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MULTIPLICATIVE BIJECTIONS BETWEEN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

Janez Mrčun and Peter Šemrl

University of Ljubljana, Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia; janez.mrcun@fmf.uni-lj.si

University of Ljubljana, Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia; peter.semrl@fmf.uni-lj.si

Abstract. We show that any multiplicative bijection between the algebras of differentiable functions, defined on differentiable manifolds of positive dimension, is an algebra isomorphism, given by composition with a unique diffeomorphism.

1. Introduction

In the theory of classical algebras, the problem of characterizing automorphisms is of fundamental importance. In the case of the algebra of smooth functions on a smooth manifold, every automorphism is a composition operator.

It has been known for a long time that the linear structure of an algebra is often completely determined by the multiplicative one. Already in 1940, Eidelheit [3] observed that the multiplicative bijective maps on real operator algebras are automatically linear. An interested reader can find a pure ring-theoretic result on automatic additivity of multiplicative maps in [5]. However, this result is not relevant for the case of commutative rings. In the commutative case, the situation is more complicated. For example, suppose that $\tau\colon X \to Y$ is a homeomorphism of compact Hausdorff spaces, and let p be a positive continuous function on X. Then the map $\mathscr{T}\colon C(Y) \to C(X)$ between the corresponding algebras of real continuous functions, given by

$$\mathscr{T}(g)(x) = |g(\tau(x))|^{p(x)} \operatorname{sign}(g(\tau(x))), \quad x \in X,\ g \in C(Y),$$

is a bijective multiplicative map, non-linear if $p \neq 1$. It turns out [6] that in the absence of isolated points, every semigroup isomorphism of $C(Y)$ onto $C(X)$ is of this form. This result shows that the multiplicative semigroup structure of $C(X)$ completely determines the underlying space X. Namely, if $C(X)$ and $C(Y)$ are isomorphic as multiplicative semigroups, then by the above result the spaces X and Y are homeomorphic.

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The aim of this note is to show that the result is much nicer in the case of real differentiable functions on differentiable manifolds. Here, any semigroup isomorphism is automatically linear. More precisely:

Theorem 1. *Let M and N be Hausdorff C^r-manifolds of positive dimension, $1 \leq r < \infty$. Then for any multiplicative bijection $\mathscr{B}\colon C^r(N) \to C^r(M)$ there exists a unique C^r-diffeomorphism $\phi\colon M \to N$ such that*

$$\mathscr{B}(g)(x) = g(\phi(x))$$

for any $g \in C^r(N)$ and any $x \in M$. In particular, the map $\mathscr{B}$ is an algebra isomorphism.

It should be mentioned that the manifolds M and N appearing in Theorem 1 are not assumed to be second-countable, paracompact or connected. Even in the presence of the linearity assumption, the above result has been proved in full generality only very recently [4, 7]. As pointed out by Weinstein in 2003, all the earlier proofs (e.g. [2]) depend heavily on the second-countability assumption.

Unfortunately our methods do not work for the case $r = \infty$. However, it is tempting to believe that the result holds in this case as well.

2. Proof of the main theorem

Throughout this paper, we assume that M and N are Hausdorff C^r-manifolds, not necessarily second-countable, paracompact or connected, $r = 0, 1, \ldots, \infty$, and that

$$\mathscr{B}\colon C^r(N) \to C^r(M)$$

is a multiplicative bijection between the algebras of real C^r-functions on N, respectively M.

First, we will show that $\mathscr{B}$ induces a homeomorphism $M \to N$ in a natural way, by means of characteristic sequences of functions introduced in [7]. Recall that a sequence (f_i) of C^r-functions on M is called *characteristic at $x \in M$* if

(i) $f_i f_{i+1} = f_{i+1}$ for any i, and
(ii) the associated sequence of supports $(\mathrm{supp}(f_i))$ is a fundamental system of neighbourhoods of x in M.

In particular, if (f_i) is a characteristic sequence of C^r-functions on M at x, then $\bigcap_i \mathrm{supp}(f_i) = \{x\}$, f_i equals 1 on $\mathrm{supp}(f_{i+1})$ and $\mathrm{supp}(f_i)$ is compact for any i large enough [7, Lemma 3].

For any point $x \in M$ we can choose a characteristic sequence (f_i) of C^r-functions on M at x. It follows from the proof of [7, Lemma 4] (which is stated for isomorphisms of algebras, but its proof only uses the fact that the map between algebras is a multiplicative bijection) that the sequence $(\mathscr{B}^{-1}(f_i))$ of C^r-functions on N is characteristic at a point $\phi(x) \in N$, and that this point is independent on the choice of the sequence (f_i). In particular, we obtain a map

$$\phi\colon M \to N.$$

By symmetry, the same construction can be applied to $\mathscr{B}^{-1}$, and the associated map $N \to M$ is clearly the inverse of ϕ. In particular, the map ϕ is a bijection.

We would like to show that for any $g \in C^r(N)$, the value of $\mathscr{B}(g)$ at $x \in M$ depends only on the value of g at $\phi(x)$. Actually, this is the core of our problem, and we will need to do several small steps before achieving this goal. First, we will see that the value $\mathscr{B}(g)(x)$ depends only on the germ of g at $\phi(x)$. We shall denote by $\mathrm{germ}_{\phi(x)}(g) = g_{\phi(x)}$ the germ of g at $\phi(x)$, and by C^r_N the sheaf of germs of all C^r-functions on N.

Lemma 2. *For any $g, h \in C^r(N)$ and any $x \in M$ we have:*

(i) $g_{\phi(x)} = h_{\phi(x)}$ *if and only if* $\mathscr{B}(g)_x = \mathscr{B}(h)_x$,
(ii) $g_{\phi(x)} = 1$ *if and only if* $\mathscr{B}(g)_x = 1$,
(iii) $g_{\phi(x)} = -1$ *if and only if* $\mathscr{B}(g)_x = -1$,
(iv) $g(\phi(x)) = 0$ *if and only if* $\mathscr{B}(g)(x) = 0$,
(v) $g(\phi(x)) > 0$ *if and only if* $\mathscr{B}(g)(x) > 0$,
(vi) $g(\phi(x)) < 0$ *if and only if* $\mathscr{B}(g)(x) < 0$.

In particular $\mathscr{B}(1) = 1$, $\mathscr{B}(-1) = -1$, $\mathscr{B}(-g) = -\mathscr{B}(g)$, *and* $\mathscr{B}(0) = 0$.

Proof. First note that by symmetry it is sufficient to prove only one of the implications in all of the equivalences above.

(i) Suppose that $g_{\phi(x)} = h_{\phi(x)}$. Choose a characteristic sequence (f_i) of C^r-functions on M at x, and put $g_i = \mathscr{B}^{-1}(f_i)$ for any i. By the definition of ϕ, the sequence (g_i) is characteristic at $\phi(x)$. In particular, the sequence $(\mathrm{supp}(g_i))$ is a fundamental system of neighbourhoods of $\phi(x)$, hence there exists k large enough so that

$$gg_k = hg_k.$$

Since $\mathscr{B}$ is multiplicative, this implies

$$\mathscr{B}(g)f_k = \mathscr{B}(h)f_k. \tag{1}$$

Because (f_i) is characteristic at x, the function f_k equals 1 on a neighbourhood of x, so (1) gives $\mathscr{B}(g)_x = \mathscr{B}(h)_x$.

(ii) The map $\mathscr{B}$ is a multiplicative bijection, therefore it preserves the unit, i.e. $\mathscr{B}(1) = 1$. If $g_{\phi(x)} = 1$, it follows from (i) that $\mathscr{B}(g)_x = \mathscr{B}(1)_x = 1$.

(iii) If $g_{\phi(x)} = -1$, then $g^2_{\phi(x)} = 1$, so by (ii) we have $\mathscr{B}(g)^2_x = \mathscr{B}(g^2)_x = 1$. Because (ii) implies that $\mathscr{B}(g)_x \neq 1$, it follows that $\mathscr{B}(g)_x = -1$.

(iv) If $g(\phi(x)) \neq 0$, there exists $w \in C^r(N)$ such that $(gw)_{\phi(x)} = g_{\phi(x)} w_{\phi(x)} = 1$. It follows from (i),(ii) and the multiplicativity of $\mathscr{B}$ that $\mathscr{B}(gw)_x = \mathscr{B}(g)_x \mathscr{B}(w)_x = 1$, and in particular $\mathscr{B}(g)(x) \neq 0$.

(v) If $g(\phi(x)) > 0$, there exists $w \in C^r(N)$ such that $w^2_{\phi(x)} = g_{\phi(x)}$. It follows that $\mathscr{B}(w)^2_x = \mathscr{B}(w^2)_x = \mathscr{B}(g)_x$, so $\mathscr{B}(g)(x) \geq 0$. On the other hand, we know from (iv) that $\mathscr{B}(g)(x) \neq 0$.

(vi) This follows directly from (iv) and (v). $\square$

Lemma 3. *The map $\phi\colon M \to N$ is a homeomorphism. Furthermore, the map $\mathscr{B}$ induces a multiplicative homeomorphism of sheaves $C^r_N \to C^r_M$ over ϕ^{-1}, again denoted by $\mathscr{B}$, which is given by*

$$\mathscr{B}(g_{\phi(x)}) = \mathscr{B}(g)_x$$

for any $x \in M$ and $g \in C^r(N)$.

Proof. Take any open subset V of N and let $x \in \phi^{-1}(V)$. Choose a function $g \in C^r(N)$ with $\mathrm{supp}(g) \subset V$ such that $g(\phi(x)) \neq 0$, and put

$$U = \{x' \in M \mid \mathscr{B}(g)(x') \neq 0\}.$$

First observe that U is open in M because $\mathscr{B}(g)$ is continuous. It follows from Lemma 2 (iv) that $U \subset \phi^{-1}(V)$ and that $x \in U$. This shows that $\phi^{-1}(V)$ is open. We therefore conclude that ϕ is continuous.

A symmetrical argument shows that ϕ^{-1} is continuous as well, so ϕ is a homeomorphism. The rest of the statement follows from Lemma 2 (i). □

As a consequence of Lemma 3, the manifolds M and N have the same dimension. From now on we will assume that $n = \dim M = \dim N \geq 1$.

It also follows from Lemma 3 that for any open subset V of N we have the multiplicative bijection $\mathscr{B}_V\colon C^r(V) \to C^r(\phi^{-1}(V))$ such that $\mathscr{B}_V(h)_x = \mathscr{B}(h_{\phi(x)})$ for any $h \in C^r(V)$ and for any $x \in \phi^{-1}(V)$. Observe that to prove Theorem 1 it is sufficient to find an open cover (V_j) of N such that the multiplicative bijection $\mathscr{B}_{V_j}$ is given by composition with $\phi|_{\phi^{-1}(V_j)}$, for any j. For instance, it is suitable to choose the cover so that V_j and $\phi^{-1}(V_j)$ are coordinate charts on N, respectively M.

Define a map $\mathscr{A}\colon C^r(N) \to C^r(M)$ by

$$\mathscr{A}(g) = \ln \circ \mathscr{B}(\exp \circ g).$$

Note that $\mathscr{A}$ is an additive bijection, with inverse $\mathscr{A}^{-1}(f) = \ln \circ \mathscr{B}^{-1}(\exp \circ f)$. The properties of $\mathscr{B}$, stated in Lemma 2 and Lemma 3, obviously translate into analogous properties of the map $\mathscr{A}$.

Corollary 4. *For any $g, h \in C^r(N)$ and any $x \in M$ we have $g_{\phi(x)} = h_{\phi(x)}$ if and only if $\mathscr{A}(g)_x = \mathscr{A}(h)_x$. Furthermore, the map $\mathscr{A}$ induces an additive homeomorphism of sheaves $\mathscr{A}\colon C^r_N \to C^r_M$ over ϕ^{-1}, and extends to additive bijections $\mathscr{A}_V\colon C^r(V) \to C^r(\phi^{-1}(V))$ by $\mathscr{A}_V(h)_x = \mathscr{A}(h_{\phi(x)})$, for all open subsets V of the manifold N.*

Let f be a real C^r-function defined on an open subset U of M. Suppose that $(x_1, \ldots, x_n)\colon W \to \mathbf{R}^n$ are local coordinates on an open subset W of U, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq r$ ($\alpha_i \in \mathbf{N} \cup \{0\}$). We use the standard notation

$$D^\alpha(f) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for the partial α-derivative of f of order $|\alpha|$ on W. We shall write $j^k_x(f)$ for the k-jet of f at a point $x \in U$, $k = 0, 1, \ldots, r$. It is an equivalence class of real C^r-functions defined on open neighbourhoods of x, with two functions belonging to the same k-jet at x if and only if they have the same partial derivatives at x of orders $0, 1, \ldots, k$ with respect to some (or any) local coordinates around x (see [8]).

Lemma 5. *If $g, h \in C^r(N)$ satisfy $j^r_{\phi(x)}(g) = j^r_{\phi(x)}(h)$ for some $x \in M$, then $j^r_x(\mathscr{A}(g)) = j^r_x(\mathscr{A}(h))$.*

Proof. By additivity of $\mathscr{A}$ we can assume without loss of generality that $h = 0$. Choose a C^r-diffeomorphism $\psi \colon V \to \mathbf{R}^n$, defined on an open neighbourhood $V \subset N$ of $\phi(x)$, such that $\psi(\phi(x)) = 0$. The function $g \circ \psi^{-1} \in C^r(\mathbf{R}^n)$ satisfies $j^r_0(g \circ \psi^{-1}) = j^r_0(0)$ because $j^r_{\phi(x)}(g) = j^r_{\phi(x)}(0)$. Hence, by the Whitney extension theorem [9], there exists a function $w \in C^r(\mathbf{R}^n)$ such that

$$w|_{[-1,0]^n} = 0$$

and

$$w|_{[0,1]^n} = (g \circ \psi^{-1})|_{[0,1]^n}.$$

Since $g|_V = w \circ \psi + (g|_V - w \circ \psi)$, it follows that

$$\mathscr{A}(g)_x = \mathscr{A}(g_{\phi(x)}) = \mathscr{A}((w \circ \psi)_{\phi(x)}) + \mathscr{A}(g_{\phi(x)} - (w \circ \psi)_{\phi(x)}). \tag{2}$$

By construction of w we have $w|_{[-1,0]^n} = 0$ and $(g \circ \psi^{-1} - w)|_{[0,1]^n} = 0$. Therefore, Corollary 4 implies that

$$\mathscr{A}_V(w \circ \psi)_{\phi^{-1}(\psi^{-1}(u))} = 0$$

and

$$\mathscr{A}_V(g|_V - w \circ \psi)_{\phi^{-1}(\psi^{-1}(v))} = 0$$

for any $u \in (-1, 0)^n$ and any $v \in (0, 1)^n$. In particular, all the partial derivatives of order $0, 1, \ldots, r$ of $\mathscr{A}_V(w \circ \psi)$ and $\mathscr{A}_V(g|_V - w \circ \psi)$ are zero arbitrary close to x. Since these two are both C^r-functions on a neighbourhood of x, this implies

$$j^r_x(\mathscr{A}_V(w \circ \psi)) = j^r_x(0)$$

and

$$j^r_x(\mathscr{A}_V(g|_V - w \circ \psi)) = j^r_x(0),$$

therefore $j^r_x(\mathscr{A}(g)) = j^r_x(0)$ by (2). $\square$

Lemma 6. *Suppose that $r < \infty$ and that M and N are open subsets of $\mathbf{R}^n$. There exist an open subset R with discrete complement in M and continuous real functions p_α on R, for any multi-index α of order $|\alpha| \le r$, such that*

$$\mathscr{A}(g)(x) = \sum_{|\alpha| \le r} p_\alpha(x) D^\alpha(g)(\phi(x))$$

for any $g \in C^r(N)$ and any $x \in R$.

Proof. Denote by $\Pi_r = \mathbf{R}_r[t_1, \ldots, t_n]$ the finite dimensional subspace of $C^r(N)$ of polynomials of order at most r. Recall that any polynomial in Π_r is uniquely determined by the values of all its partial derivatives of order at most r at any fixed point $y \in N$, and that this parameterization of Π_r is linear.

For any $x \in M$ define an additive map $\Phi_x \colon \Pi_r \to \mathbf{R}$ by

$$\Phi_x(P) = \mathscr{A}(P)(x),$$

and let

$$R = \{x \in M \mid \Phi_x \text{ is bounded on a neighbourhood of } 0 \in \Pi_r\}.$$

First, we will show that $M \setminus R$ is closed and discrete in M. To this end, suppose that (x_i) is an injective sequence of points in $M \setminus R$ which converges to $x \in M$. Put $y_i = \phi(x_i)$, choose a positive decreasing sequence (ε_i) converging to 0 such that the open balls $K(y_i, \varepsilon_i)$ are pairwise disjoint subsets of N, and choose $h_i \in C^\infty(N)$ with compact support in $K(y_i, \varepsilon_i)$ such that

$$(h_i)_{y_i} = 1$$

for any i. By the definition of R we can find for every i a polynomial $P_i \in \Pi_r$ such that

$$|D^\alpha(h_i P_i)(y)| < 1/i$$

for any $y \in N$ and any multi-index α of order $|\alpha| \leq r$, and

$$\mathscr{A}(P_i)(x_i) \geq i.$$

These assumptions imply that the sum

$$h = \sum_i h_i P_i$$

and all its partial derivatives of order at most r converge uniformly on N, thus $h \in C^r(N)$. On the other hand, by Corollary 4 we have

$$\mathscr{A}(h)(x_i) = \mathscr{A}(P_i)(x_i) \geq i,$$

which contradicts the continuity of $\mathscr{A}(h)$ at x. The set $M \setminus R$ is therefore closed and discrete in M.

Take any $x \in R$. By [1, page 35] it follows that the additive map Φ_x is in fact linear. If we linearly parameterise Π_r by the partial derivatives of the polynomials in Π_r at $\phi(x)$, we obtain unique real numbers $p_\alpha(x)$, for $|\alpha| \leq r$, such that

$$\Phi_x(P) = \mathscr{A}(P)(x) = \sum_{|\alpha| \leq r} p_\alpha(x) D^\alpha(P)(\phi(x)) \tag{3}$$

for any $P \in \Pi_r$. By induction on $|\alpha|$ we can check that all the functions p_α are continuous on R. Indeed, if we take P to be the homogeneous polynomial $P(t_1, \ldots, t_n) = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, we obtain from (3) an explicit polynomial expression for $p_\alpha(x)$ in terms of $\mathscr{A}(P)(x)$, $\phi(x)$ and $p_\beta(x)$, for $|\beta| < |\alpha|$.

Take any $g \in C^r(N)$. Let $P_g \in \Pi_r$ be the Taylor polynomial of g of order r around $\phi(x)$, i.e. $j^r_{\phi(x)}(P_g) = j^r_{\phi(x)}(g)$. It follows from Lemma 5 that $\mathscr{A}(P_g)(x) = \mathscr{A}(g)(x)$. The equation (3) therefore implies

$$\mathscr{A}(g)(x) = \sum_{|\alpha| \le r} p_\alpha(x) D^\alpha(g)(\phi(x))$$

for any $g \in C^r(N)$ and any $x \in R$. $\square$

Lemma 7. *Suppose that $r < \infty$. Then*

$$\mathscr{A}(g)(x) = \mathscr{A}(1)(x) g(\phi(x))$$

for any $g \in C^r(N)$ and any $x \in M$.

Proof. By Corollary 4 we can assume without loss of generality that M and N are open subsets of $\mathbf{R}^n$. By Lemma 6 there exist an open subset R with discrete complement in M and continuous real functions p_α on R, for any multi-index α of order $|\alpha| \le r$, such that

$$\mathscr{A}(g)(x) = \sum_{|\alpha| \le r} p_\alpha(x) D^\alpha(g)(\phi(x)) \tag{4}$$

for any $g \in C^r(N)$ and any $x \in R$. Observe that $p_{(0,\dots,0)} = \mathscr{A}(1)$.

Let I be the set of all multi-indices α of order $|\alpha| \le r$ such that p_α is not identically zero on R. Note that $I \neq \emptyset$ because $\mathscr{A} \neq 0$. Choose $\alpha \in I$ with a component α_i which is maximal among all the components of all multi-indices in I, i.e. $\alpha_i \ge \beta_j$ for any $\beta \in I$ and any $1 \le j \le n$. We will show that $\alpha_i = 0$, which implies $I = \{(0, \dots, 0)\}$.

Suppose that $\alpha_i > 0$. Observe that $k = r + 1 - \alpha_i$ satisfies $0 < k \le r$. Since p_α is not identically zero on R, there exists an open non-empty connected subset U of R such that p_α has no zeros on U. Take any $a \in U$ and choose $\varepsilon > 0$ so small that $K(a, \varepsilon) \subset U$. Define a multi-index α' by $\alpha'_i = r + 1$ and $\alpha'_l = \alpha_l$ for any $l \neq i$.

Let $P \in C^r(N)$ be the homogeneous polynomial function given by

$$P(t) = (t - \phi(a))^{\alpha'} = (t_1 - \phi_1(a))^{\alpha'_1} \cdots (t_n - \phi_n(a))^{\alpha'_n},$$

where $t = (t_1, \dots, t_n) \in N \subset \mathbf{R}^n$ and $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ for any $x \in M$.

Take any $\beta \in I$. We write $\beta \le \alpha'$ if $\beta_l \le \alpha'_l$ for all l. For the derivative $D^\beta(P)$ we have the following two possibilities:

(i) If $\beta \not\le \alpha'$, then $D^\beta(P) = 0$.

(ii) If $\beta \le \alpha'$, then we have in fact $\beta \le \alpha$ (because $\alpha_i \ge \beta_i$ by the maximality of the component α_i of α) and

$$D^\beta(P)(t) = c_\beta\, (t - \phi(a))^{\alpha' - \beta} = c_\beta\, (t - \phi(a))^{\alpha - \beta} (t_i - \phi_i(a))^k$$

for some non-zero $c_\beta \in \mathbf{R}$. The polynomial $c_\beta\, (t - \phi(a))^{\alpha-\beta}$ is constant and non-zero only in the case $\beta = \alpha$. In all other cases it has value 0 at $\phi(a)$. Put $I' = \{\beta \in$

$I \mid \beta \leq \alpha\,,\ \beta \neq \alpha\}$ and denote

$$w(t) = \sum_{\beta \in I'} c_\beta\, p_\beta(\phi^{-1}(t))(t - \phi(a))^{\alpha-\beta}.$$

This is a continuous function of $t \in \phi(R)$ which equals 0 at $\phi(a)$.

Note that $j^r_{\phi(a)}(P) = j^r_{\phi(a)}(0)$, so Lemma 5 implies that $j^r_a(\mathscr{A}(P)) = j^r_a(0)$. Choose any $j = 1, \ldots, n$, and denote by e_j the j-th vector of the standard basis of $\mathbf{R}^n$. The Taylor formula for $\mathscr{A}(P)$ at a gives for any $|h| < \varepsilon$ a real number $0 < \vartheta < 1$ such that

$$\mathscr{A}(P)(a + he_j) = \frac{1}{k!}\frac{\partial^k \mathscr{A}(P)}{\partial t_j^k}(a + \vartheta h e_j) h^k. \tag{5}$$

Note that the function

$$z(h) = \frac{1}{k!}\frac{\partial^k \mathscr{A}(P)}{\partial t_j^k}(a + \vartheta h e_j)$$

is continuous in $h = 0$ and satisfies $z(0) = 0$.

On the other hand, from (4) it follows that

$$\begin{aligned} \mathscr{A}(P)(a + he_j) &= \sum_{|\beta| \leq r} p_\beta(a + he_j) D^\beta(P)(\phi(a + he_j)) \\ &= \sum_{\beta \in I} c_\beta p_\beta(a + he_j)(\phi(a + he_j) - \phi(a))^{\alpha' - \beta} \\ &= (w(\phi(a + he_j)) + c_\alpha\, p_\alpha(a + he_j))\,(\phi_i(a + he_j) - \phi_i(a))^k. \end{aligned} \tag{6}$$

By combining (5) and (6) we obtain for $0 < |h| < \varepsilon$

$$(w(\phi(a + he_j)) + c_\alpha\, p_\alpha(a + he_j)) \left(\frac{\phi_i(a + he_j) - \phi_i(a)}{h}\right)^k = z(h).$$

When h approaches 0, the right hand side converges to 0, but the first factor of the left hand side converges to $c_\alpha\, p_\alpha(a) \neq 0$, which is possible only if the limit

$$\lim_{h \to 0} \frac{\phi_i(a + he_j) - \phi_i(a)}{h} = \frac{\partial \phi_i}{\partial t_j}(a)$$

exists and equals 0. Since this is true for any j and any point $a \in U$, it follows that ϕ_i is constant on U. In particular, the restriction $\phi|_U$ is not open, which is in contradiction with the fact that ϕ is a homeomorphism.

We can therefore conclude that $I = \{(0, \ldots, 0)\}$ and hence

$$\mathscr{A}(g)(x) = \mathscr{A}(1)(x) g(\phi(x))$$

for any $g \in C^r(N)$ and any $x \in R$. Because R is dense in M and both sides of the last equation are continuous functions of x, defined on all of M, it follows that the equality holds true for any $x \in M$. $\square$

Proof of Theorem 1. For any positive $g \in C^r(N)$ we have $\mathscr{B}(g) = \exp\circ\mathscr{A}(\ln\circ g)$, so Lemma 7 implies that

$$\mathscr{B}(g)(x) = g(\phi(x))^{p(x)} \tag{7}$$

for any $x \in M$, where $p = \mathscr{A}(1) = \ln \circ \mathscr{B}(\mathrm{e}1) \in C^r(M)$. Since the inverse ϕ^{-1} corresponds to the multiplicative bijection $\mathscr{B}^{-1}$, it follows analogously from Lemma 7 that there exists $q \in C^r(N)$ such that

$$\mathscr{B}^{-1}(f)(y) = f(\phi^{-1}(y))^{q(y)} \tag{8}$$

for any positive $f \in C^r(M)$ and for any $y \in N$. Direct computation of the composition of $\mathscr{B}$ and $\mathscr{B}^{-1}$ shows that $(p \circ \phi^{-1})q = 1$, so in particular both p and q are nowhere zero. Furthermore, it follows from (7) that the composition of a positive C^r-function with ϕ is again a C^r-function, thus ϕ is of class C^r. Analogously, by (8) the map ϕ^{-1} is of class C^r, thus we may conclude that ϕ is a C^r-diffeomorphism.

We will now show that both $p = 1$ and $q = 1$. Because $(p \circ \phi^{-1})q = 1$, it is sufficient to show that $p \geq 1$ and $q \geq 1$. By symmetry it is enough to prove $p \geq 1$. So assume that there is a point $x \in M$ such that $p(x) < 1$. Since ϕ is a C^r-diffeomorphism, we may choose a C^r-path $\gamma\colon (-1,1) \to M$ with $\gamma(0) = x$ and a function $g \in C^r(N)$ such that $g(\phi(\gamma(t))) = t$ for any $t \in (-1,1)$. Note that $\sigma = p\circ\gamma$ and $u = \mathscr{B}(g) \circ \gamma$ are C^r-functions on $(-1,1)$. It follows from (7) and Lemma 2 (i) that

$$u(t) = t^{\sigma(t)}$$

for any $t > 0$. Since u is continuous on $(-1,1)$, this is possible only if $\sigma(0) \geq 0$. As p has no zeros, this implies $p(x) = \sigma(0) > 0$. By the continuity of σ we may choose $0 < a < b < 1$ and $0 < \varepsilon < 1$ such that $a < \sigma(t) < b$ for any $t \in (-\varepsilon, \varepsilon)$.

The derivative of u at a point $t \in (0,1)$ equals

$$\frac{t^{\sigma(t)}\sigma(t)}{t} + t^{\sigma(t)} \ln(t) \frac{d\sigma}{dt}(t). \tag{9}$$

Since the derivative of σ is bounded on a small neighbourhood of 0 and $\sigma(-\varepsilon,\varepsilon) \subset [a,b] \subset (0,1)$, it follows that the second summand of (9) converges to 0 as $t > 0$ approaches 0. On the other hand, the fact that $\sigma(-\varepsilon,\varepsilon) \subset [a,b] \subset (0,1)$ implies that the first summand of (9) is unbounded on any neighbourhood of 0. Hence $u \in C^r(-1,1)$ has unbounded derivative on any neighbourhood of 0, which is a contradiction.

Thus $p = 1$, and therefore

$$\mathscr{B}(g)(x) = g(\phi(x))$$

for any positive $g \in C^r(N)$ and any $x \in M$. Finally, it follows from Lemma 2 that this formula actually holds for any $g \in C^r(N)$. $\square$

References

[1] Aczél, J., and J. Dhombres: Functional equations in several variables. - Encyclopedia Math. Appl. 31, Cambridge Univ. Press, Cambridge, 1989.

[2] Bkouche, R.: Idéaux mous d'un anneau commutatif. Applications aux anneaux de fonctions. - C. R. Math. Acad. Sci. Paris 260, 1965, 6496–6498.

[3] Eidelheit, M.: On isomorphisms of rings of linear operators. - Studia Math. 9, 1940, 97–105.

[4] Grabowski, J.: Isomorphisms of algebras of smooth functions revisited. - Arch. Math. (Basel) 85, 2005, 190–196.

[5] Martindale, W. S., III: When are multiplicative mappings additive? - Proc. Amer. Math. Soc. 21, 1969, 695–698.

[6] Milgram, A. N.: Multiplicative semigroups of continuous functions. - Duke Math. J. 16, 1949, 377–383.

[7] Mrčun, J.: On isomorphisms of algebras of smooth functions. - Proc. Amer. Math. Soc. 133, 2005, 3109–3113.

[8] Saunders, D. J.: The geometry of jet bundles. - London Math. Soc. Lecture Note Ser. 142, Cambridge Univ. Press, Cambridge, 1989.

[9] Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. - Trans. Amer. Math. Soc. 36, 1934, 63–89.

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