \geq 0$ so that the recursive evaluation (4.3) of the $a_{n,k}$ only uses additions and multiplications of non-negative numbers. This is a numerically very robust procedure. The same is true for the recursion (4.13) to compute the $q_{n,k}$. But the recursion (4.12) for the $p_{n,k}$ contains subtractions so that a loss of accuracy might possibly occur. The same is true for the formula (4.19) for the $b_{n,k}$.

5.2. The random time R when ruin occurs was defined in (2.6). It follows from (2.7) and $S_0 \geq 0$ that

$$g_n(1) = \sum_{k=0}^{\infty} \mathbf{P}(S_n = k, R > n) = \mathbf{P}(R > n) \quad \text{for } n \in \mathbf{N}_0, \tag{5.4}$$

and since $\mathbf{P}(R > n-1) = \mathbf{P}(R = n) + \mathbf{P}(R > n)$ we deduce that

$$\mathbf{P}(R = n) = g_{n-1}(1) - g_n(1) \quad \text{for } n \in \mathbf{N}. \tag{5.5}$$

Let $p_{n,k}$ $(1 \le k \le m)$ and $q_{n,k}$ $(k \ge 0)$ be given by (4.11).

Theorem 5.2. *If $S_0 = l \in \mathbf{N}_0$ and $m \in \mathbf{N}$ then the generating function is*

$$\sum_{n=1}^{\infty} \mathbf{P}(R = n)\, s^n = 1 - (1-s) g(1,s) = h(1,s), \tag{5.6}$$

see (3.5). Furthermore, see (4.11),

$$\mathbf{P}(R = n) = -\sum_{\nu=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} p_{\nu,j} q_{n-\nu,k} \quad \text{for } n \in \mathbf{N}. \tag{5.7}$$

In the special case $m = 1$ we have

$$\mathbf{P}(R = n) = \frac{l+1}{n} a_{n,n-l-1} = \frac{l+1}{n} \mathbf{P}(S_n = -1). \tag{5.8}$$

Proof. Since $\varphi(1) = 1$ it follows from (4.14) for $z \to 1$ that

$$g(1,s) = \frac{p(1,s)}{1-s} \sum_{k=0}^{l} q_k(s) \quad \text{for } s \in \mathbf{D}. \tag{5.9}$$

We see from (5.5), (3.4) and (3.5) that

$$\sum_{n=1}^{\infty} \mathbf{P}(R = n)\, s^n = \sum_{n=1}^{\infty} \big(g_{n-1}(1) - g_n(1)\big) s^n = 1 - (1-s) g(1,s) = h(1,s)$$

which proves (5.6). Furthermore it follows, by (4.14) for $z = 1$, that

$$\sum_{n=1}^{\infty} \mathbf{P}(R = n)\, s^n = 1 - p(1,s) \sum_{k=0}^{l} q_k(s) = 1 - \sum_{n=0}^{\infty} \left(\sum_{j=0}^{m} p_{n,j} \right) s^n \sum_{n=0}^{\infty} \left(\sum_{k=0}^{l} q_{n,k} \right) s^n$$

because of (4.11). This equation implies (5.7).

Now let $m = 1$. We obtain from Corollary 3.2 for $z \to 1$ and from (4.22) that

$$1 - (1-s) g(1,s) = f(s)^{l+1} = \sum_{n=l+1}^{\infty} \frac{l+1}{n} a_{n,n-l-1} s^n.$$

This implies the first equation (5.8), and the second one follows because $S_0 = l$ and $m = 1$ and thus, by (2.3),

$$\mathbf{P}(S_n = -1) = \mathbf{P}(X_1 + \ldots + X_n + l - n = -1) = a_{n,n-l-1}. \qquad \square$$

In the special case $S_0 = 0$, we have $l = 0$ and (5.7) simplifies to

$$\mathbf{P}(R = n) = -\sum_{j=1}^{m} p_{n,j} \quad (n \ge 1)$$

because $q_{n-\nu,0} = 0$ for $\nu < n$. In terms of risk theory [Asm00, Section IV.2], we are in the case of no initial reserve.

We have always kept $S_0 = l$ fixed. In risk theory [Asm00] the initial capital reserve for future claims is an important variable. Compare [Asm00, Chapter IV]. For a moment we allow l to vary and denote by R_l the time when ruin occurs for an initial capital $S_0 = l$.

Proposition 5.3. *The double generating function is*

$$\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\mathbf{P}(R_l = n)\, s^n w^l = \frac{1}{1-w}\left(1 - \frac{p(1,s)}{w^m p(w^{-1},s)}\right), \tag{5.10}$$

where the polynomial p(1,s) is defined by (3.8).

Proof. We obtain from (5.6) and (4.15) that

$$\sum_{n=1}^{\infty}\mathbf{P}(R_l = n)\, s^n = p(1,s)\sum_{k=l+1}^{\infty} q_k(s).$$

We multiply by w^l and sum over l. Inverting the summation order we obtain on the right-hand side

$$p(1,s)\sum_{k=1}^{\infty}\frac{1-w^k}{1-w}\, q_k(s) = \frac{p(1,s)}{1-w}\sum_{k=1}^{\infty}(1-w^k)q_k(s) = \frac{p(1,s)}{1-w}\left(\frac{1}{p(1,s)} - \frac{w^{-m}}{p(w^{-1},s)}\right)$$

because of (4.7). This implies (5.10), see (3.8). □

5.3. The random variable E_n was defined in (2.5). It has the value 1 before ruin occurs and the value 0 thereafter. Note that S_n may have any sign after ruin whereas S_nE_n is always non-negative. The polynomial $p(z,s)$ was introduced in (3.8).

Theorem 5.4. *Let $S_0 = l \in \mathbf{N}_0$. If $s \in \mathbf{N}$ then*

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{E}(S_nE_n)s^n = \sum_{k=0}^{l}&\frac{p'(1,s)+(l-k-m)p(1,s)}{1-s}\, q_k(s)\\ &+ \frac{\big(\mathbf{E}(X)-m\big)s\, p(1,s)}{(1-s)^2}\sum_{k=0}^{l} q_k(s).\end{aligned} \tag{5.11}$$

Proof. We obtain from (2.7) that

$$g_n'(z) = \sum_{k=1}^{\infty} k\,\mathbf{P}(S_n = k, E_n = 1)\, z^{k-1}.$$

If $k \ge 1$ then $S_nE_n = k$ holds if and only if $S_n = k$ and $E_n = 1$. It follows that $g_n'(1) = \mathbf{E}(S_nE_n)$. Hence

$$\sum_{n=0}^{\infty}\mathbf{E}(S_nE_n)s^n = g'(1,s) := \lim_{z\to 1-0}\frac{\partial}{\partial z}g(z,s). \tag{5.12}$$

In (4.14) we replace k by $l-k$, differentiate with respect to z and then let $z \to 1-0$. We obtain

$$g'(1,s) = \left(\frac{p'(1,s)}{1-s} + \frac{p(1,s)\bigl(\varphi'(1)s-m\bigr)}{(1-s)^2}\right)\sum_{k=0}^{l} q_k(s) + \frac{p(1,s)}{1-s}\sum_{k=0}^{l}(l-k)q_k(s).$$

Now we write $\varphi'(1)s - m = \bigl(\varphi'(1)-m\bigr)s - m(1-s)$ and rearrange. Using (5.12) we obtain (5.11) because $\varphi'(1) = \mathbf{E}(X)$. □

An explicit formula for $\mathbf{E}(S_n E_n)$ can easily be obtained from (4.7) and (4.11). Without using any of our function-theoretic results, we now derive a double inequality which becomes an equality for the special case $m = 1$.

Proposition 5.5. *Let $m \in \mathbf{N}$. If $0 < \mathbf{E}(X) < \infty$ and $0 \le \mathbf{E}(S_0) < \infty$ then*

$$b_n + \mathbf{P}(R \le n) \le \mathbf{E}(S_n E_n) \le b_n + m\,\mathbf{P}(R \le n), \tag{5.13}$$

where

$$b_n = \mathbf{E}(S_0) + \bigl(\mathbf{E}(X) - m\bigr)\sum_{\nu=0}^{n-1}\mathbf{P}(R > \nu). \tag{5.14}$$

Proof. By (2.5) and (2.6), we have $E_n = 1$ if $R > n$ and $E_n = 0$ otherwise. Hence we see that

$$\alpha_n := \mathbf{E}(E_n) = \mathbf{P}(R > n). \tag{5.15}$$

Furthermore we have $E_n - E_{n-1} = 0$ except if $E_{n-1} = 1$ and $E_n = 0$, in which case $S_{n-1} \ge 0$ and $-m \le S_n < 0$. It follows that $S_n(E_n - E_{n-1}) = 0$ if $R \ne n$ and $1 \le S_n(E_n - E_{n-1}) \le m$ if $R = n$. We conclude that

$$\mathbf{P}(R = n) \le \beta_n := \mathbf{E}\bigl(S_n(E_n - E_{n-1})\bigr) \le m\,\mathbf{P}(R = n). \tag{5.16}$$

We obtain from (2.2) that, for $n \in \mathbf{N}$,

$$S_n E_n = (S_{n-1} + X_n - m)E_{n-1} + S_n(E_n - E_{n-1})$$

with X_n independent of E_{n-1} and therefore, by (5.15) and (5.16), that

$$\mathbf{E}(S_n E_n) = \mathbf{E}(S_{n-1}E_{n-1}) + \bigl(\mathbf{E}(X) - m\bigr)\alpha_{n-1} + \beta_n$$

for $n \in \mathbf{N}$. Since $E_0 = 1$ we conclude that

$$\mathbf{E}(S_n E_n) = \mathbf{E}(S_0) + \bigl(\mathbf{E}(X) - m\bigr)\sum_{\nu=0}^{n-1}\alpha_\nu + \sum_{\nu=1}^{n}\beta_\nu, \tag{5.17}$$

and we have $\mathbf{P}(R \le n) \le \sum_{\nu=1}^{n}\beta_\nu \le m\,\mathbf{P}(R \le n)$ because of (5.16). Hence (5.13) together with (5.14) follows from (5.17). □

5.4. The formulas in this section can also be used to analyze dividend policies as described in Section 2.1, for instance to calculate the expectation G_N of the company's net result after N periods. If $\tilde{b}_{n,k}$ ($n \in \mathbf{N}$, $k \in \mathbf{Z}$, $k \le l$) denotes the probability of ending with a capital k after n periods without crossing the barrier l

before, then $\tilde{b}_{n,k} = b_{n,l-k}$. The probability that the capital first exceeds l at time n is the same as the probability $\mathbf{P}(R = n)$ that ruin occurs at time n in the original setting. Then

$$G_N = \sum_k \tilde{b}_{N,k} k + \sum_n \mathbf{P}(R = n)(r + G_{N-n}),$$

where the company keeps r ($r < l$) of the surplus and pays out the rest to the policy holder. From this the G_N can be calculated recursively, starting with $G_0 = 0$.

6. Limit results in the probabilistic case

6.1. We continue to study the case that φ is the generating function of the random variable X, see (2.1). First we establish some properties of the polynomial $p(z, s)$ defined in (3.8) and the functions $p_k(s)$ and $q_k(s)$ defined in (4.7).

Proposition 6.1. *The function $p(1, s)$ is positive and decreasing in $s \in (0, 1)$. The limit satisfies $p(1, 1) > 0$ if and only if $m < \mathbf{E}(X) \leq \infty$.*

Proof. Since $a_{n,k} \geq 0$ by (5.1), it follows from Proposition (4.1) with $z \to 1$ that $p(1, s)$ is positive and decreasing. We consider the convex function

$$\psi(x) = x^{-m}\varphi(x) \quad \text{for } 0 < x \leq 1,$$

which satisfies $\psi'(1) = \mathbf{E}(X) - m$ because $\varphi(1) = 1$ and $\varphi'(1) = \mathbf{E}(X)$.

First let $\mathbf{E}(X) \leq m$. Then $\psi'(1) \leq 0$ so that $\psi(x) = 1/s$ has a solution $x = x(s) \in (0, 1)$, which satisfies $x(s) \to 1$ as $s \to 1$. Since $x^m - \varphi(x)s = 0$ it follows from (3.8) that $p(x, s) = 0$. Bernstein's inequality [Mit70, p. 228] shows that $|p'(z, s)| \leq m2^m$ for $|z| \leq 1$ and $|s| \leq 1$, so that

$$|p(1, s)| = |p(1, s) - p(x(s), s)| \leq m2^m\big(1 - x(s)\big) \to 0 \quad \text{as } s \to 1.$$

Now let $\mathbf{E}(X) > m$. Then $\psi'(1) > 0$ so that there exists $r \in (0, 1)$ with $\psi(r) < 1$ and thus $r^m > \varphi(r)$. If $0 < s \leq 1$ and $|z| = r$ then

$$|z|^m = r^m > \varphi(r) \geq |\varphi(z)| \geq |\varphi(z)s|$$

and it follows from Rouché's theorem that all m zeros $z_\mu(s)$ of $z^m - \varphi(z)s$ in $\mathbf{D}$ satisfy $|z_\mu(s)| < r$. We conclude that

$$p(1, s) = \prod_{\mu=1}^{m} |1 - z_\mu(s)| \geq (1 - r)^m. \qquad \square$$

Proposition 6.2. *If $0 < \varphi(0) < 1$ then $p_k(s)$ and $q_k(s)$ have continuous extensions to $\overline{\mathbf{D}}$ for each k, furthermore $q_k(s) \geq 0$ for $0 \leq s \leq 1$.*

Proof. The variance satisfies $0 < \mathbf{V}(X) \leq \infty$ because $0 < \mathbf{P}(X = 0) < 1$. It was shown in [MP05, Theorem 7.1] that

$$\limsup_{n\to\infty} \sqrt{n} \sup_j a_{n,j} < \infty$$

holds under the assumption that φ does not have the special form $\varphi(z) = \varphi^*(z^\mu)$ for some (maximal) $\mu \geq 2$. If φ does have this form then [MP05, Theorem 7.1] is applied to φ^* instead of φ. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n}\, a_{n,mn-k} < \infty$$

and we see from (4.6) that $f_k(s)$ has a continuous extension to $\overline{\mathbf{D}}$ which is non-negative for $0 \leq s \leq 1$. Hence our assertions follow from the recursion formulas (4.8) and (4.9); we have $q_k(s) \geq 0$ because $q_0(s) = 1$. □

6.2. Let $R \leq \infty$ be the time when ruin occurs, where $R = \infty$ means that ruin never occurs; see (2.6). In the following theorem, it is well-known when $\mathbf{P}(R = \infty)$ is zero or positive, see e.g. Theorems VI.10.3 and XII.2.2 in [Fel71].

Theorem 6.3. *Let $S_0 = l \in \mathbf{N}_0$. If $\mathbf{E}(X) \leq m$ then $\mathbf{P}(R = \infty) = 0$, and if $m < \mathbf{E}(X) \leq \infty$ then*

$$\mathbf{P}(R = \infty) = p(1,1) \sum_{k=0}^{l} q_k(1) > 0. \tag{6.1}$$

Proof. From (5.4) we obtain that

$$\beta := \mathbf{P}(R = \infty) = \lim_{n\to\infty} \mathbf{P}(R > n) = \lim_{n\to\infty} g_n(1).$$

If $\varepsilon > 0$ then $\beta - \varepsilon < g_n(1) < \beta + \varepsilon$ for $n \geq n_0$ so that

$$\frac{(\beta-\varepsilon)s^{n_0}}{1-s} \leq g(1,s) = \sum_{n=0}^{\infty} g_n(1)s^n \leq \frac{(\beta+\varepsilon)s^{n_0}}{1-s} + n_0.$$

Together with (5.9) this implies that

$$\beta = \lim_{s\to 1-0} (1-s)\, g(1,s) = p(1,1) \sum_{k=0}^{l} q_k(1); \tag{6.2}$$

the limits $p(1,1)$ and $q_k(1)$ exist by Proposition 6.2.

If $\mathbf{E}(X) \leq m$ then $p(1,1) = 0$ by Proposition 6.1 so that $\mathbf{P}(R = \infty) = \beta = 0$ by (6.2). Now let $m < \mathbf{E}(X) \leq \infty$. Then $p(1,1) > 0$ by Proposition 6.1 and $q_k(1) \geq 0$ by Proposition 6.2. Since $q_0(1) = 1$ it follows from (6.2) that

$$\mathbf{P}(R = \infty) = p(1,1) \sum_{k=0}^{l} q_k(1) \geq p(1,1) > 0. \qquad \square$$

Theorem 6.4. *Let $S_0 = l \in \mathbf{N}_0$.*

(i) *If $\mathbf{E}(X) < m$ then*

$$\mathbf{P}(R > n) = o\Big(\frac{1}{n}\Big) \quad \text{as } n \to \infty. \tag{6.3}$$

(ii) *If* $\mathbf{E}(X) = m$ *then*

$$\mathbf{E}(S_n E_n) \to p'(1,1) \sum_{k=0}^{l} q_k(1) \geq l + 1 \quad \text{as } n \to \infty \tag{6.4}$$

with equality if $m = 1$.

(iii) *If* $m < \mathbf{E}(X) < \infty$ *then*

$$\mathbf{E}(S_n E_n) \sim \big(\mathbf{E}(X) - m\big)\mathbf{P}(R = \infty)n \quad \text{as } n \to \infty. \tag{6.5}$$

Remarks. (i) The relation in (6.3) strengthens Theorem 6.3 for the case $\mathbf{E}(X) \leq m$. The proof of (6.3) does not use our function-theoretic results. We do not say anything about the limit behaviour of $\mathbf{E}(S_n E_n)$ for the case $\mathbf{E}(X) < m$.

(ii) The limit $p'(1,1) = \lim_{s \to 1-0} \frac{\partial}{\partial z} p(z,s)\Big|_{z=1}$ exists by Proposition 6.2. It follows from (6.4) that $p'(1,1) > 0$.

(iii) If $\mathbf{E}(X) > m$ then $\mathbf{P}(R = \infty) > 0$ by Theorem 6.3 and its value is given by (6.1).

Proof. We define α_n and β_n as in (5.15) and (5.16). Since $S_n E_n \geq 0$ and $S_0 = l$, we see from (5.17) that

$$0 \leq \mathbf{E}(S_n E_n) = l + \big(\mathbf{E}(X) - m\big) \sum_{\nu=0}^{n-1} \alpha_\nu + \sum_{\nu=1}^{n} \beta_\nu. \tag{6.6}$$

(i) Let $\mathbf{E}(X) < m$. Then it follows from (6.6) and (5.16) that

$$\sum_{\nu=0}^{\infty} \alpha_\nu \leq \big(m - \mathbf{E}(X)\big)^{-1} \Big(l + m \sum_{\nu=0}^{\infty} \mathbf{P}(R = \nu)\Big) < \infty.$$

Since $\alpha_n = \mathbf{P}(R > n)$ is decreasing we deduce [Kno64, p. 125] that $\mathbf{P}(R > n) = o(1/n)$.

(ii) Let $\mathbf{E}(X) = m$. By (5.11) in Theorem 5.4 we have

$$\sum_{n=0}^{\infty} \mathbf{E}(S_n E_n) s^n = \frac{1}{1-s} \sum_{k=0}^{l} \big(p'(1,s) + (l-k-m)p(1,s)\big) q_k(s). \tag{6.7}$$

We have $p(1,1) = 0$ by Proposition 6.1 and $\gamma := \lim \mathbf{E}(S_n E_n) < \infty$ exists by (6.6) and (5.16). Hence we can argue as in the proof of Theorem 6.3 to deduce from (6.7) that

$$\gamma = p'(1,1) \sum_{k=0}^{l} q_k(1).$$

It follows from (5.13) in Theorem 5.5 that

$$l + \mathbf{P}(R \leq n) \leq \mathbf{E}(S_n E_n) \leq l + m\,\mathbf{P}(R \leq n).$$

Since $\mathbf{P}(R \leq n) \to \mathbf{P}(R < \infty) = 1$ by Theorem 6.3, it follows that $l + 1 \leq \gamma \leq l + m$, in particular $\gamma = l + 1$ if $m = 1$.

(iii) Let $m < \mathbf{E}(X) < \infty$. We have $\alpha_n = \mathbf{P}(R > n) \to \mathbf{P}(R = \infty) > 0$ $(n \to \infty)$ by (5.15) and (6.1). Since $\sum \beta_\nu$ converges we deduce from (6.6) that

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{E}(S_n E_n) = \big(\mathbf{E}(X) - m\big) \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \alpha_\nu$$
$$= \big(\mathbf{E}(X) - m\big)\mathbf{P}(R = \infty) > 0. \qquad \square$$

The explicit formulas in Theorem 6.3 and 6.4 contain $p(1,1)$, $p'(1,1)$ and the $q_k(1)$. Contrary to the situation in Section 5, these values cannot be obtained from the coefficients a_k of φ by finitely many additions and multiplications. Now we indicate how to compute these values.

First we compute the zeros z_μ $(\mu = 1, \ldots, m)$ of $z^m - \varphi(z)$ with $|z_\mu| < 1$. Then we form

$$p(z,1) = \prod_{\mu=1}^{m} (z - z_\mu) = \sum_{k=0}^{m} p_k(1) z^{m-k}$$

as in (3.8). This allows us to compute $p(1,1)$ and $p'(1,1)$. It follows from (4.7) that

$$\sum_{k=0}^{m} p_k(1) z^{-k} \sum_{k=0}^{\infty} q_k(1) z^{-k} = 1, \quad p_0(1) = q_0(1) = 1,$$

which leads to the recursion formula

$$q_k(1) = -\sum_{j=1}^{\min(m,k)} p_j(1) q_{k-1}(1) \quad \text{for } k \in \mathbf{N}. \tag{6.8}$$

We need to compute the $q_k(1)$ only for $k \leq l$.

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