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# HYPERBOLIC DISTANCE, $\lambda$-APOLLONIAN METRIC AND JOHN DISKS

**X. Wang, M. Huang, S. Ponnusamy* and Y. Chu**

Hunan Normal University, Department of Mathematics
Changsha, Hunan 410081, P. R. China; xtwang@hunnu.edu.cn

Hunan Normal University, Department of Mathematics
Changsha, Hunan 410081, P. R. China

Indian Institute of Technology Madras, Department of Mathematics
Chennai-600 036, India; samy@iitm.ac.in

Huzhou Teachers College, Department of Mathematics
Huzhou, Zhejiang 313000, P. R. China

**Abstract.** In this paper, by using the hyperbolic distance and the $\lambda$-Apollonian metric, we establish a sufficient condition for a simply connected proper subdomain $D \subset \mathbf{C}$ to be a John disk. We also construct two examples to show that the converse of this result does not necessarily hold. As a consequence the answer to Conjecture 6.2.12 in the Ph.D. thesis of Broch [2] is negative.

## 1. Introduction and main results

As in [9] and [13], a simply connected proper domain $D$ of the complex plane $\mathbf{C}$ is called *a $b$-John disk* if for any two points $z_1$, $z_2 \in D$, there is a rectifiable arc $\alpha \subset D$ joining them with

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \le b \operatorname{dist}(z, \partial D) \quad \text{for all } z \in \alpha,$$

where $b$ is a positive constant. Here $\ell(\alpha[z_j, z])$ denotes the Euclidean arclength of the subarc of $\alpha$ with the endpoints $z_j$ and $z$; $\operatorname{dist}(z, \partial D)$ denotes the Euclidean distance from $z$ to $\partial D$ which is the boundary of $D$. We call a domain $D$ a *John disk* if it is a $b$-John disk for some positive constant $b$.

It has been known that a Jordan domain $D \subset \mathbf{C}$ is a quasidisk if and only if both $D$ and $D^* = \mathbf{C}\backslash\overline{D}$ are John disks (cf. [10]), and every quasidisk is a John disk (see [5]). Hence John disks can be thought of as "one-sided quasidisks". Also several other necessary and sufficient conditions for $D$ to be a John disk have been given. For example, Näkki and Väisälä obtained the following

**Theorem A.** [13] *Let $D$ be a simply connected proper subdomain in $\mathbf{C}$. Then the following conditions are equivalent.*

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*Corresponding author.

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(1) *$D$ is a $b$-John disk.*
(2) *For each $x \in \mathbf{R}^2$ and $r > 0$, any two points in $D \setminus \overline{\mathbf{B}}(x, r)$ can be joined by an arc in $D \setminus \overline{\mathbf{B}}(x, \frac{r}{c})$, where the constants $b$ and $c$ depend only on each other and $\mathbf{B}(x, r)$ denotes the disk with the center $x$ of radius $r$.*
(3) *For every straight crosscut $\alpha$ of $D$ dividing $D$ into subdomain $D_1$ and $D_2$, we have $\min_{j=1,2} \operatorname{diam}(D_j) \le c \ \operatorname{diam}(\alpha)$, where the constants $b$ and $c$ depend only on each other and $\operatorname{diam}(\alpha)$ means the diameter of $\alpha$.*

John disks appear naturally in many areas of analysis (see [12, 13]). In [10], Kim and Langmeyer presented a number of results characterizing $b$-John disk (see [10, Theorem 2.3]). To present our main result, we need some preparation. The following result, which is actually planar version of [10, Theorem 4.1], characterizes $b$-John disk, in terms of a bound for hyperbolic distance.

**Theorem B.** *A simply connected proper subdomain $D \subset \mathbf{C}$ is a $b$-John disk if and only if there exists a constant $c \ge 1$ such that*

$$h_D(z_1, z_2) \le c \, j'_D(z_1, z_2)$$

*for all $z_1$, $z_2 \in D$. Here the constants $b$ and $c$ depend only on each other.*

The definitions of $h_D$ and $j'_D$ are presented in Section 2.

By using the hyperbolic distance $h_D$ and the $\lambda$-Apollonian metric $a'_D$ (see again Section 2 for its definition) in $D$, Broch [2, Theorem 6.2.9] obtained the following result which again provides a necessary and sufficient condition for a Jordan domain to be a $b$-John disk.

**Theorem C.** [2] *A Jordan proper subdomain $D \subset \mathbf{C}$ is a $b$-John disk if and only if there are constants $\mu$ and $\nu$ such that*

$$h_D(z_1, z_2) \le \mu \, a'_D(z_1, z_2) + \nu$$

*for all pairs $z_1$, $z_2 \in D$, where $\mu$ and $\nu$ depend only on $b$, and $b$ depends only on $\mu$ and $\nu$.*

By comparing Theorem C with Theorem B, Broch [2, Conjecture 6.2.12] raised the following conjecture.

**Conjecture 1.1.** *A simply connected (Jordan) domain $D \subset \mathbf{C}$ is a $b$-John disk if and only if there is a constant $c$ such that*

$$h_D(z_1, z_2) \le c \, a'_D(z_1, z_2)$$

*for all pairs $z_1$, $z_2 \in D$. Here the constants $b$ and $c$ depend only on each other.*

For a discussion on related problems, we refer to [1, 3, 4, 7, 8, 9, 11, 12].

In this paper, we mainly consider Conjecture 1.1. Our main results follow.

**Theorem 1.2.** *Let $L_1 = \{z : |z+1| = 1, \ \operatorname{Im} z \ge 0\}$, $L_2 = \{-2 + iy : -1 \le y \le 0\}$, $L_3 = \{x - i : -2 \le x \le 2\}$, $L_4 = \{2 + iy : -1 \le y \le 0\}$ and $L_5 = \{z : |z-1| = 1, \ \operatorname{Im} z \ge 0\}$. Let $D$ be the bounded domain bounded by $L_j$,*

$j = 1, 2, \ldots, 5$. *Then $D$ is a $b$-John disk with $0 < b \le 6(2+\sqrt{2})$, but there does not exist a constant $c$ such that*

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2)$$

*for all $z_1, z_2 \in D$.*

**Theorem 1.3.** *Let $D^* = \{x + iy : x > 0,\ |y| < \frac{1}{2}\}$ and $D = \mathbf{C}\backslash\overline{D^*}$. Then $D$ is a $b$-John disk for some $b = 6$, but there does not exist any constant $c$ such that*

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2)$$

*for all $z_1, z_2 \in D$.*

Theorems 1.2 and 1.3 show that the answer to Conjecture 1.1 is negative irrespective of whether $D$ is bounded or unbounded.

## 2. Preliminary material

Throughout the discussion we restrict ourselves to simply connected proper subdomains $D$ of the complex plane $\mathbf{C} = \mathbf{R}^2$. The hyperbolic density at $z \in D$ is given by

$$\rho_D(z) = \rho_{\mathbf{B}}(g(z))|g'(z)|,$$

where $\rho_{\mathbf{B}}(z) = 2/(1 - |z|^2)$ and $g$ is a conformal mapping of $D$ onto the unit disk $\mathbf{B} \subset \mathbf{C}$. Then for any pair of points $z_1$ and $z_2$ in $D$, we define

$$h_D(z_1, z_2) = \inf_{\alpha} \int_{\alpha} \rho_D(z)\,|dz| \quad \text{and} \quad k_D(z_1, z_2) = \inf_{\alpha} \int_{\alpha} \frac{|dz|}{\operatorname{dist}(z, \partial D)},$$

where each infimum in the above is taken over all rectifiable curves $\alpha$ in $D$ from $z_1$ to $z_2$. The quantities $h_D(z_1, z_2)$ and $k_D(z_1, z_2)$ are called the *hyperbolic distance* and *quasihyperbolic distance* between $z_1, z_2$, respectively. The idea of quasihyperbolic distance was introduced in [7] and developed in [7, 6]. Also, it is well known that for all pairs $z_1$ and $z_2$ in $D$ there exists a unique hyperbolic geodesic curve $\beta$ from $z_1$ to $z_2$, i.e. a curve $\beta$ along which the above infimum is obtained, and

$$h_D(z_1, z_2) = \int_{\beta} \rho_D(z)\,|dz|.$$

It follows from [2, 14, 6] that

**Lemma 2.1.** *For all $z_1$, $z_2 \in D$,*

$$\frac{1}{2}\, k_D(z_1, z_2) \le h_D(z_1, z_2) \le 2\, k_D(z_1, z_2)$$

*and*

$$k_D(z_1, z_2) \ge \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_j, \partial D)}\right) \quad (j = 1, 2).$$

For a pair of points $z_1, z_2$ in $D$, the *inner distance* between them is defined by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\alpha) : \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

We call $\lambda_D$ the inner metric on $D$. A point $w$ in the boundary $\partial D$ of $D$ is said to be *rectifiably accessible* if there is a half open rectifiable arc $\alpha$ in $D$ ending at $w$. Let $\partial_r D$ denote the subset of $\partial D$ which consists of all the rectifiably accessible points, that is

$$\partial_r D = \{w \in \partial D : w \text{ is rectifiably accessible}\}.$$

Further, as in [2], we define the $\lambda$-*Apollonian metric* $a'_D$, in terms of inner distances, by

$$a'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log(|z_1, z_2, w_1, w_2|_\lambda),$$

where

$$|z_1, z_2, w_1, w_2|_\lambda = \frac{\lambda_D(z_1, w_1)\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)\lambda_D(z_2, w_1)}.$$

At this place, it might be important to point out that $a'_D \neq a''_D$, if $a''_D$ denotes the inner Apollonian metric defined by

$$a''_D(z_1, z_2) = \inf\{\ell_a(\alpha) : \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

Also as in [2], we define a metric $j'_D(z_1, z_2)$ for any $z_1$, $z_2 \in D$ by

$$j'_D(z_1, z_2) = \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right).$$

This version of the metric is obtained by replacing the Euclidean distances in the $j_D$-metric introduced by Gehring and Osgood [6] by *inner* distances. Note that sometimes, our $j'_D$ is multiplied by a factor 1/2. We also recall that the relation between $k_D$ and $j'_D$ in John disks is stated in Theorem B and Lemma 2.1. We end the section with the following result.

**Lemma 2.2.** *Suppose that $D$ is a simply connected proper subdomain in* $\mathbf{C}$. *Then*

$$a'_D(z_1, z_2) \leq j'_D(z_1, z_2)$$

*for all $z_1, z_2 \in D$.*

*Proof.* For any $w \in \partial_r D$, we know

$$\frac{\lambda_D(z_1, w)}{\lambda_D(z_2, w)} \leq \frac{\lambda_D(z_1, z_2) + \lambda_D(z_2, w)}{\lambda_D(z_2, w)} = 1 + \frac{\lambda_D(z_1, z_2)}{\lambda_D(z_2, w)} \leq 1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}.$$

Thus, by the symmetry and the arbitrariness of $w_1, w_2 \in \partial_r D$ in the definition of $a'_D$ , we have

$$a'_D(z_1, z_2) \leq \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right) = j'_D(z_1, z_2). \qquad \square$$

**Corollary 2.3.** *Suppose that $D$ is a simply connected proper subdomain in* $\mathbf{C}$ *and that there is a constant $c$ such that*

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2)$$

*for all $z_1, z_2 \in D$. Then $D$ is a $b$-John disk, where $b$ depends only on $c$.*

*Proof.* By Lemma 2.2, we see that there exists a constant $c$ such that

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2) \le c\, j'_D(z_1, z_2).$$

It follows from Theorem B that $D$ is a $b$-John disk, where $b$ depends only on $c$. □

## 3. Proof of main theorems

For the proof of Theorem 1.2, we need the following lemma.

**Lemma 3.1.** *Let $D$ be the bounded domain bounded by $L_j$ $(j = 1, 2, \dots, 5)$ as in Theorem 1.2. Let $z_1 = -t^2 - it \in D$ with $0 < t < 1/2$, and $z_2 = -\overline{z_1}$. Then we have*

$$\frac{\lambda_D(z_2, p)}{\lambda_D(z_1, p)} \le 1 + 4t^2 \tag{3.2}$$

*and*

$$\frac{\lambda_D(z_1, p)}{\lambda_D(z_2, p)} \le 1 + 4t^2 \tag{3.3}$$

*for all $p \in \partial D$.*

*Proof.* Obviously, it is sufficient to prove (3.2). The proof of (3.3) easily follows from the proof of (3.2).

Let $L_j$ $(j = 1, 2, \dots, 5)$ be defined in Theorem 1.2. Then $\partial D = \bigcup_{j=1}^{5} L_j$, see Figure 1. For any $p = (x, y) \in L_1$, we divide the proof into several steps.

*Case* (i). Let $x > -2t^2/(1 + t^2)$. Then $(1 + x/2)(1 + t^2) > 1$. Now we compute

$$\begin{aligned}
&(1+t^2)\lambda_D^2(z_1,p) - \lambda_D^2(z_2,p)\\
&= (1+t^2)|z_1 - p|^2 - (|z_2| + |p|)^2\\
&= (1+t^2)|z_1 - p|^2 - (|z_1| + |p|)^2\\
&= (1+t^2)\left[(x+t^2)^2 + (y+t)^2\right] - \left(\sqrt{x^2+y^2} + \sqrt{t^2+t^4}\right)^2,\\
&= 2t\left[(1+t^2)y - \sqrt{-2x}\sqrt{1+t^2} + t^3(x+1/2) + t^5/2\right] \quad (\text{since } x^2+y^2 = -2x)\\
&= 2t\left[\sqrt{1+t^2}\sqrt{-2x}\left(\sqrt{1+t^2}\sqrt{1+x/2} - 1\right) + t^3(x+1/2) + t^5/2\right]
\end{aligned}$$

which is clearly nonnegative and hence, we have

$$\frac{\lambda_D(z_2, p)}{\lambda_D(z_1, p)} \le \sqrt{1 + t^2}.$$

Note that $\sqrt{1 + t^2} < 1 + 4t^2$ for $0 < t < 1/2$.

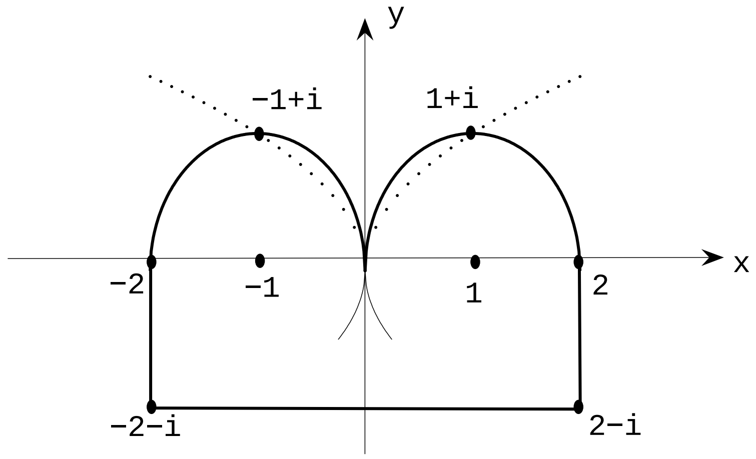


Figure 1. The domain $D$ bounded by five curves $L_j$ $(j = 1, 2, \ldots, 5)$.

*Case* (ii). Let $x \le -2t^2/(1+t^2)$. Then $0 \le (1 + x/2)(1 + t^2) \le 1$. In this case, we see that

$$\begin{aligned}(1+4t^2)\lambda_D^2(z_1,p) - \lambda_D^2(z_2,p) &= (1+4t^2)|z_1-p|^2 - |z_2-p|^2 \\ &= (1+4t^2)[(x+t^2)^2+(y+t)^2] \\ &\quad - [(x-t^2)^2+(y+t)^2] \\ &= 4t^2[-x(1-2t^2)+t^4+t^2+2ty] \ge 0,\end{aligned}$$

since $x^2 + y^2 = -2x$ and $0 < t < 1/2$. It follows that

$$\frac{\lambda_D(z_2,p)}{\lambda_D(z_1,p)} \le \sqrt{1+4t^2}.$$

Thus, (3.2) holds when $p \in L_1$.

If $p \in L_2$, then we see that $\lambda_D(z_1, p) \ge 2 - t^2$. If $p \in L_3$, then we find that $\lambda_D(z_1, p) \ge 1 - t$. Consequently, we easily obtain that

$$\frac{\lambda_D(z_2,p)}{\lambda_D(z_1,p)} \le \frac{\lambda_D(z_1,p)+|z_1-z_2|}{\lambda_D(z_1,p)} = 1 + \frac{2t^2}{\lambda_D(z_1,p)} \le \begin{cases} 1+\frac{16}{7}t^2 & \text{if } p \in L_2, \\ 1+4t^2 & \text{if } p \in L_3. \end{cases}$$

Finally, if $p \in L_4 \cup L_5$, it is obvious that

$$\frac{\lambda_D(z_2,p)}{\lambda_D(z_1,p)} \le 1.$$

The proof of (3.2) is completed. □

*Proof of Theorem 1.2.* The equivalence of (1) and (3) in Theorem A implies that $D$ is a $b$-John disk for some constant $b$ with $0 < b \le 6(2 + \sqrt{2})$.

Suppose, on the contrary, that there exists a constant $c$ such that

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2) \tag{3.4}$$

for all $z_1, z_2 \in D$.

Let $z_1$ and $z_2$ be as in Lemma 3.1. Then (3.2) and (3.3) imply that

$$a'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log\left(\frac{\lambda_D(z_1, w_1)\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)\lambda_D(z_2, w_1)}\right) \le 2\log(1 + 4t^2).$$

By Lemma 2.1, we deduce that

$$h_D(z_1, z_2) \ge \frac{1}{2}\log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right) = \frac{1}{2}\log\left(1 + \frac{2t}{\sqrt{t^2+1}}\right).$$

Thus, by assumption (3.4), these two inequalities yield that

$$\log\left(1 + \frac{2t}{\sqrt{t^2+1}}\right) \le 2\, h_D(z_1, z_2) \le 2c\, a'_D(z_1, z_2) \le 4c\, \log(1 + 4t^2).$$

But, on the other hand,

$$\lim_{t \to 0} \frac{\log\left(1 + 2t/\sqrt{t^2+1}\right)}{\log(1 + 4t^2)} = \infty$$

which is a contradiction and we complete the proof of Theorem 1.2. □

Before the proof of Theorem 1.3, we prove the following lemma.

**Lemma 3.5.** *Let $D$ be as in Theorem 1.3 and let $z_1 = x_0 + \frac{1}{2}i$, $z_2 = \overline{z_1}$ with $x_0 < 0$. Then we have*

$$\frac{\lambda_D(z_1, p)}{\lambda_D(z_2, p)} \le \sqrt{1 + \frac{1}{\operatorname{dist}(z_2, \partial D)^2}} \tag{3.6}$$

*and*

$$\frac{\lambda_D(z_2, p)}{\lambda_D(z_1, p)} \le \sqrt{1 + \frac{1}{\operatorname{dist}(z_1, \partial D)^2}} \tag{3.7}$$

*for all $p \in \partial D$.*

*Proof.* Let $L_1 = \{x + i/2 : \ x > 0\}$, $L_2 = \{iy : \ -1/2 \le y \le 1/2\}$, $L_3 = \{x - i/2 : \ x > 0\}$, see Figure 2.

We need only to prove (3.6) as the proof of (3.7) easily follows from that of (3.6).

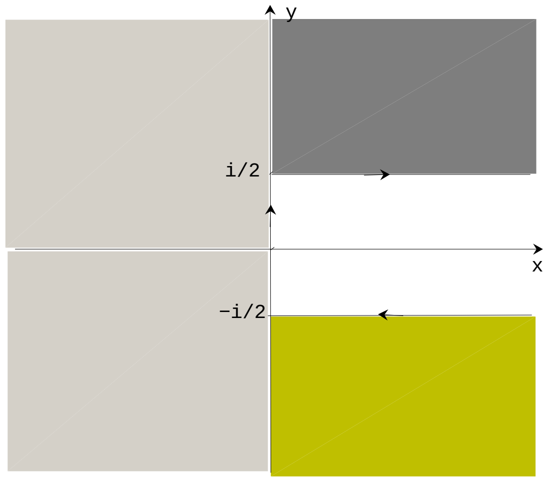


Figure 2. $D$, the exterior of the region bounded by $L_1, L_2, L_3$.

If $p \in L_1$, then we have

$$\lambda_D(z_1, p) = |z_1 - p| = \text{dist}(z_1, \partial D) + |p - i/2|$$

and

$$\lambda_D(z_2, p) = \lambda_D(z_2, i/2) + |p - i/2| = \sqrt{1 + \text{dist}(z_1, \partial D)^2} + |p - i/2|$$

so that (3.6) becomes obvious.

If $p \in L_2$, then

$$\lambda_D(z_1, p) = |z_1 - p| \quad \text{and} \quad \lambda_D(z_2, p) = |z_2 - p|.$$

Obviously,

$$\frac{\lambda_D(z_1, p)}{\lambda_D(z_2, p)} \le \frac{\lambda_D(z_1, -i/2)}{\lambda_D(z_2, -i/2)} \le \frac{|z_1 + i/2|}{|z_2 + i/2|} = \sqrt{1 + \frac{1}{\text{dist}(z_2, \partial D)^2}}.$$

If $p \in L_3$, then

$$\lambda_D(z_2, p) = |z_2 - p| = \text{dist}(z_2, \partial D) + |p + i/2|$$

and

$$\lambda_D(z_1, p) = \lambda_D(z_1, -i/2) + |p + i/2| = \sqrt{1 + \text{dist}(z_2, \partial D)^2} + |p + i/2|.$$

We now obtain that

$$\frac{\lambda_D(z_1, p)}{\lambda_D(z_2, p)} \le \frac{\lambda_D(z_1, -i/2)}{\lambda_D(z_2, -i/2)} = \frac{\sqrt{1 + \text{dist}(z_2, \partial D)^2}}{\text{dist}(z_2, \partial D)} = \sqrt{1 + \frac{1}{\text{dist}(z_2, \partial D)^2}}.$$

The proof is completed. □

*Proof of Theorem 1.3.* Clearly, (2) in Theorem A holds with $c = 1$ and so, one could get an explicit constant, for instance $b = 6$. Now, the equivalence of (1) and (2) in Theorem A implies that $D$ is a $b$-John disk with $b = 6$.

Suppose, on the contrary, that there exists a constant $c$ such that

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2) \tag{3.8}$$

for all $z_1$, $z_2 \in D$, where $D$ is as in Theorem 1.3.

Now, let $z_1$ and $z_2$ be as in Lemma 3.5. Then

$$\begin{aligned} a'_D(z_1, z_2) &= \sup_{w_1, w_2 \in \partial_r D} \log \left( \frac{\lambda_D(z_1, w_1)\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)\lambda_D(z_2, w_1)} \right) \\ &\le \log \left( \sqrt{1 + \frac{1}{\operatorname{dist}(z_2, \partial D)^2}} \sqrt{1 + \frac{1}{\operatorname{dist}(z_1, \partial D)^2}} \right). \end{aligned}$$

We know that

$$a'_D(z_1, z_2) \le \log \left( 1 + \frac{1}{\operatorname{dist}(z_1, \partial D)^2} \right), \tag{3.9}$$

since $\operatorname{dist}(z_1, \partial D) = \operatorname{dist}(z_2, \partial D)$.

It follows from Lemma 2.1, (3.8) and (3.9) that

$$\log \left( 1 + \frac{1}{\operatorname{dist}(z_1, \partial D)} \right) \le 2c \log \left( 1 + \frac{1}{\operatorname{dist}(z_1, \partial D)^2} \right).$$

But

$$\lim_{x_0 \to -\infty} \frac{\log \left( 1 + \frac{1}{\operatorname{dist}(z_1, \partial D)} \right)}{\log \left( 1 + \frac{1}{\operatorname{dist}(z_1, \partial D)^2} \right)} = \infty.$$

This is the desired contradiction. □

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