Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 32, 2007, 381–408

ON THE NEGATIVE CONVERGENCE OF THURSTON'S STRETCH LINES TOWARDS THE BOUNDARY OF TEICHMÜLLER SPACE

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Abstract. Stretch lines are geodesics for Thurston's asymmetric metric on Teichmüller space [10]. Each stretch line is directed by a complete geodesic lamination. An anti-stretch line directed by the complete geodesic lamination μ is a stretch line directed by μ traversed in the opposite direction. It is not necessarily a geodesic. In this paper, we tackle the problem of the convergence (or non-convergence) of anti-stretch lines towards a point of Thurston's boundary of Teichmüller space. We show that an anti-stretch line directed by a complete geodesic lamination μ which is made up of a compact and uniquely ergodic measured sublamination γ, with its other leaves spiraling around it, converges to the projective class of γ.

Introduction

Let S be a surface obtained by removing finitely many points—called the *punctures*—from an orientable closed surface $\hat{S}$, in such a way that the Euler characteristic of S is negative. The *Teichmüller space* $\mathscr{T}(S)$ of S is the set of isotopy classes of complete hyperbolic metrics with finite area on S. If S is endowed with such a complete hyperbolic metric of finite area, then each puncture has a neighborhood which is isometric to the quotient of $\{z = x + iy \in \mathbf{C} \ : \ y > a > 0\} \subset \mathbf{H}^2$ by the group generated by the translation $z \mapsto z+1$. Such a neighborhood is called a *cusp*.

In what follows, *hyperbolic structure* shall stand for an isotopy class of complete hyperbolic metrics with finite area on S, that is, an element of $\mathscr{T}(S)$.

This paper is about a geometry on $\mathscr{T}(S)$. This geometry is defined by an asymmetric Finsler metric L which measures the smallest Lipschitz constant of homeomorphisms isotopic to the identity from a hyperbolic structure on S to another one.

Some (oriented) geodesic rays for this metric are obtained by *stretching* a given hyperbolic structure along a given *complete* geodesic lamination, that is, along a geodesic lamination μ such that every component of $S \setminus \mu$ is isometric to the interior of an ideal triangle. When the surface S has punctures, simple examples of complete geodesic laminations are provided by *ideal triangulations* of S, that is, by

2000 Mathematics Subject Classification: Primary 30F60, 57M50, 53C22.

Key words: Teichmüller space, hyperbolic structure, geodesic lamination, stretch, Thurston's boundary, measured foliation.

triangulations of the surface $\hat{S}$ whose vertices are the punctures and whose edges in S are (infinite) geodesics. Then stretching a hyperbolic structure on S along such a kind of complete geodesic lamination μ amounts to increasing by the same multiplicative factor the shifts between adjacent ideal triangles of $S \setminus \mu$ (see Figure 1). We shall recall the general definition of stretching in the next section.

The *stretch line* directed by μ and passing through $g \in \mathscr{T}(S)$ is an oriented geodesic line in $\mathscr{T}(S)$ extending naturally the oriented ray obtained by stretching the structure g along the complete geodesic lamination μ. An *anti-stretch line* is a stretch line with the reverse orientation. For instance, if S has punctures and if μ is an ideal triangulation, then following the anti-stretch line corresponding to a stretch line directed by μ, accordingly to its orientation, amounts to decreasing by the same factor the shifts between adjacent ideal triangles of $S \setminus \mu$. As aforesaid, one of the main features concerning the metric L is that it is not symmetric. Therefore, an anti-stretch line is not necessarily a geodesic. (See [10], [6] and [9] for an account of this geometry.)

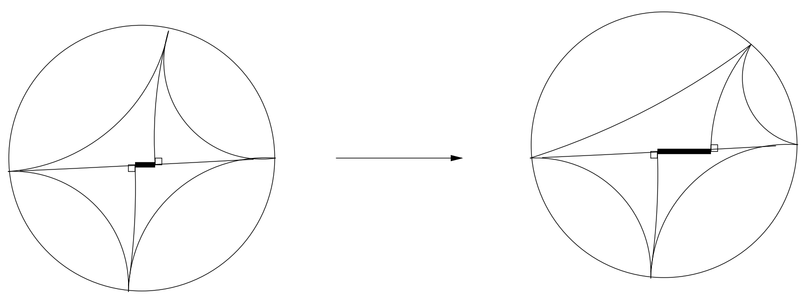

Figure 1. A stretch along an ideal triangulation of an ideal square in $\mathbf{H}^2$. The shift between the two adjacent ideal triangles is the signed length of the segment in bold line.

The Teichmüller space $\mathscr{T}(S)$, endowed with the topology making close two hyperbolic structures $g, g' \in \mathscr{T}(S)$ for which the g-length and the g'-length of any simple closed geodesic are close, has a celebrated compactification by $\mathscr{PL}_0(S)$, the space of projective classes of measured geodesic laminations with compact support. The boundary of $\mathscr{T}(S)$ provided by this compactification is called *Thurston's boundary* of Teichmüller space.

We are interested here in the convergence (or non convergence) of stretch lines, in both directions, towards Thurston's boundary of Teichmüller space. Specifically, consider the stretch line in $\mathscr{T}(S)$ directed by μ and passing through the point g. Let denote it by $t \mapsto g_t$, $t \in \mathbf{R}$, where $g_0 = g$ and where t is the signed arc-length parameter for which the orientations of $\mathbf{R}$ and of the stretch line match. We shall say that the stretch line *positively converges* towards a point of Thurston's boundary if g_t converges to a point of $\mathscr{PL}_0(S)$ as $t \to +\infty$, and *negatively converges* if g_t converges as $t \to -\infty$. Our problem is to determine whether the stretch line positively and negatively converges and, in that case, to recognize the limit points.

The positive convergence has been fully solved by Papadopoulos in [6], where a partial answer to the negative convergence has also been given. He first showed that any stretch line directed by a complete geodesic lamination μ positively converges to a point on the boundary and gave this limit point (see Theorem 1.9 below). Moreover, he proved that when μ supports a unique (up to scalar multiples) transverse measure of full support, the stretch line directed by μ negatively converges and the negative limit point is the projective class of μ. One of our main results relaxes Papadopoulos' hypothesis, namely, the negative convergence is shown for all stretch lines directed by complete geodesic laminations μ whose maximal (with respect to inclusion) compact measured part—called the *stump* of μ—is not empty and supports a unique transverse measure, up to scalar multiples (see Theorem 3.2). In particular, our theorem also deals with punctured surfaces. As one may expect, the limit point is the projective class of the stump.

1. A short geometric account

In this section, we briefly recall and define the notions we are going to use throughout our paper. We first give some basic facts concerning geodesic laminations and measured foliations and then recall the definition of stretches.

Let S be endowed with a fixed hyperbolic metric. A *geodesic lamination* λ on S is a union of pairwise disjoint simple geodesics—the *leaves* of λ—forming a closed subset of S. A *transverse measure* (of full support) on a geodesic lamination is a positive Radon measure defined on each compact arc a transverse to λ, whose support is exactly $a \cap \lambda$ and which is invariant if we slide a along the leaves of λ by an isotopy respecting these leaves. A geodesic lamination carrying a transverse measure is called a *measured geodesic lamination*. The set of all measured geodesic laminations of compact support is denoted by $\mathscr{ML}_0(S)$. Since $\mathbf{R}_+^*$ acts on $\mathscr{ML}_0(S)$ by multiplying transverse measures by positive scalars, it is natural to consider the associated projective space $\mathscr{PL}_0(S)$. Thurston showed that $\mathscr{PL}_0(S)$, endowed with the quotient topology coming from that of $\mathscr{ML}_0(S)$ (see below), is compact and he used this space to compactify $\mathscr{T}(S)$ (see [4] for an equivalent description of Thurston's compactification using measured foliations, in the case where S is compact).

Definition 1.1. (Spiral) An infinite half-leaf l of a geodesic lamination λ is said to *spiral around* a leaf l' of λ if it does not go out to a cusp and if there are lifts $\tilde{l}$ and $\tilde{l}'$ of l and l' to the universal covering which have a common endpoint on the circle at infinity. A leaf l of λ is said to spiral around a geodesic sublamination γ of λ if there is a leaf l' of γ around which a half-leaf of l spirals.

An *isolated spiral* of a geodesic lamination λ is an isolated leaf that spirals around some sublamination of λ.

The existence of a transverse measure on λ rules out isolated spirals. Moreover, leaves of λ going out to cusps prohibit the existence of compactly supported transverse measures on λ. Thus, a geodesic lamination does not always carry a transverse measure of compact support. Nevertheless, when the geodesic lamination λ is not exclusively made up of leaves going in both directions towards cusps, there always exists a non-empty compact sublamination of λ admitting a transverse measure.

Definition 1.2. (Stump) Let λ be a geodesic lamination. The *stump* of λ is the maximal, with respect to inclusion, compact sublamination of λ admitting a transverse measure.

Note that the stump may carry a whole family of transverse measures, distinct even up to positive scalar multiplication. Let us now check that the stump is well-defined, and let us give a criterion for it to be non-empty.

Lemma 1.3.
(1) *Any geodesic lamination has a well-defined stump (which might be empty).*
(2) *The stump of a geodesic lamination is empty if and only if the leaves of the lamination all go in both directions towards cusps.*

Proof. (1) Suppose that a geodesic lamination λ admits two stumps γ_1 and γ_2. By the uniqueness of the decomposition of λ as a union of leaves, γ_1 and γ_2 cannot intersect transversely (see [2]). Therefore, $\gamma_1 \cup \gamma_2$ is a measured compact sublamination of λ. By maximality of the stumps γ_i, $i = 1, 2$, we have $\gamma_1 = \gamma_2$ setwise. This proves the first assertion.

(2) As previously pointed out, if there exists a leaf of λ admitting one end not converging to a cusp, then this end spirals around a compact measured geodesic sublamination of λ. The stump of λ contains that sublamination and is therefore non-empty. Conversely, if all the leaves of λ go at both ends towards cusps, then none of them can belong to a compact sublamination. Therefore, the stump of λ is empty. $\square$

Consequently, a geodesic lamination λ is the union of its compact stump (which might be empty) and of finitely many isolated and infinite leaves whose ends either spiral around γ or go towards cusps (see [3], [2]).

A priori, a geodesic lamination has been defined using a fixed hyperbolic metric on S. It turns out that there is a natural correspondence between the geodesic laminations associated to any two hyperbolic metrics. This correspondence enables us to define a geodesic lamination without specifying any underlying hyperbolic metric. In fact, corresponding geodesic laminations defined for various hyperbolic metrics on S are isotopic on S (see [11]). This correspondence will be extensively used throughout this paper. It stems from the fact that the circle at infinity associated to a hyperbolic universal covering of S can be defined topologically (for instance, using infinite expansions in terms of elements of the fundamental group of S), thus giving a canonical one-to-one correspondence between the circles at infinity of any two hyperbolic universal coverings over S. This correspondence restricts to the correspondence between geodesic laminations just mentioned above.

A geodesic lamination λ cuts the surface S into finitely many subsurfaces with boundary. In more rigourous terms, $S \setminus \lambda$ is a union of finitely many subsurfaces and, for any hyperbolic metric on S, the completion of such a subsurface is a complete hyperbolic surface of finite area with totally geodesic boundary. A geodesic lamination μ is *complete* if all the components of $S \setminus \mu$ are interiors of ideal triangles. This is equivalent to the fact that no extra leaf can be added to μ. The edges of the finitely many ideal triangles of $S \setminus \mu$ are leaves of μ and they are usually called the *frontier leaves* of μ. In general, the union of frontier leaves forms a proper and dense subset of μ: there are examples of geodesic laminations that possess infinitely many leaves. (In those examples, the intersection of an arc transverse to μ with the leaves of μ is a Cantor set.)

Definition 1.4. (Glued edge-to-edge) Two ideal triangles are *glued edge-to-edge* if they share a common frontier leaf.

For instance, the ideal triangles of an ideal triangulation are glued edge-to-edge.

In what follows, the transverse measures on geodesic laminations we shall consider will always be compactly supported. Consequently, when we will talk about a measured geodesic lamination λ, it will be always tacitely assumed to be compact. When we will consider a measured geodesic lamination λ forgetting its transverse measure, we shall sometimes talk about the *topological* lamination associated to λ and we shall also denote it by the same letter. For instance, a geodesic lamination λ will be said to be topologically contained in a geodesic lamination μ if the topological geodesic lamination λ is contained (as a set) in the topological geodesic lamination μ.

Given a hyperbolic structure $g \in \mathscr{T}(S)$, any measured geodesic lamination λ has a well-defined length, denoted by $\mathrm{length}_g(\lambda)$, which can be defined as follows: when λ is a simple closed geodesic, $\mathrm{length}_g(\lambda)$ equals the length of λ with respect to g. If one denotes by $k\lambda$ the simple closed geodesic λ endowed with a weight $k > 0$ (or equivalently, endowed with the transverse measure that is k times the number of intersection points with λ), then we set $\mathrm{length}_g(k\lambda) = k\,\mathrm{length}_g(\lambda)$. A fundamental theorem of Thurston asserts that the set of weighted simple closed geodesics is dense in $\mathscr{ML}_0(S)$, for a natural topology we shall recall below (see also [4]). The notion of length for measured geodesic laminations is the unique continuous extension of the length defined for weighted simple closed geodesics. Another way to define the length of a measured geodesic lamination λ is by covering λ with finitely many rectangles of disjoint embedded interiors, $R_1, \ldots, R_N$, glued along their edges, such that, in each rectangle R_i, the leaves of $\lambda \cap R_i$ join one "vertical" edge to the other. If one chooses a vertical edge ∂R_i for each rectangle R_i, $i = 1, \ldots, N$, and if $l(x)$ denotes the leaf of $\lambda \cap R_i$ passing through $x \in \lambda \cap \partial R_i$, then one has

$$\mathrm{length}_g(\lambda) = \sum_{i=1}^{N} \int_{\lambda \cap \partial R_i} \mathrm{length}_g(l(x))\, d\lambda(x),$$

where $d\lambda$ denotes the transverse measure of λ (see Figure 2).

Given two measured geodesic laminations λ and μ, one can define their *intersection number* $i(\lambda, \mu)$ as follows: Cover λ with finitely many rectangles of disjoint embedded interiors, $R_1, \dots, R_N$, glued along their edges, such that, in each rectangle R_i, the leaves of $\lambda \cap R_i$ join one vertical edge to the other, as above. Moreover, we can choose the rectangles in such a way that the leaves of μ intersect λ (if any) in the interior of those rectangles. Then,

$$i(\lambda, \mu) = \sum_{i=1}^{N} \int_{\lambda \cap \mu \cap R_i} d\lambda d\mu,$$

where $d\lambda$ and $d\mu$ denote the transverse measures of λ and μ respectively.

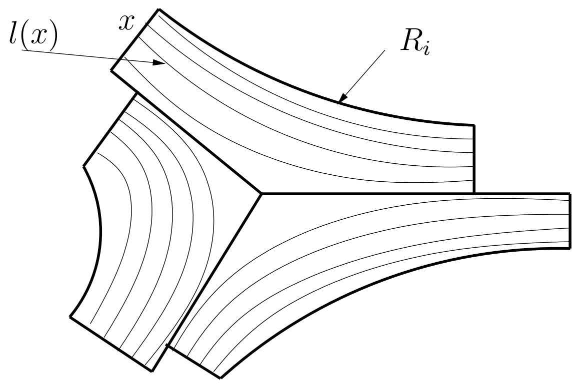

Figure 2. The picture shows three rectangles covering a (part of a) measured geodesic lamination λ. The length is computed by first summing in each rectangle R_i the lengths of all segments $l(x)$ using the transverse measure of λ and then by summing the numbers obtained for every rectangle.

When λ and μ are simple closed geodesics (seen as measured geodesic laminations endowed with the transverse measure given by the number of intersection points), one recovers the notion of geometric intersection number: Specifically, let $\mathscr{S}$ denote the set of homotopy classes of essential simple closed curves in S. Let $A, B \in \mathscr{S}$. Then one can define the *geometric intersection number* $i(A, B)$ by

$$i(A, B) = \inf_{a \in A, b \in B} \sharp a \cap b.$$

If, for a fixed hyperbolic metric on S, α and β denote the unique geodesic representatives of A and B respectively, one has $i(A, B) = i(\alpha, \beta) = \sharp \alpha \cap \beta$. (Note that there is a bijection between $\mathbf{R}_+^* \times \mathscr{S}$ and the set of weighted simple closed geodesics.) The topology on $\mathscr{M}\mathscr{L}_0(S)$ is defined by saying that two measured geodesic laminations λ and μ are close when the functions $i(\lambda, \cdot)$ and $i(\mu, \cdot)$ defined on $\mathscr{S}$ are close for the weak topology.

A measured foliation F is said to be *standard near the cusps* if every puncture has a neighborhood in which the leaves of F are homotopic to that puncture and if the transverse measure of a (non-compact) arc going out to a cusp is infinite. (Of course, if the surface has no cusp, then all the measured foliations are standard near cusps.)

There is a standard construction which gives a correspondence between the set of equivalence classes of measured foliations that are standard near the cusps and the set of measured geodesic laminations, including the empty lamination if and only if the surface S has cusps (see [5], [7]); the equivalence relation on measured foliations is generated by isotopies and Whitehead moves (see [4]). Under this correspondence, a weighted simple closed geodesic (or, equivalently, an element of $\mathbf{R}_+^* \times \mathscr{S}$) corresponds to the equivalence class of a foliation that is made up of finitely many foliated cylinders glued along their boundaries, one around each puncture and one whose leaves are homotopic to the closed geodesic.

For a given equivalence class F of measured foliations, one can also define the function $i(F, \cdot)$ on $\mathscr{S}$ which evaluates the minimal transverse intersection of an element $A \in \mathscr{S}$ with respect to any representative of F. One can extend this notion to a notion of intersection number $i(L, M)$ between two classes of measured foliations L, M (See [8]). Then one has, for a fixed hyperbolic structure on S, $i(L, M) = i(\lambda, \mu)$, where λ and μ are the measured geodesic laminations corresponding to L and M respectively. The notation $i(\lambda, M) = i(L, \mu)$ also makes sense in view of the above correspondence.

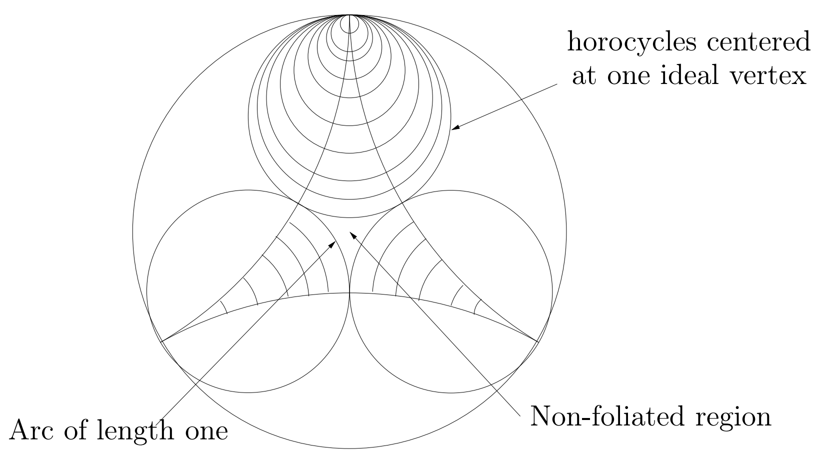

Figure 3. The horocyclic foliation in an ideal triangle.

We now recall some basic notions about stretches (see [10]). First of all, let us fix a complete geodesic lamination μ on S. To any hyperbolic metric m on S is associated a well-defined partial measured foliation $F_\mu(m)$, the latter being called the *horocyclic foliation* associated to the hyperbolic metric m and to the complete geodesic lamination μ. Let us briefly recall its construction. Let μ also denote the geodesic representative of μ with respect to the metric m. The partial foliation $F_\mu(m)$ will be first contructed in the interior of each ideal triangle of $S \setminus \mu$ and will then be extended to the whole surface S by continuity. So we consider an ideal triangle T. We partially foliate it with arcs contained in horocycles centered at the various vertices of T in such a way that arcs coming from two vertices eventually meet tangentially at some point (see Figure 3). There is only one way to do this.

Note that all these arcs are perpendicular to the edges of T. Also note that the three arcs that meet tangentially have length one and that they enclose a non-foliated triangle in T. The vertices of the non-foliated triangle are called the *distinguished points* of T. A transverse measure for this foliation of T is defined by requiring that the measure of a compact arc a contained in an edge of T is equal to the length of a. This construction defines $F_\mu(m)$ in the interior of each ideal triangle of $S \setminus \mu$. It is then extended to the whole surface; this is possible because the leaves of μ form a Lipschitz field of directions. If one wants to consider a genuine foliation (that is, not a partial foliation), one can collapse each triangular non-foliated region onto a tripod. Passing to the equivalence class $g \in \mathscr{T}(S)$ of m, this gives rise to a well-defined isotopy class of measured foliations on S denoted by $F_\mu(g)$ and also called the *horocyclic foliation* (even if it is an equivalence class).

Every horocyclic foliation is by construction transverse to μ and standard near the cusps. The latter property is equivalent to the completeness and the area-finiteness of the hyperbolic structure g the horocyclic foliation $F_\mu(g)$ comes from. Hence, one has a map

$$\phi_\mu \colon \mathscr{T}(S) \to \mathscr{MF}_0(S), \quad g \mapsto F_\mu(g),$$

where $\mathscr{MF}_0(S)$ denotes the *space of equivalence classes of measured foliations*. A fundamental result of Thurston (see [10] Proposition 9.2 and Proposition 10.9) is the following

Theorem 1.5. (Thurston [10]) *The map ϕ_μ is a homeomorphism onto its image, which is made up of all classes of measured foliations that are transverse to μ and standard near the cusps.*

The *stretch line* directed by μ and passing through $g \in \mathscr{T}(S)$ is the curve

$$t \mapsto g_t := \phi_\mu^{-1}(e^t F_\mu(g)),\ t \in \mathbf{R},\ g = g_0,$$

where the notation $kF_\mu(g)$, $k \in \mathbf{R}_+^*$, means that the transverse measure of the (class of the) foliation $F_\mu(g)$ has been multiplied by the scalar k. The *stretch ray* directed by μ and emanating from $g \in \mathscr{T}(S)$ is the curve

$$t \mapsto g_t := \phi_\mu^{-1}(e^t F_\mu(g)),\ t \geq 0,\ g = g_0.$$

Now recall that there is a natural correspondence between equivalence classes of measured foliations that are standard near the cusps and measured geodesic laminations (including the empty lamination if and only if the surface has cusps). We shall often prefer considering the measured geodesic lamination $\lambda_\mu(g)$ associated to the horocyclic foliation $F_\mu(g)$ under this correspondence instead of the foliation itself. We shall call it the *horocyclic measured geodesic lamination* or the *horocyclic lamination* for short. We emphasize right now that the horocyclic measured geodesic lamination might be empty.

Lemma 1.6. *Let S be a surface equipped with a complete geodesic lamination μ. There exists a hyperbolic structure $g \in \mathscr{T}(S)$ such that $\lambda_\mu(g) = \emptyset$ if and only if S has at least one cusp and μ is an ideal triangulation.*

Remark 1.7. Let γ denote the stump of μ. Using Lemma 1.3, another way to state the lemma above would be

$$\gamma \neq \emptyset \iff \lambda_\mu(g) \neq \emptyset,\ \forall g \in \mathscr{T}(S).$$

Proof. First note that, in the case where S has cusps and the complete geodesic lamination μ is an ideal triangulation, it quite is easy to produce a hyperbolic structure g for which $\lambda_\mu(g) = \emptyset$: It is obtained by gluing the ideal triangles in such a way that the distinguished points of two adjacent triangles coincide. This construction gives rise to a horocyclic foliation made up of foliated cylinders, glued together along their boundaries, each of them representing a foliated neighborhood of a cusp. The geodesic lamination corresponding to that (class of) foliation is the empty geodesic lamination.

Conversely, assume that there exists $g \in \mathscr{T}(S)$ such that $\lambda_\mu(g) = \emptyset$. Consider the horocyclic foliation $F_\mu(g)$ associated to $\lambda_\mu(g)$. As above, S must have at least one cusp and $F_\mu(g)$ is made up of foliated cylinders, each representing a foliated neighborhood of a cusp. By construction, the leaves of $F_\mu(g)$ are transverse to μ and $F_\mu(g)$ is standard near cusps. In particular, any leaf of μ entering a foliated cylinder of $F_\mu(g)$ cannot escape from that cylinder, and therefore goes out to a cusp. This proves that μ is an ideal triangulation. □

If μ is a complete geodesic lamination of non-empty stump γ, then one has by construction

$$i(F_\mu(g), \gamma) = i(\lambda_\mu(g), \gamma) = \mathrm{length}_g(\gamma).$$

Let $\mathscr{MF}_0(\mu) = \phi_\mu(\mathscr{T}(S))$, that is, let $\mathscr{MF}_0(\mu)$ be the set of equivalence classes of measured foliations transverse to μ and standard near the cusps. Let $\mathscr{ML}_0(\mu)$ denote the corresponding set of measured geodesic laminations (the empty lamination belongs to it if and only if μ is an ideal triangulation). We shall say that a geodesic lamination λ is *totally transverse* to μ if each leaf of λ intersects μ transversely infinitely many times and if each leaf of μ that does not go to a cusp meets λ transversely infinitely many times. (In counting intersections, the leaves are parametered by reals. With this convention, a simple closed geodesic meets transversely infinitely many times any compact transverse arc.) Thurston proved ([10], Proposition 9.4) that the non-empty measured geodesic laminations contained in $\mathscr{ML}_0(\mu)$ are exactly those measured geodesic laminations that are totally transverse to μ. We want to understand how the stump γ of μ is intersected by such a geodesic lamination $\lambda \in \mathscr{ML}_0(\mu)$. Assume γ to be non-empty and let $\mathscr{ML}_0(\gamma)$ denote the subset of $\mathscr{ML}_0(S)$ made up of those compact measured geodesic laminations that meet every component of γ transversely. We have

Lemma 1.8. $\mathscr{ML}_0(\gamma) = \mathscr{ML}_0(\mu)$.

Proof. Let us first show that $\mathscr{ML}_0(\gamma) \subset \mathscr{ML}_0(\mu)$. Let $\lambda \in \mathscr{ML}_0(\gamma)$. First note that no leaf of λ can be contained in μ: By contradiction, assume that there exists a leaf of λ contained in μ. This leaf cannot be a leaf of γ since λ meets every components of γ transversely. Therefore, it is contained in an isolated leaf of μ

which, because of the compactness of λ, spirals at both ends around γ. But such a spiral eventually meets another leaf of λ transversely, which is a contradiction. (Recall that the leaves of $\mu \setminus \gamma$ that spiral recursively come back around leaves of γ.) Since μ is complete and λ is compact, this implies that each leaf of λ meets infinitely many times μ transversely (with the convention above). Now, if l is a leaf of μ that spirals around the stump γ, then l meets infinitely many times λ transversely. Therefore, each leaf of μ which does not go to a cusp meets λ transversely infinitely many times. Thus, λ is totally transverse to μ, that is, $\lambda \in \mathscr{ML}_0(\mu)$.

Let us show the reverse inclusion. Let $\lambda \in \mathscr{ML}_0(\mu)$. Since every leaf of $\gamma \subset \mu$ does not go to a cusp and since λ is totally transverse to μ, each leaf of γ meets infinitely many times λ transversely, which implies that λ meets every component of γ transversely. Therefore, $\lambda \in \mathscr{ML}_0(\gamma)$.

This concludes the proof. $\square$

We close this section with Papadopoulos' theorem about positive convergence of stretch lines ([6], Theorem 5.1 p.169).

Theorem 1.9. (Papadopoulos [6]) *A stretch line passing through a point at which the horocyclic lamination is not empty positively converges to Thurston's boundary of Teichmüller space. The positive limit point is the projective class of the horocyclic lamination.*

2. Asymptotic behavior of lengths along an anti-stretch line

2.1. Classification theorem. In this section we describe the asymptotic behavior of the lengths of measured geodesic laminations as one follows an anti-stretch line. The classification we obtain depends upon the intersection pattern of the measured geodesic laminations with the stump, and is similar to the classification obtained in [9] as one follows a stretch line, the roles of the stump and of the horocyclic lamination having been interchanged. As a by-product of this classification, we can push further an analysis first made by Papadopoulos in [6] (Lemma 5.3 p.170) on the properties of cluster points of an anti-stretch line (See Corollary 3.1). This enables us to solve the negative convergence question for a whole family of stretch lines, namely those directed by complete geodesic laminations whose stumps are uniquely ergodic (See Theorem 3.2).

Theorem 2.1. (Classification Theorem) *Let μ be a complete geodesic lamination of stump γ and let $t \mapsto g_t$, $t \in \mathbf{R}$, be a stretch line directed by μ. The length of the measured geodesic lamination α of compact support behaves asymptotically according to the cases enumerated below:*

(1) *If α is topologically contained in γ, then* $\lim_{t\to-\infty} \mathrm{length}_{g_t}(\alpha) = 0$.
(2) *If α has a nonempty transverse intersection with γ, then* $\lim_{t\to-\infty} \mathrm{length}_{g_t}(\alpha) = +\infty$.
(3) *If α is disjoint from γ, then* $\{\mathrm{length}_{g_t}(\alpha) \ : \ t \le 0\}$ *is bounded in* $\mathbf{R}_+^*$.

Remark 2.2.

(1) Putting Theorem 2.1 and Theorem 2 of [9] together, we obtain the following table which gives the asymptotic behavior of $\mathrm{length}_{g_t}(\alpha)$, $\alpha \in \mathscr{ML}_0(S)$, along a stretch line, as t converges to $+\infty$ and $-\infty$. This behavior depends on the intersection pattern of α with the stump γ and the horocyclic lamination λ.

	$\alpha\cap\gamma=\emptyset$ $\alpha\cap\lambda=\emptyset$	$\alpha\cap\gamma=\emptyset$ $\alpha\cap\lambda\neq\emptyset$	$\alpha\cap\gamma\neq\emptyset$ $\alpha\cap\lambda=\emptyset$	$\alpha\cap\gamma\neq\emptyset$ $\alpha\cap\lambda\neq\emptyset$
$t\to+\infty$	$<\infty$	0 if $\alpha\subseteq\lambda$ ∞ if $\alpha\nsubseteq\lambda$	$<\infty$	0 if $\alpha\subseteq\lambda$ ∞ if $\alpha\nsubseteq\lambda$
$t\to-\infty$	$<\infty$	$<\infty$	0 if $\alpha\subseteq\gamma$ ∞ if $\alpha\nsubseteq\gamma$	0 if $\alpha\subseteq\gamma$ ∞ if $\alpha\nsubseteq\gamma$

(2) If the stump γ is empty, then μ is an ideal triangulation of S. In this case, for any $\alpha \in \mathscr{ML}_0(S)$, α is disjoint from γ and the set $\{\mathrm{length}_{g_t}(\alpha) \,:\, t \leq 0\}$ is bounded in $\mathbf{R}_+^*$. In fact, there exists a hyperbolic structure $g_\infty \in \mathscr{T}(S)$ such that all the stretch lines directed by μ and passing through points g different from g_∞ negatively converge towards g_∞. The hyperbolic structure g_∞ is obtained by gluing the ideal triangles of $S \setminus \mu$ edge-to-edge in such a way that the distinguished points of any two adjacent ideal triangles coincide. Using the homeomorphism ϕ_μ of Theorem 1.5, this structure is equivalently described by the fact that $\lambda_{g_\infty}(\mu)$ is empty (see Lemma 1.6). Stretching g_∞ along μ does not change anything, that is, the stretch line directed by μ passing through g_∞ is a point.

2.2. Proof of the Classification Theorem. Before proving Theorem 2.1, we make a preliminary discussion about rectangular coverings.

Let μ be a complete geodesic lamination of stump γ and let $t \mapsto g_t$, $t \in \mathbf{R}$, be a stretch line directed by μ.

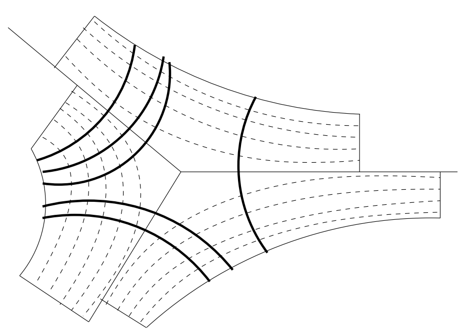

Figure 4. The picture shows three rectangles covering a (part of a) measured geodesic lamination α (in dotted lines). Some leaves of μ are represented in bold lines.

Let α be a measured geodesic lamination and assume that α has a nonempty transverse intersection with μ. Cover α by finitely many rectangles, $R_1, \ldots, R_N$, of disjoint embedded interiors, glued along their edges, such that, in each rectangle R_i, the leaves of $\alpha \cap R_i$, $i = 1, \ldots, N$, all join the vertical edges of R_i. Moreover, we can require that the leaves of μ cross the rectangles from the interior of one horizontal edge to the interior of the other (see Figure 4).

This construction can be made independent of the hyperbolic structure g_t: Consider the preimage $\widetilde{\mu}$ of μ in the universal covering $\widetilde{S}$ of S. Any endpoint of a geodesic not contained in $\widetilde{\mu}$ is the limit of a nested family of half-spaces bounded by edges of ideal triangles of $\widetilde{S} \backslash \widetilde{\mu}$. Since a geodesic of $\widetilde{S}$ is determined by its endpoints and since it intersects a leaf of $\widetilde{\mu}$ at most once, each leaf of α is determined by its intersection pattern with the leaves of $\widetilde{\mu}$ (see Figure 5). This pattern does not depend upon the underlying hyperbolic structure. Now it is easy to construct a rectangular cover $R_1, \ldots, R_N$ of α as above such that the intersection pattern of each leaf of $\alpha \cap R_i$ is independent of t.

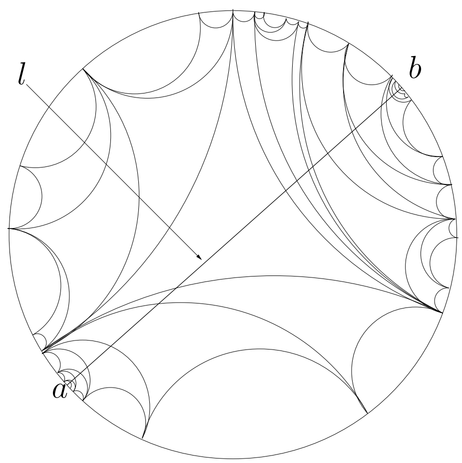

Figure 5. The picture shows a part of $\widetilde{\mu}$ and a geodesic l transverse to $\widetilde{\mu}$ with endpoints a, b. The ideal triangles bound nested half-spaces and the geodesic l and its endpoints are determined by the intersection pattern with $\widetilde{\mu}$.

In each rectangle R_i, let m_1^i and m_2^i be respectively the leftmost and the rightmost segment of $\mu \cap R_i$. These two segments of μ are well-defined, independently of t, and they belong to leaves of μ we also denote by m_1^i and m_2^i.

For each $i = 1, \ldots, N$ and for each $t \in \mathbf{R}$, consider a lift $\widetilde{R}_i$ of R_i to the universal covering and denote by $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ the leftmost and the rightmost segments of $\widetilde{\mu} \cap \widetilde{R}_i$ respectively. Denote by $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ the geodesics containing those segments as well. These geodesics are leaves of $\widetilde{\mu}$ that project on the leaves m_1^i and m_2^i.
If the geodesics $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ have no common endpoint, let δ_t^i be the unique geodesic segment joining them perpendicularly. Otherwise, they have only one common endpoint. For each $i = 1, \ldots, N$ and for each $t \in \mathbf{R}$, set

$$w_t^i := \begin{cases} \text{length}(\delta_t^i) > 0 & \text{if } \widetilde{m}_1^i \text{ and } \widetilde{m}_2^i \text{ have no common endpoint,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that w_t^i bounds from below the length of any curve joining $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$. In particular, w_t^i bounds from below the length of any curve in S joining the segments m_1^i and m_2^i. Consequently, if $\alpha(x)$ is the leaf of $\alpha \cap R_i$ passing through $x \in \alpha \cap \partial R_i$, we have, for all $t \in \mathbf{R}$, $\mathrm{length}_{g_t}(\alpha(x)) \geq w_t^i$. Hence, for all $t \in \mathbf{R}$,

$$\text{(2.1)} \qquad \mathrm{length}_{g_t}(\alpha) = \sum_{i=1}^{N} \int_{\partial R_i \cap \alpha} \mathrm{length}_{g_t}(\alpha(x))\, d\alpha(x) \geq \sum_{i=1}^{N} w_t^i \Big(\int_{\partial R_i \cap \alpha} d\alpha(x) \Big).$$

Note that, for all $i = 1, \ldots, N$, $\int_{\partial R_i \cap \alpha} d\alpha(x)$ is strictly positive and does not depend upon t.

The point now is to study the variations and the behavior of the functions $t \mapsto w_t^i$ as t converges to $-\infty$. This is the object of the following lemma whose proof is postponed to the next section. In order to state it, we first give some definitions.

Definition 2.3. (Separating geodesic, width) Let γ_1 and γ_2 be two disjoint geodesics of the hyperbolic plane. These two geodesics bound a closed infinite *strip*. A strip is also called a *wedge* if γ_1 and γ_2 have one common endpoint, and this common endpoint is the *vertex* of that wedge.

A geodesic γ is said to *separate* γ_1 and γ_2 if it is contained in that strip and if it intersects any curve joining one point of γ_1 to a point of γ_2 .

Likewise, a wedge, an ideal triangle, is said to *separate* γ_1 and γ_2 if it is contained in the strip and if two of its edges separate γ_1 and γ_2.

Two disjoint geodesics that have no common endpoint are said to be *ultraparallel*.

The *width* of a strip is, in the case where the edges of that strip are ultraparallel, the length of the unique geodesic segment joining them perpendicularly, or is zero otherwise.

Let $\widetilde{\mu}$ be a complete geodesic lamination of the hyperbolic plane. Two geodesics γ_1 and γ_2 are said to be *strongly separated by* $\widetilde{\mu}$ if they are ultraparallel and if there are infinitely many leaves of $\widetilde{\mu}$ that separate them.

Two geodesics γ_1 and γ_2 are said to be *weakly separated by* $\widetilde{\mu}$ if they are ultraparallel and if they are not strongly separated.

According to the preceding definitions, for each $i = 1, \ldots, N$, the function $t \mapsto w_t^i$ is the width of the strip bounded by the geodesics $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ in the hyperbolic plane.

Now we can state the width lemma which describes the behavior of those width functions.

Lemma 2.4. (Width lemma) *Let* $i \in \{1, \ldots, N\}$. *The function* $t \mapsto w_t^i$ *is either constantly equal to zero or is positive and strictly decreasing. Moreover, in the latter case, we have*

- $\lim_{t \to -\infty} w_t^i = +\infty \iff \widetilde{m}_1$ *and* $\widetilde{m}_2$ *are strongly separated by* $\widetilde{\mu}$.
- $\{w_t^i\}_{t \leq 0}$ *is bounded in* $\mathbf{R}_+^* \iff \widetilde{m}_1$ *and* $\widetilde{m}_2$ *are weakly separated by* $\widetilde{\mu}$.

Assuming the validity of the width lemma, let us start the proof of the classification theorem.

Proof of Theorem 2.1. The first assertion is easy since

$$\forall t \in \mathbf{R},\ \mathrm{length}_{g_t}(\gamma) = i(\lambda_\mu(g_t), \gamma) = e^t i(\lambda_\mu(g), \gamma) = e^t \mathrm{length}_g(\gamma).$$

Moreover, if γ is not empty, then $\mathrm{length}_g(\gamma) > 0$. Therefore, $\lim_{t\to-\infty} \mathrm{length}_{g_t}(\gamma) = 0$.

Let us prove Assertions (2) and (3).

Let α be a measured geodesic lamination which is not contained in μ. We shall use inequality 2.1 to prove Assertion (2) and the positive lower bound in Assertion (3). However, one has to choose carefully the rectangular covering $R_1, \ldots, R_N$ to insure the existence of at least one non-vanishing width function.

Lemma 2.5. *The rectangular cover $R_1, \ldots, R_N$ can be chosen in such a way that there exists an index $j \in \{1, \ldots, N\}$ so that $w_t^j > 0$ for one, hence all, $t \in \mathbf{R}$.*

Proof of Lemma 2.5. It is convenient to lift the situation to the universal covering. We keep here the notations we have already used above. We shall show that, given a rectangular cover $R_1, \ldots, R_N$ of α, we can modify it slightly so that there exists an index j for which the geodesics $\widetilde{m}_1^j$ and $\widetilde{m}_2^j$ are ultraparallel.

Suppose that for every i, the leaves $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ crossing $\widetilde{R}_i$ have a common endpoint p_i. (Here, $\widetilde{R}_i$ denotes a lift of R_i.) Consider any two adjacent rectangles $\widetilde{R}_i$ and $\widetilde{R}_j$ (that is, rectangles having intersecting vertical edges) and assume that the vertices p_i and p_j are different. Then we can widen a little bit the rectangle $\widetilde{R}_i$ so that $\widetilde{m}_2^i = \widetilde{m}_1^j$ (see Figure 6). (We do this operation in an equivariant way on the preimage of the rectangular cover so that it projects onto a rectangular cover in S.) The geodesics $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ are separated by leaves with different endpoints, which implies that $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ are ultraparallel.

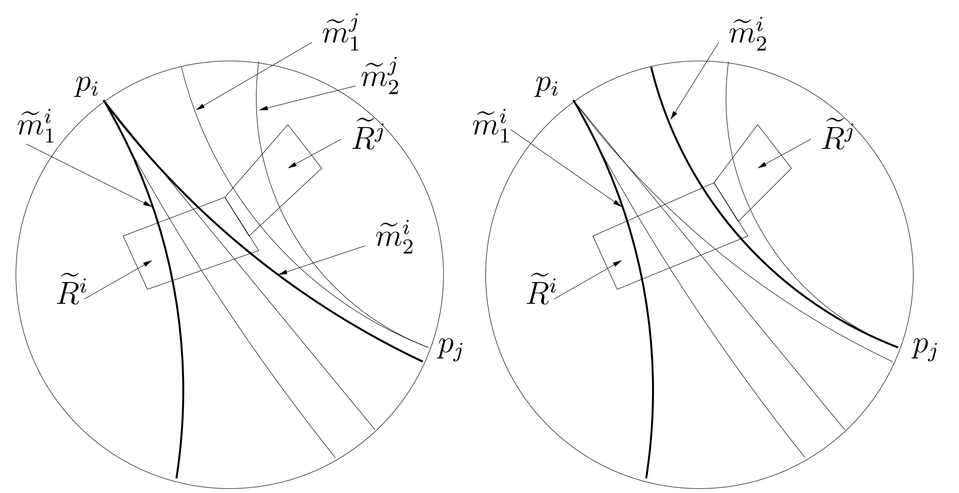

Figure 6. The picture shows how to widen $\widetilde{R}_i$ so that $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ are ultraparallel.

Now assume that any two adjacent rectangles $\widetilde{R}_i$ and $\widetilde{R}_j$ satisfy $p_i = p_j$. Take any leaf a of α and consider a lift $\widetilde{a}$ of it; this geodesic passes from one rectangle $\widetilde{R}_i$ to another adjacent one, etc. Let us cover $\widetilde{a}$ by those successive rectangles and let $\widetilde{\rho}$ denote their union. $\widetilde{\rho}$ contains $\widetilde{a}$ and is the union of infinitely many rectangles glued along their vertical edges. By assumption, all leaves of $\widetilde{\mu}$ intersecting $\widetilde{\rho}$ have a common endpoint p (see Figure 7). The rectangular cover $R_1, \ldots, R_N$ being compact, any lift of it is compact and, therefore, $\widetilde{\rho}$ is comprised between two horocycles centered at p. This implies that the endpoints of $\widetilde{a}$ are equal to p, which is absurd. This concludes the proof of the lemma. □

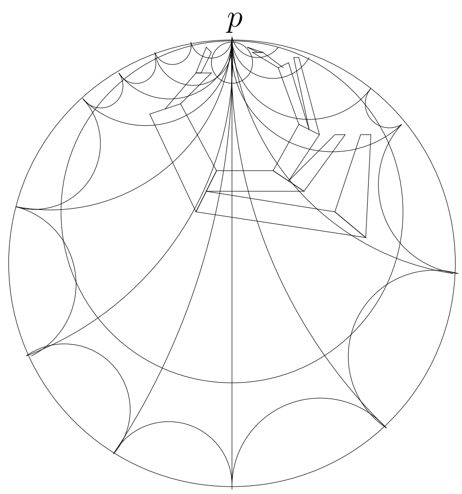

Figure 7. The picture shows a part of $\widetilde{\rho}$. All the leaves of $\widetilde{\mu}$ crossing it have a common endpoint p. The set $\widetilde{\rho}$ is comprised between two horocycles centered at p. Therefore, the endpoints of a lift of a leaf of α must be p.

Let us continue the proof of Theorem 2.1.

Assume that the rectangular covering has been chosen according to Lemma 2.5. By inequality 2.1 we have, for all $t \in \mathbf{R}$,

$$\text{length}_{g_t}(\alpha) \geq \sum_{i=1}^{N} w_t^i L^i(\alpha),$$

where $L^i(\alpha) = \int_{\partial R_i \cap \alpha} d\alpha(x)$. Recall that $L^i(\alpha)$ is strictly positive and does not depend upon t. Moreover, there exists an index j such that, for all $t \in \mathbf{R}$, w_t^j is strictly positive.

Suppose that α intersects γ transversely. Let $k \in \{1, \ldots, N\}$ such that $\gamma \cap R_k \neq \emptyset$. The leaves of γ are not isolated in μ; therefore, $\widetilde{m}_1^k$ and $\widetilde{m}_2^k$ are separated by infinitely many leaves of μ. The endpoints of $\widetilde{m}_1^k$ and $\widetilde{m}_2^k$ are distinct, except in one particular case where γ is a union of disjoint simple closed geodesics and the other half-leaves of μ that do not go out to a cusp spiral around the components of γ in opposite directions, for an observer lying on γ. Such a geodesic lamination μ is said to be particular. We will come back to that case later. For the moment, we assume that μ is not particular. Therefore, the geodesics $\widetilde{m}_1^k$ and $\widetilde{m}_2^k$ are strongly separated. Hence, by Lemma 2.4, $\lim_{t\to-\infty} w_t^k = +\infty$. Therefore, $\lim_{t\to-\infty} \text{length}_{g_t}(\alpha) = \infty$, which proves Assertion (2) when μ is not particular. If μ is particular, then Assertion (2) is a consequence of the Collar Lemma, since we know from Assertion (1) that the stump converges to zero. Thus, Assertion (2) is proved.

Suppose that α is disjoint from γ. Then, for all $i = 1, \dots, N$, the geodesics $\widetilde{m}_1^i$ and $\widetilde{m}_2^i$ are not strongly separated. Moreover, by our choice of the rectangular covering, there exists $j \in \{1, \dots, N\}$ such that the geodesics $\widetilde{m}_1^j$ and $\widetilde{m}_2^j$ are weakly separated, i.e., $w_t^j > 0$ for all $t \in \mathbf{R}$. By Lemma 2.4, $\{w_t^i : t \leq 0\}$ is bounded in $\mathbf{R}_+^*$ for all $i = 1, \dots, N$. This proves that $\{\text{length}_{g_t}(\alpha) : t \leq 0\}$ is bounded from below in $\mathbf{R}_+^*$.

It remains to prove that $\{\text{length}_{g_t}(\alpha) : t \leq 0\}$ is bounded from above. This case follows from the following double inequality, which has been established in [9]:

$$i(\alpha, \lambda_\mu(g_t)) \leq \text{length}_{g_t}(\alpha) \leq i(\alpha, \lambda_\mu(g_t)) + L(g_t, \alpha).$$

Let us explain the term $L(g_t, \alpha)$ in the right-hand inequality. To do this, we briefly explain how this inequality is established. The idea is to deform each leaf of the measured geodesic lamination α into some curve which has minimal transverse intersection with respect to μ and the horocyclic foliation $F_\mu(g_t)$. One way to do this is to cover the measured geodesic lamination α with rectangles $R_1, \dots, R_N$ of disjoint embedded interiors. Each leaf of α intersects a rectangle R_i from one vertical side to the other. If a is a component of $\alpha \cap R_i$, then replace it by a curve a^* with the same endpoints which is made up of a segment contained in $F_\mu(g_t)$ followed by a segment contained in μ. Such a curve is called *horogeodesic*. Doing this replacement for every component in every rectangle, one eventually obtains a family of curves whose union α^* is called a *horogeodesic lamination* associated to α. The transverse measure of α carries out to a transverse measure on α^*. It is then possible to compute the length $L(\alpha^*)$ of α^* by summing in each rectangle the lengths of the horogeodesic curves a^* replacing the geodesic segments a as above and by using the transverse measure on α^* induced by that of α. The triangle inequality readily implies that this length $L(\alpha^*)$ bounds the length of α from above. The length $L(\alpha^*)$ splits into two parts, namely, the length of the part of α^* contained in μ (the "geodesic part" of α^*) and the length of the part contained in $F_\mu(g_t)$ (the "horocyclic part" of α^*). The horogeodesic lamination α^* might have some superfluous intersections with μ, materialized by disks bounded by a segment contained in a^* and a segment contained in $F_\mu(g_t)$. One can erase them using homotopies and eventually obtains a horogeodesic lamination associated to α which has minimal variation with respect to $F_\mu(g_t)$ and to μ. One also shows that the inequality concerning the lengths remains valid as one erases those useless disks. Once α^* has been put in the right position, the length of the geodesic part of α^* turns out to be equal to $i(\alpha, \lambda_\mu(g_t))$. The term $L(g_t, \alpha)$ represents the length of the horocyclic part of α^*.

If α is disjoint from the stump γ, then the horocyclic part of α^* is a union of disjoint curves contained in $F_\mu(g_t)$, each one made up of a concatenation of *finitely* *many* horocyclic segments. Since the length of every such segment is bounded form above by one, this shows that the length of all the segments are uniformly bounded independently of t, and so is $L(g_t, \alpha)$. In fact, one possible upper bound for $L(g_t, \alpha)$ would be a constant times the number of spikes a rectangular covering of α crosses.

Moreover, one has $i(\alpha, \lambda_\mu(g_t)) = 0$. This proves that the length of α is bounded from above in the case where α is disjoint from γ. This proves Assertion (3) and concludes the proof of Theorem 2.1. □

2.3. Proof of the width lemma. In this section we give the proof of the width lemma. This lemma, reformulated as a general statement, claims the following

Lemma 2.6. (Width lemma) *Let μ be a complete geodesic lamination on a hyperbolic surface. Let $\widetilde{\mu}$ be the preimage of μ in the universal covering and let γ_1 and γ_2 be two leaves of $\widetilde{\mu}$. Then, the width $t \mapsto w_t$ of the strip bounded by those two geodesics is either strictly decreasing or constantly equal to zero as one stretches the underlying hyperbolic metric along μ with the parameter t. The first case occurs if and only if γ_1 and γ_2 are ultraparallel or, equivalently, the last case occurs if and only if the strip is a wedge. Moreover, in the first case, one has*

- $\lim\limits_{t\to-\infty} w_t = +\infty \iff \gamma_1$ *and* γ_2 *are strongly separated by* $\widetilde{\mu}$.
- $\{w_t\}_{t\le 0}$ *is bounded in* $\mathbf{R}_+^* \iff \gamma_1$ *and* γ_2 *are weakly separated by* $\widetilde{\mu}$.

The proof is rather simple but also rather long (in a written form). The idea is to first prove it in the particular case where the lamination $\widetilde{\mu}$ restricts in the strip bounded by γ_1 and γ_2 to a lamination with a countable number of leaves (see Lemma 2.11). The general case follows by a simple geometric argument.

For the moment, we first describe a deformation of the hyperbolic plane which is in some intuitive sense a limit case of shear deformations (also called earthquakes) in the same sense that parabolic transformations can be seen as limits of hyperbolic transformations in the hyperbolic plane.

Definition 2.7. (Spreading out) Assume once for all that the hyperbolic plane is oriented. Let γ be an oriented geodesic of $\mathbf{H}^2$. It bounds two open half-planes H_+ and H_-, the former lying on the right-hand side of γ.

A *(right) spreading out along* γ is an injective discontinuous map $\iota\colon \mathbf{H}^2 \to \mathbf{H}^2$ of the following form

$$\begin{cases} \iota_{|\overline{H_-}} = \mathrm{id}_{|\overline{H_-}} \\ \iota_{|H_+} = \big(P_d\big)_{|H_+} \end{cases}$$

where P_d is a parabolic isometry centered at one of the endpoints of γ.

A *left spreading out* along γ may be defined similarly.

A *spreading out* is a choice of an oriented geodesic and of a right spreading out along that geodesic. Note that a spreading out is a local isometry of the hyperbolic plane.

Let $\mathscr{C}$ be a strip, bounded by the geodesics γ_1 and γ_2. Let ι be a spreading out along some geodesic separating γ_1 and γ_2. Let us say that the strip $\mathscr{C}'$ bounded by

$\iota(\gamma_1)$ and $\iota(\gamma_2)$ has been obtained by *spreading out* the strip $\mathscr{C}$. In what follows, we shall always assume that, if the strip $\mathscr{C}$ is a wedge, then every spreading out of $\mathscr{C}$ is performed so that the image strip $\mathscr{C}'$ is again a wedge. This is equivalent to assuming that the parabolic isometry entering the definition of a spreading out fixes the vertex of the wedge.

The process of *spreading out indefinitely* corresponds to choosing an infinite sequence ι_n of right spreading out's along a fixed geodesic γ such that $\iota_n(H_+) \subsetneq \iota_{n+1}(H_+)$ and for all compact subsets K of $\mathbf{H}^2$, there is an n such that $\iota_n(H_+) \cap K = \emptyset$.

Lemma 2.8. (Spreading out lemma) *Let C be a strip and let C' be a strip obtained by spreading out C. If the width of C is equal to zero then so is the width of C'. Otherwise, the width of C' is strictly greater than the width of C. Furthermore, in the latter case, if one spreads out C indefinitely, then the width converges to infinity.*

Proof. The case where the width of C equals zero or, in other words, the case where C is a wedge follows at once from the assumption made in the definition above: The strip C', which is the image of C by a spreading out, is still a wedge. Therefore, the width remains equal to zero.

Now assume that the geodesic boundary of C is made up of two ultraparallel geodesics γ_1 and γ_2. Let δ be the unique geodesic segment joining these geodesics perpendicularly. Let γ be the oriented geodesic along which C is spread out into C'. By definition, γ separates γ_1 and γ_2. Therefore, γ intersects (transversely) δ. We now refer the reader to the left-hand picture of Figure 8. In this picture, we have used, in order to represent the situation, the upper half-plane model of $\mathbf{H}^2$. The geodesic γ corresponds in this picture to the vertical geodesic. The geodesics γ_1 and γ_2 are respectively the left-hand and the right-hand geodesics. We also have represented a part of the hypercycle κ around γ_1 which meets γ_2 tangentially; This point of tangency is the point where the geodesic δ meets γ_2. The points of the hypercycle κ all lie at the same distance from γ_1, namely the width of C.

Now consider the right-hand picture of Figure 8. It represents the previous situation after having spread out C on the right along γ. The right-hand half-plane determined by γ has been shifted to the right by a translation of positive distance. The positions of γ, γ_2 and κ before the spreading out ι are recalled in dashed points and the corresponding positions after the spreading out ι are drawn in continuous line. Consider now the hypercycle κ' around $\iota(\gamma_1) = \gamma_1$ which meets $\iota(\gamma_2)$ tangentially. Since $\iota(\gamma_2)$ has been obtained by translating γ_2, we have $\iota(\gamma_2) \cap \kappa = \emptyset$. Therefore, κ is contained in the domain bounded by the hypercycle κ' and containing γ_1. Since the points of κ' all lie at the same distance from γ_1, namely, the width of C', this proves that the width of C' is strictly greater than the width of C.

If one spreads out C along γ indefinitely, that is, if the distance of translation converges to infinity, then the width also converges to infinity and the lemma is proved. $\square$

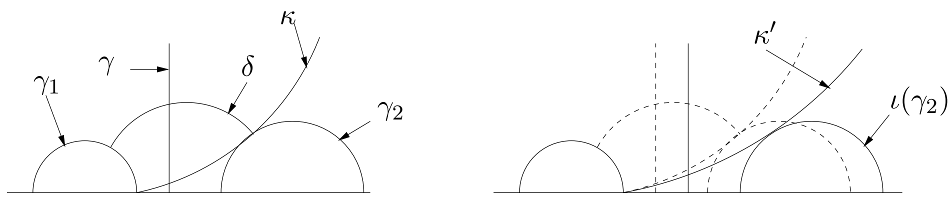

Figure 8.

We now clarify the link between stretching and spreading out.

Let us recall the situation we have been so far dealing with. Let μ be a complete geodesic lamination on the surface S. Assume that S is endowed with some fixed hyperbolic metric m and denote by m_t the hyperbolic metric obtained by (anti)stretching m along μ at time $t \in \mathbf{R}$. Now consider the universal coverings over m and m_t. Those can be identified with two copies of the hyperbolic plane, each one being endowed with the complete geodesic lamination $\widetilde{\mu}$ and $\widetilde{\mu}_t$ respectively, which are both the preimages of μ. Now select two leaves γ_1 and γ_2 of $\widetilde{\mu}$ and consider the corresponding leaves in $\widetilde{\mu}_t$, denoted by $\gamma_{1,t}$ and $\gamma_{2,t}$. These two pairs of leaves bound two strips C and C_t. (Both can be wedges if γ_1 and γ_2 share a common endpoint.) The restrictions of $\widetilde{\mu}$ and $\widetilde{\mu}_t$ to C and C_t respectively give rise to complete geodesic laminations $\widetilde{\mu}_{|C}$ and $\widetilde{\mu}_{|C_t}$ in C and C_t. There is a prefered homeomorphism S_t between the two strips C and C_t which respects the geodesic laminations $\widetilde{\mu}_{|C}$ and $\widetilde{\mu}_{|C_t}$ and the horocyclic foliations $F_{\widetilde{\mu}}(m)$ and $F_{\widetilde{\mu}}(m_t)$ associated to them. As a matter of fact, it sends the interior of one ideal triangle coming from $\widetilde{\mu}_{|C}$ to the interior of an ideal triangle from $\widetilde{\mu}_{|C_t}$, respecting the non-foliated regions.

Choose a leaf of $F_{\widetilde{\mu}}(m)$ that crosses γ_1 and γ_2 and denote by α the intersection of that leaf with C. (Note that such a leaf always exists.) This segment α is called a *cross segment*. Give α an orientation and denote by α_t the corresponding oriented cross segment in C_t. (This segment is contained in the leaf of $F_{\widetilde{\mu}}(m_t)$ that corresponds to the leaf chosen to define α.) The segment α crosses all the ideal triangles coming from $\widetilde{\mu}_{|C}$ that separate γ_1 and γ_2. Moreover, α crosses each of these ideal triangles through the spike determined by the two edges separating γ_1 and γ_2. Of course, the same holds for α_t. Thus, the set $\alpha \setminus \alpha \cap \widetilde{\mu}_{|C}$ is a union of countably many open disjoint intervals I_j, $j \in J$, ordered following the orientation of α, each one being a horocyclic arc contained in one spike of a separating ideal triangle. Let T_j denote the ideal triangle containing the arc I_j.

What is the effect of a stretch along μ on that arc α? In other words, how can one describe the arc α_t? The arc α_t crosses the ideal triangles corresponding to the triangles T_j in the same order. Thus the set $\alpha_t \setminus \alpha_t \cap \widetilde{\mu}_{|C_t}$ is an ordered union of countably many open disjoint intervals $I_{j,t}$, $j \in J$, each one of them being a horocyclic arc contained in one spike of the ideal triangle $T_{j,t}$ corresponding to T_j. The only difference stems on the lengths l_j and $l_{j,t}$ of those horocyclic segments I_j and $I_{j,t}$. Indeed, one has the relation

$$l_{j,t} = (l_j)^{e^t}, \text{ for all } j \in J.$$

This relation comes from the very definition of a stretch deformation: if a horocyclic arc contained in a spike of an ideal triangle has length $l \leq 1$, then it lies at a distance $-\log(l)$ from the non-foliated region. Its image is a horocyclic arc of the same spike lying at a distance $-e^t \log(l) = -\log(l^{e^t})$, since the stretch deformation multiplies the lengths of arcs contained in the sides of the ideal triangle by the factor e^t. Therefore, the length of the image arc is l^{e^t}.

Now consider the following situation: erase all non separating leaves of $\widetilde{\mu}_{|C}$ and $\widetilde{\mu}_{|C_t}$. The strips C and C_t are now divided by those remaining leaves into countably many wedges. More precisely, if L denotes the union of those remaining leaves, then each component of $C \setminus L$ is the interior of a wedge. Among those separating leaves L, let L_{front} denote those leaves that bound a wedge. (The leaves of L_{front} are frontier leaves of $\widetilde{\mu}$.) There is of course a corresponding situation in C_t. Now we claim that, if t is negative, one can pass from C to C_t by performing a spreading out along each geodesic of L_{front}.

Lemma 2.9. *Let t be a negative real number. Then, the strip C_t is congruent to a strip obtained from C by a sequence of spreading outs along all the leaves of $\widetilde{\mu}$ contained in L_{front}. In particular, the width function $t \mapsto w_t$, $t \in \mathbf{R}$, is either constantly equal to zero or is strictly decreasing, depending on whether the strip C is a wedge or not.*

Proof. We keep the notations of the preceding discussion. Set, for $i = 1, 2$, $p_i = \alpha \cap \gamma_i$ and $p_{i,t} = \alpha_t \cap \gamma_{i,t}$. Consider the unit disk model for both universal coverings. With these identifications, one can assume, up to some isometry, that $\gamma_1 = \gamma_{1,t}$ and $p_1 = p_{1,t}$.

Let I_j be a component of $\alpha \setminus \alpha \cap \widetilde{\mu}_{|C}$ and let W_j be the wedge containing I_j. Let E_- be the leftmost edge of W_j with respect to the orientation of α. Apply a right spreading out ι along E_- in such a way that the vertex of the wedge W_j is fixed by ι. The image curve $\iota(\alpha)$ is now disconnected; reconnect it using a horocyclic segment contained in the horocycle prolonging $\iota(I_j)$. The upshot is a new strip $\iota(C)$ separated by the wedges $\iota(W_j)$ and traversed by a cross segment α' whose intersection with each wedge is a horocyclic arc. Moreover, exactly one of the wedges, namely W_j, has been replaced by the wedge $\iota(W_j)$, which is bigger in the sense that the length l' of $\alpha' \cap \iota(W_j)$ is strictly greater than the length l_j of I_j. Now specify the spreading out ι so that $l' = (l_j)^{e^t}$.

By doing a spreading out for every geodesic of L_{front}, one eventually obtains a strip C'_t which is separated by a family of wedges $W'_{j,t}$, $j \in J$, and which is crossed by a segment α'_t which satisfies, for every $j \in J$,

$$\operatorname{length}(\alpha'_t \cap W'_{j,t}) = l_{j,t} = (l_j)^{e^t}.$$

The two strips C'_t and C_t are isometric since the prefered map sending the geodesic lamination $\widetilde{\mu} \cap C_t$ onto $\widetilde{\mu} \cap C'_t$ and the corresponding horocyclic foliation F_t onto F'_t as well is an isometry. This concludes the proof of the first assertion.

Since C and t have been chosen arbitrarily, the first assertion together with Lemma 2.8 imply at once that, if C is not a wedge, then, for all $t, t' \in \mathbf{R}$ such that $t < t'$, we have $w_t > w_{t'}$. In other words, the width function $t \mapsto w_t$ is strictly decreasing. If C is a wedge, then using the first assertion together with Lemma 2.8 again, the width function is constantly equal to zero, and the proof of the lemma is complete. □

The preceding lemma almost implies the width lemma: Indeed, we have already shown that

(1) If γ_1 and γ_2 bound a wedge, then the width function $t \mapsto w_t$ is constantly equal to zero, and
(2) If γ_1 and γ_2 are ultraparallel, then the width function $t \mapsto w_t$ restricted to $\mathbf{R}_-$ is bounded from below by $w_0 = w > 0$ and then strictly increases as t decreases.

It therefore remains to show that, if the geodesics γ_1 and γ_2 are weakly separated, then the width function is bounded from above and to show that, if the geodesics γ_1 and γ_2 are strongly separated, then $\lim_{t\to-\infty} w_t = +\infty$. The assertion about the upper bound is easy. Indeed, for each $t \in \mathbf{R}$, the length l_t of the cross segment α_t is an upper bound to the width w_t. We have $l_t = \sum_{|J|} l_{j,t}$, where $|J|$ is the cardinality of J. The horocyclic arc $I_{j,t}$ being contained in a spike of an ideal triangle for every t and j, one has the constraint $l_{j,t} \leq 1$. Therefore,

$$\forall t \in \mathbf{R},\ w_t \leq l_t \leq |J|.$$

If the geodesics γ_1 and γ_2 are weakly separated, the cardinality $|J|$ of J is finite and of course independent of t. This proves the assertion about the upper bound. Therefore, it remains to prove that the width function converges to infinity as t converges to $-\infty$ when the geodesics γ_1 and γ_2 are strongly separated. We shall first show this assertion in the case where the separating ideal triangles coming from $\widetilde{\mu}_{|C}$ are glued edge-to-edge. We first give a definition.

Definition 2.10. (Discrete approximation) Let γ_1 and γ_2 be two ultraparallel leaves of a complete geodesic lamination $\widetilde{\mu}$ of the hyperbolic plane and let C denote the strip bounded by those two leaves. Let $\widetilde{\mu}_{|C}$ be the restriction of $\widetilde{\mu}$ to C and let α be an oriented cross segment in C.

A *discrete approximation* $\widetilde{\mu}^D$ of $\widetilde{\mu}_{|C}$ is a geodesic lamination of some strip C^D obtained as follows:

The set $\alpha \setminus \alpha \cap \widetilde{\mu}_{|C}$ is a union of countably many open disjoint intervals I_j, $j \in J$, ordered following the orientation of α, each one of them being a horocyclic arc contained in one spike of a separating ideal triangle. Let T_j denote the ideal triangle containing the arc I_j and W_j the wedge bounded by the edges of T_j that meet α.

Choose any subset J_0 of J, with the ordering induced by that of J. Glue the wedges W_j, $j \in J_0$, edge-to-edge following the ordering of J_0, in such a way that the horocyclic arcs I_{j_1} and I_{j_2} of two adjacent wedges W_{j_1} and W_{j_2}, $j_1, j_2 \in J_0$, coalesce into one arc. One eventually obtains a union of wedges glued edge-to-edge, together with an arc α' which crosses each wedge. Consider the closure C^D of that union; it is a strip bounded by two outermost geodesics γ_1^D and γ_2^D. The union of those two geodesics together with the edges of the wedges form a geodesic lamination $\widetilde{\mu}^D$ of C^D whose leaves all separate γ_1^D and γ_2^D. It is easy to see that this definition is, up to congruence, independent from the choice of the cross segment α or from its orientation.

Note that there is a canonical correspondence between the leaves of $\widetilde{\mu}^D \backslash (\gamma_1^D \cup \gamma_2^D)$ and their images in $\widetilde{\mu}_{|C}$.

Lemma 2.11. *Let C be a strip bounded by two ultraparallel leaves of a complete geodesic lamination $\widetilde{\mu}$ of the hyperbolic plane. Let $\widetilde{\mu}^D$ be a discrete approximation of $\widetilde{\mu}_{|C}$. Then the strip C^D is, up to congruence, contained into the strip C. In particular, if w^D and w denote the width of C^D and of C respectively, then one gets*

$$w^D \leq w.$$

Proof. The idea of the proof is based on the observation that, in order to recover from a discrete approximation $\widetilde{\mu}^D$ the original strip C (up to congruence), one has to insert (possibly infinitely many) wedges. Inserting wedges is done by spreading out the strip C^D along the leaves of $\widetilde{\mu}^D$. Then, a simple application of Lemma 2.8 concludes the proof. We now formalize this idea.

We are using the same notations as in the definition.

Let $J_0 \subset J$ be the set of indices defining $\widetilde{\mu}^D$. Consider an increasing sequence $J_0 \subset J_1 \subset \cdots J_n \subset \cdots$ whose union is equal to J. Since J is countable, so are all the J_i's and we can assume that J_{i+1} has been obtained from J_i by adding one element. Consider the sequence C_i^D of strips associated to the sequence J_i. By construction, C_{i+1}^D has been obtained by using one more wedge than for C_i^D and gluing all these wedges edge-to-edge. Arguing exactly like in the proof of Lemma 2.9, this amounts to spreading out C_i^D along some geodesic (exactly where the new wedge has to be inserted). Again, following the proof of that lemma, one recovers after those spreading outs a strip which is congruent to C. This concludes the proof.

□

Now that we have defined a discrete approximation of a geodesic lamination, we explain how we propagate the stretch deformation done along the initial lamination to its discrete approximation.

Let us fix t negative and let C_t be the strip obtained from C after a stretch along μ of time t. By Lemma 2.9, this strip C_t is also obtained by spreading out C along the leaves contained in L_{front}. We *propagate* those spreading outs to spreading outs done in C^D. Among the spreading outs used to pass from C to C_t, retain those that are performed along the geodesics that correspond to the geodesics in the discrete approximation $\widetilde{\mu}^D$. Now apply these spreading outs along the corresponding geodesics in C^D and denote by C_t^D the obtained strip. This strip

C_t^D is the discrete approximation of C_t obtained by selecting the same subset J_0 of indices as for C^D. By Lemma 2.11, one has the following

Lemma 2.12. (Propagation) *Let C be a strip bounded by two ultraparallel leaves of a complete geodesic lamination $\widetilde{\mu}$ of the hyperbolic plane. Let $\widetilde{\mu}^D$ be a discrete approximation of $\widetilde{\mu}_{|C}$ and let w_t and w_t^D denote the widths of the strips C_t and C_t^D, respectively obtained by anti-stretching C and propagating this anti-stretching to C^D. Then one has, for all $t < 0$,*

$$w_t^D \leq w_t.$$

We are now ready to prove the width lemma for a discrete approximation.

Lemma 2.13. (Glued edge-to-edge case) *The width lemma is true for geodesics γ_1 and γ_2 separated by ideal triangles glued edge-to-edge.*

Proof. First remark that, using the notations above, we have $L_{\text{front}} = L$. This is due to our glued-edge-to-edge assumption.

Next, recall we only have to show that if γ_1 and γ_2 are strongly separated, then $\lim\limits_{t\to-\infty} w_t = +\infty$. Consequently, assume that γ_1 and γ_2 are strongly separated.

We first show that it suffices to prove the lemma in the following case:

All leaves of L_{front}, except γ_2, have a common endpoint (which is an endpoint of γ_1 as a matter of fact).

The reason for this claim goes as follows. The geodesics γ_1 and γ_2 are ultraparallel. Therefore, the strip C is bounded in the circle at infinity by two disjoint segments c_+ and c_- of open interiors. Each leaf of L_{front} has an endpoint in c_+ and an endpoint in c_-. Moreover, the closures of the components of $C \setminus L_{\text{front}}$ are wedges; denote by W_j, $j \in J$, those wedges. Since γ_1 and γ_2 are strongly separated, the cardinal of J is infinite and, therefore, one segment, say c_+, contains the vertices of infinitely many wedges among the W_j. Let W_j^+, $j \in J^+ \subset J$, $|J^+| = \infty$, denote those wedges with vertices in c_+. Choose any wedge W with vertex in c_-. (Such a wedge exists since γ_1 and γ_2 are ultraparallel.)

Consider the discrete approximation $\widetilde{\nu}^D$ made up of those geodesics of L_{front} that bound the wedges W_j^+, $j \in J^+$, and the wedge W. In other words, the corresponding strip C'^D is obtained by gluing edge-to-edge the wedges W and W_j^+, $j \in J^+$, and then by taking the closure. (The extra wedge W has been added to insure that the strip C'^D is not a wedge itself.) This discrete approximation is almost of the type prescribed above, except that the wedge W might separate C'^D into two connected components. Necessarily, one of these connected components is a wedge which has been obtained by gluing infinitely many wedges of W_j^+, $j \in J^+$. Consider then the subwedge composed by that component and by W and call it C^D. Then this discrete approximation is of the desired form. If w^D and w denote the widths of C^D and of C respectively, then, by Lemma 2.11, one has $w^D \leq w$. By propagating the anti-stretch deformation on the discrete approximation, we get, for all $t < 0$, $w_t^D \leq w_t$, and the claim follows.

Assume now that all leaves of L_{front}, except γ_2, have a common endpoint a. There are infinitely many leaves having a as an endpoint and all these leaves are edges of infinitely many wedges glued edge-to-edge. The union $\cup W_i$ of these wedges is again a wedge we denote by W_1. This big wedge W_1 is bounded by γ_1 and a geodesic γ which joins one endpoint of γ_1 to the opposite endpoint of γ_2. It is glued edge-to-edge along γ to a single wedge W_2 bounded by γ and γ_2.

Let α be a cross segment in C. The segment $\alpha \cap W_1$ is a horocyclic segment which is the union of the closures of the infinitely many horocyclic segments $\alpha \cap W_i$. Let l_i be the length of the segment $\alpha \cap W_i$ and l the length of $\alpha \cap W_1$. Then $l = \sum_i l_i$.

For each $t \in \mathbf{R}$, one has

$$l_t = \sum_i l_{i,t} = \sum_i (l_i)^{e^t},$$

where l_t and $l_{i,t}$ denote the length of the image of $\alpha \cap W_1$ and the length of the image of $\alpha \cap W_i$ after propagating the stretch deformation. Therefore, $\lim\limits_{t\to-\infty} l_t = +\infty$. By Lemma 2.9, this implies that, as t converges to $-\infty$, the strip C_t is indefinitely spread out. By Lemma 2.8, it follows that $\lim\limits_{t\to-\infty} w_t = +\infty$, which concludes the proof. □

Proof of the width lemma. It remains to prove the assertion about the limit of the width function. Choose any discrete approximation $\widetilde{\mu}^D$ of $\widetilde{\mu}_{|C}$ with infinitely many ideal triangles. Then, by Lemma 2.11 and Lemma 2.12, one has

$$w_t^D \leq w_t,$$

where w_t^D denotes the width of C_t^D, obtained by propagating the stretch deformation on C^D. By Lemma 2.13,

$$\lim_{t\to-\infty} w_t^D = +\infty,$$

which concludes the proof of the width lemma. □

3. On negative convergence of stretch lines

Corollary 3.1. (Cluster points) *Let $t \mapsto g_t$, $t \in \mathbf{R}$, be a stretch line directed by μ with non-empty stump γ. Then, every point of $\overline{\{g_t\,:\,t \leq 0\}} \setminus \{g_t\,:\,t \leq 0\}$ (if any) is a projective class of a measured geodesic lamination which is topologically contained in γ.*

Proof. Let $[\alpha] \in \mathscr{PL}_0(S)$ be a point of $\overline{\{g_t,\, t \leq 0\}} \setminus \{g_t,\, t \leq 0\}$. Let $\{g_n\}$ be a sequence of $\{g_t,\, t \leq 0\}$ converging to $[\alpha] \in \mathscr{PL}_0(S)$. Then we have $i(\alpha, \gamma) = 0$, that is, α and γ have no transverse intersection. Indeed, there exists a sequence $\{x_n\}$ of $\mathbf{R}_+^*$ such that, for all $\beta \in \mathscr{ML}_0(S)$,

$$\lim_{n\to\infty} x_n \operatorname{length}_{g_n}(\beta) = i(\alpha, \beta).$$

Since $\{g_n\}$ is a sequence of $\mathscr{T}(S)$ converging to infinity, we must have $\lim_{n\to\infty} x_n = 0$. Now taking $\beta = \gamma$ and using Theorem 2.1 (1) we get the desired equality.

Since $i(\alpha, \gamma) = 0$, it therefore remains to show that α cannot have any component disjoint from γ.

Suppose that α is not contained in γ, that is, α has a component α_0 which is disjoint from γ.

Two mutually exclusive cases occur: Either

(1) there exists an essential simple closed curve β such that
- $\beta \cap \gamma = \emptyset$ and
- $\beta \cap \alpha_0 \neq \emptyset$,

Or

(2) such a closed curve β does not exist, in which case
- α_0 is a simple closed geodesic that separates the surface, and
- $\forall \beta \in \mathscr{S},\ \beta \cap \alpha_0 \neq \emptyset \Longrightarrow \beta \cap \gamma \neq \emptyset$.

Indeed, consider the component C of $S \setminus \gamma$ that contains α_0. The metric completion $\bar{C}$ of C (with respect to any hyperbolic metric) is a hyperbolic surface with totally geodesic boundary. First remark that $\bar{C}$ is not simply connected, because of the existence of α_0. In other words, $\bar{C}$ contains at least one essential simple closed curve. The classification of surfaces, associated with hyperbolic geometry, asserts that there exist finitely many pairwise disjoint simple closed geodesics κ in $\bar{C}$ such that the completion of each component of $\bar{C} \setminus \kappa$ is either a degenerated pair of pants or a crown. Recall that a degenerated pair of pants is a complete hyperbolic surface of finite area whose interior is homeomorphic to a three punctured sphere and a crown is a complete hyperbolic surface whose boundary is a disjoint union of finitely many infinite geodesics and whose interior is homeomorphic to a once punctured disk. Note that the sphere with three cusps and the crown with one cusp cannot occur in the decomposition given by κ. Furthermore, the existence of α_0 implies that the completion P of each component of $\bar{C} \setminus \kappa$ contains at least one boundary component which does not belong to γ in the surface S. Due to the topology of those components and to the fact that α_0 is connected, the geodesic lamination α_0 either intersects κ transversely or is contained in κ. If α_0 has a nonempty transverse intersection with a component β of κ, then β is not contained in γ and therefore belongs to Case (1). Now assume that α_0 is contained in κ. In particular, α_0 is a simple closed geodesic. Let P_1 and P_2 be the completions of the two components of $\bar{C} \setminus \kappa$ whose boundaries contain α_0. (The surfaces P_1 and P_2 might be equal.) If one of them, say P_1, is a crown, then the boundary component a of P_1 that is different from α_0 belongs to γ (in the surface S) and every geodesic segment of S which has a nonempty transverse intersection with a has a nonempty transverse intersection with α_0 as well, and conversely. Moreover, in this case, α_0 separates the surface S, so we are in Case (2).

If P_1 and P_2 are degenerated pairs of pants, then there exists at least one simple closed geodesic β satisfying the conditions of Case (1). All the cases are treated and the claim is proved.

Now, let us deal with the first case and choose an essential simple closed curve $\beta \in \mathscr{S}$ such that $\beta \cap \gamma = \emptyset$ and $\beta \cap \alpha_0 \neq \emptyset$. Using Theorem 2.1, the two requirements on β imply

(1) $\mathrm{length}_{g_n}(\beta)$ is bounded from above, and
(2) $i(\alpha, \beta) > 0$.

This is in contradiction with the fact that $x_n \to 0$. We conclude that α is topologically contained in γ, which proves the corollary in that case.

Now, let us deal with the second case and consider an essential simple closed curve $\beta \in \mathscr{S}$ such that $\beta \cap \gamma \neq \emptyset$ and $|\beta \cap \alpha_0| = 2$. (Such a simple closed geodesic β exists because α_0 is a separating simple closed geodesic.) The set $\beta \setminus \beta \cap \alpha_0$ has two components, β_1 and β_2, at least one intersecting γ. For the moment, assume that only one of them, say β_1, intersects γ. Let β' be the geodesic representative of the simple closed curve obtained by joining the two endpoints of β_1 using one of the two geodesic segments contained in α_0. There always exists a choice a of such a subsegment of α_0 so that β' is essential. To see this, consider the pair of pants of S containing α_0 and β_1. Then β' is one of the boundary components of that pair of pants. Therefore, β' is essential if and only if this boundary component is not a cusp. But there are at least one boundary component which is not a cusp, otherwise γ would be empty.

Now we claim that

$$\lim_{n\to\infty} x_n(\mathrm{length}_{g_n}(\beta) - \mathrm{length}_{g_n}(\beta')) = 0.$$

The proof of this claim is easy: First note that $\lim_{n\to\infty} \mathrm{length}_{g_n}(\beta_1) = +\infty$. This is due to the fact that the endpoints of any lift of the geodesic segment β_1 to the universal covering belong to two lifts of α_0 which are strongly separated by γ. The width lemma implies that $\lim\limits_{n\to\infty} \mathrm{length}_{g_n}(\beta_1) = \infty$. Next, by Theorem 2.1, $\{\mathrm{length}_{g_n}(\alpha_0)\}$ is bounded from above. Therefore, $\{\mathrm{length}_{g_n}(a)\}$ is also bounded from above. Now consider a lift of $\beta_1 \cup a$ in the universal covering. This lift is a curve c which is the concatenation of two geodesic segments and the endpoints of that curve are identified through the action of the holonomy group of S. Therefore, the geodesic segment d joining them projects on S to a simple closed curve homotopic to β' and its length is an upper bound for $\mathrm{length}_{g_n}(\beta')$. Using hyperbolic trigonometry in the triangle bounded by c and d and replacing $\mathrm{length}(x)$ by x in order to lighten notations, one gets

$$\begin{aligned}\cosh(\beta') &\leq \cosh(d) = \cosh(\beta_1)\cosh(a) - \sinh(\beta_1)\sinh(a)\cos(\theta)\\ &\leq \cosh(\beta_1)(\cosh(a) - \tanh(\beta_1)\sinh(a)\cos(\theta)) \leq \cosh(\beta_1)C_n,\end{aligned}$$

where θ denotes the angle between β_1 and a. From the preceding discussion, the sequence C_n is bounded. Moreover, we have $\lim_{n\to\infty} \mathrm{length}_{g_n}(\beta') = \infty$. This can also be seen using the pair of pants P of S containing α_0 and β_1. At least one geodesic of the boundary of the crown crosses P from one boundary component to β'. The infinite geodesics of the boundary of the crown are leaves of γ.

Therefore, β' has a nonempty transverse intersection with γ. Theorem 2.1 implies $\lim_{n\to\infty} \mathrm{length}_{g_n}(\beta') = \infty$. It follows that $\exp(\beta' - \beta_1)$ is bounded. Therefore, since x_n converges to zero, one gets

$$\lim_{n\to\infty} x_n(\mathrm{length}_{g_n}(\beta_1) - \mathrm{length}_{g_n}(\beta')) = 0.$$

The claim follows.

It is now fairly easy to conclude since, by definition of convergence towards Thurston's boundary, one has

$$\lim_{n\to\infty} x_n \mathrm{length}_{g_n}(\beta) = i(\alpha,\beta) = i(\gamma,\beta) + 2, \text{ and}$$
$$\lim_{n\to\infty} x_n \mathrm{length}_{g_n}(\beta') = i(\alpha,\beta) = i(\gamma,\beta),$$

which raises a contradiction.

The case where both β_1 and β_2 intersect γ is handled in the same way by constructing two simple closed geodesics β' and β'' as above and by showing that the length of the union of those two geodesics, multiplied by x_n, has the same limit as the length of β multiplied by x_n.

Therefore, extra components for γ cannot exist, that is, $\gamma \subset \alpha$ and the proof is complete. □

A geodesic lamination is *uniquely ergodic* if it supports a transverse measure which is unique up to scalar multiples. (Note that a uniquely ergodic measured geodesic lamination is not empty.) An immediate corollary is the following

Theorem 3.2. (Negative convergence) *Every stretch line directed by a complete measured geodesic lamination with uniquely ergodic stump γ converges negatively to the projective class of γ.*

Proof. Suppose that the stump γ is uniquely ergodic. This implies in particular that γ is connected. From the preceding corollary, we conclude that any cluster point α (if any) of $\overline{\{g_t : t \leq 0\}} \setminus \{g_t : t \leq 0\}$ is topologically equal to γ. Hence every point of $\overline{\{g_t,\, t \leq 0\}} \setminus \{g_t,\, t \leq 0\}$ (if any) is the projective class $[\gamma] \in \mathscr{PL}_0(S)$, again by unique ergodicity.

Since the Teichmüller space bordered by $\mathscr{PL}_0(S)$ is compact, the set $\overline{\{g_t,\, t \leq 0\}} \setminus \{g_t,\, t \geq 0\}$ is non-empty: for instance, the sequence $\{g_{-n}\}$ which converges to infinity always admits a cluster point. This proves the theorem. □

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Received 17 March 2006