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ON NONLANDING DYNAMIC RAYS OF EXPONENTIAL MAPS

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Abstract. We consider the case of an exponential map $E_\kappa\colon z \mapsto \exp(z) + \kappa$ for which the singular value κ is accessible from the set of escaping points of E_κ. We show that there are dynamic rays of E_κ which do not land. In particular, there is no analog of Douady's "pinched disk model" for exponential maps whose singular value belongs to the Julia set. We also prove that the boundary of a Siegel disk U for which the singular value is accessible both from the set of escaping points and from U contains uncountably many indecomposable continua.

1. Introduction

In polynomial dynamics, *dynamic rays* provide an important tool which permits the investigation of a function's dynamics in combinatorial terms. In many important cases, the Julia set is locally connected, and all dynamic rays land. In this situation, the Julia set can be described as a "pinched disk" [Do], that is, as the quotient of $\mathbf{S}^1$ by a natural equivalence relation. These ideas form the foundation for many spectacular advances recently made in the study of polynomial maps.

In this article, we consider the family of exponential maps $E_\kappa\colon z \mapsto \exp(z) + \kappa$, which has enjoyed much attention over the past two decades as the simplest parameter space of transcendental entire functions. For such maps, the set of escaping points

$$I(E_\kappa) := \{z \in \mathbf{C} : |E_\kappa^n(z)| \to \infty\}$$

contains no open subsets, and thus belongs to the Julia set, which is not locally connected even in the simplest cases.

Nonetheless, it is known that the path-connected components of $I(E_\kappa)$ are curves to ∞ [SZ1], providing an analog for dynamic rays of polynomials. It is therefore natural to ask whether there is also an analog of the notion of local connectivity, and a corresponding topological model for the dynamics of "tame" exponential maps defined only in terms of their combinatorics.

This is indeed the case when E_κ has an attracting or parabolic cycle: here the Julia set is homeomorphic to a "pinched Cantor Bouquet"; that is, the quotient

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of a certain (universal) space by a suitable equivalence relation. Furthermore, the dynamics on $J(E_\kappa)$ is the quotient of a universal dynamical system on this Cantor Bouquet. (See [BDD, Theorem 5.7] and [R4, Corollary 9.3].) We will show that this is the only case in which this is possible.

1.1. Theorem. (Exponential maps with simple Julia sets) *Let E_κ be an exponential map. Then the following are equivalent:*

(a) *E_κ has an attracting or parabolic orbit.*
(b) *Every dynamic ray of E_κ lands in $\hat{\mathbf{C}}$ and every point of the Julia set is on a dynamic ray or the landing point of such a ray.*

This shows that there is no obvious analog of the "pinched disk model" for exponential maps whose singular value κ belongs to the Julia set. This includes "tame" examples such as the postsingularly finite ("Misiurewicz") case. In contrast, polynomial Misiurewicz maps always have locally connected Julia sets.

To prove Theorem 1.1, we will show the following result on the existence of nonlanding rays.

1.2. Theorem. (Existence of nonlanding rays) *Suppose that E_κ is an exponential map whose singular value κ is on a dynamic ray or is the landing point of such a ray.*

Then there exist uncountably many dynamic rays g whose accumulation set (on the Riemann sphere) is an indecomposable continuum containing g.

If the hypotheses of this theorem are satisfied, we say that the singular value is *accessible* (from the escaping set). By [SZ1, SZ2], all Misiurewicz parameters, as well as all parameters for which the singular value escapes (*escaping parameters*) satisfy this condition. We expect that accessibility holds for a much larger class of parameters.

The presence of indecomposable continua in exponential dynamics (albeit not as the accumulation set of a dynamic ray) was first observed by Devaney [De] when $\kappa \in (-1, \infty)$. For Misiurewicz parameters, the existence of dynamic rays whose accumulation sets are such continua (as in Theorem 1.2) was first observed by Schleicher in 2000 (personal communication; see also [DJM]). The same result was proved in the case of $\kappa \in (-1, \infty)$ by Devaney and Jarque [DJ], using similar methods. The basic idea underlying both these results, as well as our proof, is fairly simple (see "idea of the proof" below). However, we will gain control of the accumulation sets of dynamic rays using combinatorial considerations rather than the previously used expansion methods (which rely on the fact that the singular orbit is discrete). This allows us to prove our result in a considerably more general situation.

Theorem 1.2 has the interesting continuum-theoretic corollary that there is no analog of the Moore triod theorem [P, Proposition 2.18] for "Knaster-like" continua; i.e. indecomposable continua containing dense curves (compare [CP, Section 8] for a discussion of such questions).

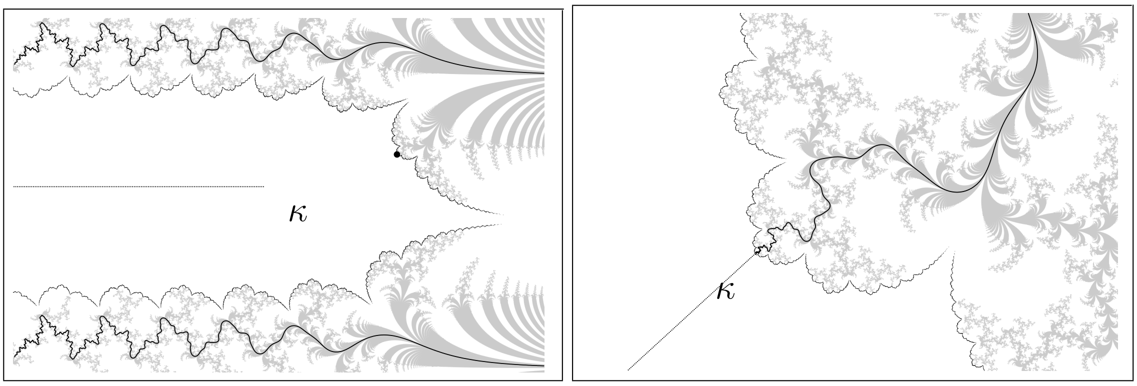

Figure 1. An exponential map with a Siegel disk U whose rotation number is the golden mean. The solid curves are dynamic rays (contained in the escaping set $I(E_\kappa)$), while the dotted line is in the Siegel disk. The singular orbit (which is dense in ∂U) is also plotted. The picture on the right is a magnification about the singular value; the curves here are the images of the corresponding curves on the left. These pictures suggest that κ is accessible both from U and from $I(E_\kappa)$.

1.3. Corollary. (Uncountable number of indecomposable continua) *There is an uncountable set of pairwise disjoint indecomposable plane continua each of which contains a dense injective curve.*

Our results also have an interesting consequence for exponential maps with Siegel disks. Suppose that E_κ has a Siegel disk U whose rotation number is of bounded type. It is conjectured that the singular value κ (and hence ∞) is accessible both from U and from the escaping set. However, it is currently unknown whether this is true for *any* rotation number, even for the golden mean (compare Figure 1).

The analogous problem for quadratic (or, more generally, unicritical) polynomials is resolved by showing that the function is quasiconformally conjugate to an appropriate model map. In particular, the Siegel disk is a quasicircle, and every boundary point is accessible. One of the problems with producing a similar proof for the exponential family is that it is not clear what the topological model for the boundary ∂U should be. One might at first expect that every component of ∂U is an arc to ∞. However, the following result shows that the model would need to be more complicated.

1.4. Theorem. (Indecomposable continua in Siegel disk boundaries) *Suppose that E_κ is an exponential map for which the singular value κ is accessible both from a Siegel disk U and from the escaping set $I(E_\kappa)$.*

Then ∂U contains uncountably many indecomposable continua.

It seems likely that, for every exponential map E_κ with $\kappa \in J(E_\kappa)$, there exists some dynamic ray whose accumulation set contains an entire dynamic ray. Towards this end, we show the following variant of Theorem 1.2.

1.5. Theorem. (Accumulation at infinity) *Let $\kappa \in \mathbf{C}$. Then either*

(a) *the accumulation set of every dynamic ray of E_κ is bounded, or*

(b) *there are uncountably many dynamic rays whose accumulation set contains a complete dynamic ray.*

Remark. In particular, case (b) holds whenever the singular value κ belongs to the accumulation set of some dynamic ray (even if the ray does not land at κ).

Idea of the proof. The proof of the main theorem requires a certain amount of combinatorial preparations, so let us sketch the underlying idea, which is fairly straightforward. Suppose that g_1 is some iterated preimage of the dynamic ray g_0 landing at the singular value κ of E_κ. Since a preimage of a curve landing at κ under E_κ will be a curve whose real parts tend to $-\infty$, it follows that the curve g_1 tends to ∞ in both directions.

Now pick some other dynamic ray g_2, very close to g_1, which is another iterated preimage of g_0. Then g_2 will stay close to g_1 for a long time, but eventually tend to ∞ in a different direction. If we repeat this process, in the limit we should end up with a dynamic ray which does not have a landing point. It is possible to ensure that this limit ray will actually accumulate on itself, and to conclude that its closure is an indecomposable continuum.

Organization of the article. In Section 2, we review the basic combinatorial concepts required for this article, while Section 3 is devoted to a short discussion of the continuity of dynamic rays with respect to their external address. (A more comprehensive discussion of these topics can be found in [RS, Section 2 and 3] and [R4, Section 3], respectively.)

In Section 4, we give a simple combinatorial condition under which the limit set of a ray which accumulates on itself is an indecomposable continuum. Section 5 contains the proof of our main result.

Finally, Section 6 discusses the accumulation of dynamic rays at infinity in a more general setting, in particular providing the proof of Theorem 1.5. Appendix A contains a brief discussion of generalizations of our results to larger classes of entire functions.

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Notation. Throughout this article, $\mathbf{C}$ and $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ denote the complex plane and the Riemann sphere, respectively. The closure of a set $A \subset \mathbf{C}$ in $\mathbf{C}$ resp. $\hat{\mathbf{C}}$ will be denoted $\overline{A}$ and $\widehat{A}$, respectively. The Fatou and Julia sets of an exponential map are denoted $F(E_\kappa)$ and $J(E_\kappa)$, as usual.

Let us also fix the function $F\colon [0,\infty) \to [0,\infty); t \mapsto \exp(t) - 1$ as a model of exponential growth. We conclude any proof by the symbol ■, while a result which is cited without proof is indicated by □. Separate steps within a proof are concluded by the symbol △.

2. Combinatorics of exponential maps

It has long been customary in exponential dynamics to encode the mapping behavior of dynamically defined curves under E_κ using symbolic dynamics. We will give a concise summary of these concepts here; see [RS, Sections 2 and 3] for a more comprehensive account.

A sequence $\underline{s} = s_1 s_2 s_3 \ldots$ of integers is called an *(infinite) external address.* If E_κ is an exponential map and $\gamma : (T, \infty) \to \mathbf{C}$ is a curve, we say that γ *has external address* $\underline{s}$ (as $t \to \infty$) if

$$\operatorname{Re} E_\kappa^j(\gamma(t)) \underset{t\to\infty}{\longrightarrow} +\infty \quad \text{and} \quad \operatorname{Im} E_\kappa^j(\gamma(t)) \underset{t\to\infty}{\longrightarrow} 2\pi s_{j+1}$$

for all $j \geq 0$. We say that an external address $\underline{s}$ is *exponentially bounded* if

$$t_{\underline{s}} := \limsup_{k\to\infty} F^{-(k-1)}(2\pi |s_k|) < \infty.$$

The space of all exponentially bounded external addresses is denoted by $\mathscr{S}_0$.

2.1. Proposition. (Classification of escaping points [SZ1]) *Let* $\kappa \in \mathbf{C}$ *and* $\underline{s} \in \mathscr{S}_0$. *Then there is a unique maximal curve* $g_{\underline{s}} \colon (t_{\underline{s}}^\kappa, \infty) \to I(E_\kappa)$ *(where* $t_{\underline{s}}^\kappa \geq t_{\underline{s}}$*) of escaping points which has external address* $\underline{s}$ *as* $t \to \infty$ *and satisfies*

$$|\operatorname{Re} E_\kappa^n(g_{\underline{s}}(t)) - F^n(t)| \to 0$$

as $n \to \infty$ *for any* $t > t_{\underline{s}}^\kappa$. *This curve is unique up to reparametrization and is called the* dynamic ray at address $\underline{s}$. *We say that the ray* $g_{\underline{s}}$ lands *at a point* $z \in \hat{\mathbf{C}}$ *if* $\lim_{t\to t_{\underline{s}}^\kappa} g_{\underline{s}}(t) = z$.

If $\kappa \notin I(E_\kappa)$, *then every escaping point is either on a dynamic ray or the landing point of a dynamic ray. If* $\kappa \in I(E_\kappa)$, *then every escaping point eventually maps to a point on a dynamic ray or to the landing point of a dynamic ray.* □

If $g_{\underline{s}}$ is a dynamic ray, we will denote its limit set by

$$L_{\underline{s}} := \bigcap_{t > t_{\underline{s}}^\kappa} \overline{g_{\underline{s}}\big((t_{\underline{s}}^\kappa, t]\big)} \subset \hat{\mathbf{C}}.$$

Intermediate external addresses. An *intermediate external address* is a finite sequence of the form

$$\underline{s} = s_1 s_2 \ldots s_{n-2} s_{n-1} \infty,$$

where $n \geq 2$, $s_k \in \mathbf{Z}$ for $k < n-1$ and $s_{n-1} \in \mathbf{Z} + \frac{1}{2}$. The space $\mathscr{S}$ of all infinite and intermediate external addresses, equipped with lexicographic order, is order-complete. Its one-point compactification is $\overline{\mathscr{S}} := \mathscr{S} \cup \{\infty\}$, which carries a complete circular ordering. (We can think of ∞ as being an intermediate external address of length 1.) The shift map $\sigma \colon \mathscr{S} \to \overline{\mathscr{S}}$ is a locally order-preserving map. We will say that $\underline{r}_1, \underline{r}_2 \in \mathscr{S}$ *surround* an address $\underline{s}$ if $\underline{s}$ belongs to the bounded component of $\mathscr{S} \setminus \{\underline{r}_1, \underline{r}_2\}$.

Addresses of connected sets. Let $\underline{r}^-, \underline{r}^+ \in \mathscr{S}_0$ with $\underline{r}^- < \underline{r}^+$, and let $R > 0$. We say that $\langle \underline{r}^-, \underline{r}^+ \rangle$ *essentially separates* the half plane $\mathbf{H}_R := \{\operatorname{Re} z > R\}$ if the set

$$\mathbf{H}_R \setminus \big(g_{\underline{r}^-}\big([t^\kappa_{\underline{r}^-} + 1, \infty)\big) \cup g_{\underline{r}^+}\big([t^\kappa_{\underline{r}^+} + 1, \infty)\big)\big)$$

has a component U with unbounded real parts but bounded imaginary parts. (In other words, both ray pieces intersect the line $\{\operatorname{Re} z = R\}$.) The component U (if it exists) is necessarily unique, and will be denoted by $U_R(\langle \underline{r}^-, \underline{r}^+ \rangle)$.

Let $A \subset \mathbf{C}$, and let $\underline{s} \in \mathscr{S}_0$. We say that A *is separated from* $\underline{s}$ if there exist $\underline{r}^-, \underline{r}^+ \in \mathscr{S}_0$ and some $R > 0$ such that $\underline{r}^-$ and $\underline{r}^+$ surround $\underline{s}$, the pair $\langle \underline{r}^-, \underline{r}^+ \rangle$ essentially separates $\mathbf{H}_R$ and $A \cap U_R(\langle \underline{r}^-, \underline{r}^+ \rangle) = \emptyset$. Similarly, we say that A is separated from the address $\infty \in \overline{\mathscr{S}}$ if $\operatorname{Im} A$ is bounded and $\operatorname{Re} A$ is bounded from below. The set

$$\operatorname{Addr}(A) := \{\underline{s} \in \overline{\mathscr{S}} : A \text{ is not separated from } \underline{s}\}$$

is clearly compact, and is empty if and only if A is bounded.

Remark. Another way to phrase this definition is as follows: we form a natural compactification of $\mathbf{C}$ by adjoining the space $\overline{\mathscr{S}}$ as a circle at ∞. The set $\operatorname{Addr}(A)$ is then exactly the set of accumulation points of A in $\overline{\mathscr{S}}$ with respect to this topology.

Two directions of dynamic rays. We will usually apply the above concepts to dynamic rays $g_{\underline{s}}$ which accumulate at ∞ as $t \searrow t^\kappa_{\underline{s}}$. In order to facilitate these discussions, let us abbreviate

$$\operatorname{Addr}^-(\underline{s}) := \operatorname{Addr}^-(g_{\underline{s}}) := \operatorname{Addr}\big(g_{\underline{s}}\big((t^\kappa_{\underline{s}}, t^\kappa_{\underline{s}} + 1]\big)\big).$$

Itineraries

2.2. Definition. (Accessible singular values) Let E_κ be an exponential map with $\kappa \in J(E_\kappa)$. We say that the singular value is *accessible* if κ is either on a dynamic ray or the landing point of a dynamic ray. If $\underline{r}$ is the external address of such a ray, we write $\underline{r} = \operatorname{addr}(\kappa)$.

Remark. 1. The address $\underline{r}$ need not be unique.

2. Misiurewicz and escaping parameters have accessible singular values, as mentioned in the introduction. Furthermore, the landing points of parameter rays at "regular" addresses in the sense of Devaney, Goldberg and Hubbard [BDG] have this property. We expect that accessibility holds in most situations in which local connectivity is known for unicritical polynomials, for example the Siegel parameters mentioned in the introduction. In particular, we believe that this condition should be generic in the bifurcation locus and includes many parameters for which the singular orbit is dense in the plane.

Suppose that E_κ is an exponential map with accessible singular value in $J(E_\kappa)$, and let $\underline{r} = \operatorname{addr}(\kappa)$. Then the preimages $E_\kappa^{-1}(g_{\underline{r}})$ (that is, the dynamic rays at addresses of the form $k\underline{r}$ for $k \in \mathbf{Z}$) cut the plane into countably many strips S_k. Let us label these such that S_k is the strip bounded by $g_{k\underline{r}}$ and $g_{(k+1)\underline{r}}$. (We refer to these strips as the *dynamic partition*.)

If $z \in \mathbf{C}$, we can assign to z an *itinerary*

$$\operatorname{itin}(z) := \operatorname{itin}_{\underline{r}}(z) := \underline{\mathtt{u}} := \mathtt{u}_1\mathtt{u}_2\mathtt{u}_3\dots$$

where $\mathtt{u}_j = k$ if $E_\kappa^{j-1}(z) \in S_k$, and $\mathtt{u}_j = \binom{\mathtt{j}}{\mathtt{j}-1}$ if $E_\kappa^{j-1}(z) \in g_{j\underline{r}}$.

If $z \in g_{\underline{s}}$ for some $\underline{s} \in \mathscr{S}_0$, then the itinerary entries of z clearly satisfy

$$\mathtt{u}_k = \begin{cases} \mathtt{j} & \text{if } \mathtt{j}\underline{r} < \sigma^{k-1}(\underline{s}) < (\mathtt{j}+1)\underline{r} \\ \binom{\mathtt{j}}{\mathtt{j}-1} & \text{if } \sigma^{k-1}(\underline{s}) = \mathtt{j}\underline{r}. \end{cases} \tag{1}$$

For any $\underline{r} \in \mathscr{S}$ and every infinite external address $\underline{s}$, we can define an address $\operatorname{itin}_{\underline{r}}(\underline{s})$ by the formula (1). If $\underline{s}$ is an intermediate external address of length n, then we similarly define

$$\operatorname{itin}_{\underline{r}}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*,$$

where $\mathtt{u}_1, \dots, \mathtt{u}_{n-1} \in \mathbf{Z}$ satisfy (1). The *kneading sequence* of $\underline{s} \in \mathscr{S}$ is the itinerary $\mathbb{K}(\underline{s}) := \operatorname{itin}_{\underline{s}}(\underline{s})$.

Finally, let us say that an itinerary entry $\mathtt{m} \in \mathbf{Z}$ is *adjacent* to the entry $\mathtt{u}$ if $\mathtt{m} = \mathtt{u}$ or $\mathtt{u}$ is a boundary sumbol $\binom{\mathtt{j}}{\mathtt{j}-1}$ with $\mathtt{m} \in \{\mathtt{j}-1, \mathtt{j}\}$.

We will frequently use the following simple observation.

2.3. Lemma. (Addresses sharing an itinerary) *Let $\underline{s} \in \mathscr{S}$ and $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$. Suppose that $\underline{r} \neq \underline{\widetilde{r}}$ are two addresses sharing the same itinerary $\underline{\tilde{\mathtt{u}}} := \operatorname{itin}_{\underline{s}}(\underline{r}) = \operatorname{itin}_{\underline{s}}(\underline{\widetilde{r}})$, and let $m \geq 1$ with $r_m \neq \widetilde{r}_m$.*

Then for every $k \geq 0$, there exists $0 \leq j \leq k$ such that $\sigma^{m+k}(\underline{r})$ and $\sigma^{m+k}(\underline{\widetilde{r}})$ surround $\sigma^j(\underline{s})$. In particular, $\tilde{\mathtt{u}}_{m+k} \in \{\mathtt{u}_1, \dots, \mathtt{u}_k\}$ for all $k \geq 1$.

Proof. Let $k \geq 0$. We may suppose without loss of generality that $r_{m+i} = \widetilde{r}_{m+i}$ for all $i \in \{1, \dots, k\}$. (Otherwise, we can replace m by $m+i$ and k by $k-i$.)

By the definition of itineraries, $\underline{s}$ belongs to the interval I of $\mathscr{S}$ between $\sigma^m(\underline{r})$ and $\sigma^m(\underline{\widetilde{r}})$. Since we assumed that the latter addresses agree in their first k entries, it follows that $\sigma^k(I) \ni \sigma^k(\underline{s})$ is the interval bounded by $\sigma^{m+k}(\underline{r})$ and $\sigma^{m+k}(\underline{\widetilde{r}})$, as required. ∎

3. Continuity among dynamic rays

The dependence of $g_{\underline{s}}(t)$ on the pair $(\underline{s}, t)$ is quite complicated in general, and will depend to some degree on the parameter κ. On the other hand, it is well-known that there is a certain continuity in the $\underline{s}$-direction, e.g. in the sense that every compact subpiece of the dynamic ray $g_{\underline{s}}$ can be approximated from above and below by suitable sequences of dynamic rays. In fact, in [R4], a *simultaneous* parametrization of all dynamic rays was given, which greatly simplifies both questions of this kind and those regarding escaping endpoints of rays. However, to reduce the prerequisites for this article, and to improve consistency among articles on exponential dynamics, we shall state the required results here using the original parametrization from [SZ1], and sketch a proof using only results from [SZ1].

3.1. Proposition. (Asymptotics of dynamic rays [SZ1, Proposition 3.4]) *Let $\kappa \in \mathbf{C}$ and $\underline{s} \in \mathscr{S}_0$, and set*

$$t^*_{\underline{s}} := \sup_{k\in\mathbf{N}} F^{-(k-1)}(2\pi|s_k|).$$

Then $t^\kappa_{\underline{s}} < T_{\underline{s}} := 2t^*_{\underline{s}} + \log^+|\kappa| + 4$, and for all $t \geq T_{\underline{s}}$,

$$\bigl|g_{\underline{s}}(t) - (t + 2\pi i s_1)\bigr| < e^{-t/2}. \qquad \square$$

Remark. This is not quite the formulation given in [SZ1], but is easily deduced from it.

3.2. Lemma. (Continuity between rays) *Let $\kappa \in \mathbf{C}$, $\underline{s} \in \mathscr{S}_0$ and $K > 0$. For $n_0 \in \mathbf{N}$, denote by $S(\underline{s}, K, n_0)$ the set of all addresses $\widetilde{\underline{s}}$ which agree with $\underline{s}$ in the first n_0 entries and satisfy $|\widetilde{s}_n| \leq K + \max_{k\leq n}|s_k|$ for all $n \geq 1$.*

Then for every $t_0 > t^\kappa_{\underline{s}}$ and every $\varepsilon > 0$, there exists n_0 such that $t_0 > t^\kappa_{\widetilde{\underline{s}}}$ and

$$|g_{\underline{s}}(t) - g_{\widetilde{\underline{s}}}(t)| < \varepsilon$$

for all $t \geq t_0$ and all $\widetilde{\underline{s}} \in S(\underline{s}, K, n_0)$.

Sketch of proof. It is easy to see (from the definitions of $t_{\underline{s}}$ and $t^*_{\underline{s}}$, together with the fact that $t_0 > t^\kappa_{\underline{s}} \geq t_{\underline{s}}$), that there is some n_1 such that

$$F^n(t_0) > 2t^*_{\sigma^n(\underline{s})} + 2K + |\kappa| + 4$$

for all $n \geq n_1$. By definition, we have $t^*_{\sigma^n(\widetilde{\underline{s}})} \leq t^*_{\sigma^n(\underline{s})} + K$ for all $\widetilde{\underline{s}} \in S(\underline{s}, K, n)$. Thus we have $F^n(t) > T_{\sigma^n(\widetilde{\underline{s}})}$ (where $T_{\sigma^n(\widetilde{\underline{s}})}$ is the number from Proposition 3.1) whenever $n \geq n_1$, $\widetilde{\underline{s}} \in S(\underline{s}, K, n+1)$ and $t \geq t_0$. In particular, since $F^n(t) > |\kappa| + 4$, we have

$$\begin{aligned}&\operatorname{Re} g_{\sigma^n(\widetilde{\underline{s}})}\bigl(F^n(t)\bigr) \geq F^n(t) - e^{-F^n(t)/2} > \operatorname{Re}\kappa + 2 \quad \text{and}\\ &|g_{\sigma^n(\widetilde{\underline{s}})}(F^n(t)) - g_{\sigma^n(\underline{s})}(F^n(t))| < 2e^{-F^n(t)/2} < 1.\end{aligned}$$

In particular, for $n \geq n_1$, the pieces $g^n := g_{\sigma^n(\underline{s})}\bigl([F^n(t_0), \infty)\bigr)$ are contained in the half-plane $H := \{z : \operatorname{Re}(z - \kappa) > 2\}$. Note that E_κ is expanding on H. It follows easily that, for sufficiently large n, there exists a branch

$$\phi : \bigl\{z \in \mathbf{C} : \operatorname{dist}(z, g^n) < 1\bigr\} \to \mathbf{C}$$

of E_κ^{-n} with $\phi(g_{\sigma^n(\underline{s})}(F^n(t))) = g_{\underline{s}}(t)$ for $t \geq t_0$ such that $|\phi'| < \varepsilon$. By the definition of dynamic rays, we have

$$\phi(g_{\sigma^n(\widetilde{\underline{s}})}(F^n(t))) = g_{\widetilde{\underline{s}}}(t)$$

for all $\widetilde{\underline{s}} \in S(\underline{s}, K, n+1)$, and the claims follow. ∎

3.3. Lemma. (Accumulation on dynamic rays) *Let E_κ be an exponential map, and suppose that $A \subset \mathbf{C}$ is connected and intersects at most countably many dynamic rays of E_κ. If $\underline{s} \in \mathscr{S}_0$ with $\underline{s} \in \operatorname{Addr}(A)$, then $g_{\underline{s}} \subset \overline{A}$ or $\overline{A} \subset g_{\underline{s}}$.*

Proof. Since A only intersects countably many dynamic rays, we can find a sequence (k_n) with $k_n \in \{1, 2\}$ such that, for all n, A does not intersect the dynamic rays $g_{\underline{r}_n^+}$ and $g_{\underline{r}_n^-}$ defined by

$$\underline{r}_n^{\pm} := s_1 \dots s_n (s_{n+1} \pm k_{n+1})(s_{n+2} \pm k_{n+1}) \dots .$$

By Lemma 3.2, $g_{\underline{r}_n^+} \to g_{\underline{s}}$ uniformly on every interval $[t, \infty)$ with $t > t_{\underline{s}}^{\kappa}$, and the same is true for $\underline{r}_n^-$.

So the rays $g_{\underline{r}_n^{\pm}}$ approximate $g_{\underline{s}}$ from above and below. Suppose that there is $t_0 > t_{\underline{s}}^{\kappa}$ such that $\varepsilon := \operatorname{dist}(g_{\underline{s}}(t_0), A) > 0$, and let U_n denote the component of

$$\mathbf{C} \setminus \big(\mathbf{D}_{\varepsilon}(g_{\underline{s}}(t_0)) \cup g_{\underline{r}_n^+} \cup g_{\underline{r}_n^-}\big)$$

which contains $g_{\underline{s}}(t)$ for large t. Then $A \subset U_n$. Clearly $\bigcap U_n \subset g_{\underline{s}}\big([t_0, \infty)\big)$, and the claim follows. ∎

3.4. Corollary. *Let E_κ have an accessible singular value, and $\underline{s} = \operatorname{addr}(\kappa)$, and let $\underline{r} \in \mathscr{S}_0$.*

(a) *Suppose that $\operatorname{Addr}^-(\underline{r})$ contains an intermediate external address of length m or a preimage of $\underline{s}$ under σ^m. Then the ray $g_{\sigma^m(\underline{r})}$ accumulates on κ. In particular, all itinerary entries of $\sigma^m(\underline{r})$ are adjacent to the corresponding entries of $\mathbb{K}(\underline{s})$.*

(b) *If the ray $g_{\underline{r}}$ lands at ∞, then $g_{\sigma^m(\underline{r})}$ lands at κ for some $m > 0$.*

Proof. If $\operatorname{Addr}^-(\underline{r})$ contains a preimage of $\underline{s}$ under σ^m, then $\underline{s} \in \operatorname{Addr}^-(\sigma^m(\underline{r}))$. By the previous lemma, $L_{\sigma^m(\underline{r})}$ contains the entire ray $g_{\underline{s}}$, and therefore also κ, its landing point.

Similarly, if $\operatorname{Addr}^-(\underline{r})$ contains an intermediate external address of length m, then $\infty \in \operatorname{Addr}^-(\sigma^{m-1}(\underline{r}))$. Since $g_{\sigma^{m-1}(\underline{r})}$ belongs to some strip of the dynamic partition, this is only possible if $\operatorname{Im}(g_{\sigma^{m-1}(\underline{r})})$ is unbounded from below, which means that $g_{\sigma^m(\underline{r})}$ accumulates at κ.

Finally, suppose that $g_{\underline{r}}$ lands at ∞. Then $\operatorname{Addr}^-(\underline{r})$ consists of a single address $\widetilde{\underline{r}}$. By the previous lemma, $\widetilde{\underline{r}} \notin \mathscr{S}_0$, as otherwise the accumulation set $L_{\underline{r}}$ would contain an entire dynamic ray. Also, $\widetilde{\underline{r}}$ cannot be an exponentially unbounded infinite address since every entry of its itinerary must be adjacent to the corresponding entry of $\operatorname{itin}_{\underline{s}}(\underline{r})$. So $\widetilde{\underline{r}}$ is an intermediate external address of length m, which means that $g_{\sigma^m(\underline{r})}$ lands at κ. ∎

4. Topological considerations

A useful tool for showing that the accumulation set of a dynamic ray is an indecomposable continuum is given by the following theorem of Curry [C].

4.1. Theorem. (Curry) *Suppose that $g \subset \hat{\mathbf{C}}$ is a ray (i.e., a continuous injective image of $[0, \infty)$), and let G denote its accumulation set. If G has topological dimension one, does not separate the Riemann sphere into infinitely many components and contains g, then G is an indecomposable continuum.* □

4.2. Lemma. (Accumulation sets of dynamic rays) *Let E_κ be an exponential map with accessible singular value. Then for every $\underline{r} \in \mathscr{S}_0$, the accumulation set $L_{\underline{r}}$ has empty interior.*

Furthermore, suppose that $\mathrm{Addr}^-(\underline{r})$ *is finite. Then* $\mathbf{C} \setminus L_{\underline{r}}$ *has only finitely many components.*

Proof. Let $\underline{s} = \mathrm{addr}(\kappa)$ and $\tilde{\mathtt{u}} := \mathrm{itin}_{\underline{s}}(\underline{r})$. Then for every $n \geq 0$, the set $E_\kappa^n(L_{\underline{r}})$ is contained in the closure of some strip $S_{\tilde{\mathtt{u}}_{n+1}}$ of the dynamic partition. Thus we can find some $\mathtt{m} \in \mathbf{Z}$ and a subsequence n_j such that $E_\kappa^{n_j}(g_{\underline{s}}) \cap S_{\mathtt{m}} = \emptyset$ for all j. If U was a component of $\mathrm{int}(L_{\underline{r}})$, then $E_\kappa^{n_j}|_U$ would omit all points of $S_{\mathtt{m}}$, and thus $U \subset F(E_\kappa)$ by Montel's theorem. This contradicts the fact that $L_{\underline{r}} \subset J(E_\kappa)$.

Assume that $\mathrm{Addr}^-(\underline{r})$ is finite. Then for each component I of $\mathscr{S} \setminus \mathrm{Addr}^-(\underline{r})$, there is a unique component V of $\mathbf{C} \setminus L_{\underline{r}}$ such that, for every $\widetilde{\underline{r}} \in I$, the ray $g_{\widetilde{\underline{r}}}$ eventually tends to ∞ in V. Since $\mathscr{S} \setminus \mathrm{Addr}^-(\underline{r})$ is finite, there are only finitely many such components V. Let U denote the union of all other components of $\mathbf{C} \setminus L_{\underline{r}}$; we claim that U has at most one component.

Similarly as above, it follows that $U \subset F(f)$. Since $\kappa \in J(f)$, it follows that every component of U is a preimage of a Siegel disk V of E_κ. By passing to a forward iterate, we may suppose without loss of generality that $V \subset U$. Since E_κ is injective on U, it is impossible for U to contain any other component which eventually maps to V. ■

5. Existence of nonlanding rays

5.1. Theorem. *Let E_κ be an exponential map with accessible singular value $\kappa \in J(E_\kappa)$. Then there exists an uncountable set $R \subset \mathscr{S}_0$ such that*

(a) $\mathrm{Addr}^-(\underline{r}) = \{\underline{r}\}$ *for all $\underline{r} \in R$, and*
(b) *no two addresses in R share the same itinerary.*

If $\mathrm{addr}(\kappa)$ *is bounded, then R can be chosen to consist only of bounded addresses.*

Proof. Let $\underline{s} := \mathrm{addr}(\kappa)$ and set

$$T_n := 2 + \max_{k \leq n} s_k$$

for every $n \in \mathbf{N}$. We define R_1 to be the set of all addresses of the form

$$\begin{aligned} \underline{r} &= \underline{r}(n_1, n_2, n_3, \dots) \\ &:= T_1 s_1 s_2 \dots s_{n_1 - 1} T_{n_1} s_1 s_2 \dots s_{n_2 - 1} T_{n_2} s_1 s_2 \dots s_{n_3 - 1} T_{n_3} \dots, \end{aligned} \tag{2}$$

where (n_k) is some sequence of natural numbers. (The set R will be a suitable subset of R_1.) Given $N_1, \dots, N_k \in \mathbf{N}$, let us also denote by $R_1(N_1, \dots, N_k)$ the subset of R_1 consisting of all sequences $\underline{r}(n_1, n_2, \dots)$ with $n_j = N_j$ for $j \leq k$.

Claim 1. For every $\underline{r} \in R_1$, $t_{\underline{r}}^* \leq t_{\underline{s}}^* + 2$. In particular, $t_{\underline{r}}^\kappa \leq T_0 := 2t_{\underline{s}}^* + \log^+|\kappa| + 8$.

Proof. This follows from the definitions and Proposition 3.1. △

Claim 2. For every $\underline{r} \in R_1$, there are no other addresses whose itinerary coincides with that of $\underline{r}$. In particular, no two addresses in R_1 share the same itinerary and $\mathrm{Addr}^-(\underline{r}) \subset \{\underline{r}\}$ for all $\underline{r} \in R_1$.

Proof. Set $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$ and $\underline{\tilde{\mathtt{u}}} := \mathrm{itin}_{\underline{s}}(\underline{r})$. By definition of R_1, for every $m \in \mathbf{N}$ there is some $k \geq 1$ such that $r_{m+k} = T_{k'}$ for some $k' \geq k$. In particular, $r_{m+k} > 2 + s_k$, and therefore $\mathtt{u}_k \neq \tilde{\mathtt{u}}_{m+k}$. By Lemma 2.3, there can be no other address with the same itinerary.

In particular, $\mathrm{Addr}^-(\underline{r})$ cannot contain any infinite external addresses other than $\underline{r}$ which are not preimages of $\underline{s}$. On the other hand, $\mathrm{Addr}^-(\underline{r})$ also cannot contain any preimages of $\underline{s}$ or any intermediate external addresses by Corollary 3.4. $\triangle$

We define R to be the subset of all $\underline{r} \in R_1$ with $\mathrm{Addr}^-(\underline{r}) \neq \emptyset$. We will show that R is nonempty by inductively constructing a sequence (n_k) for which the dynamic ray at address $\underline{r}(n_1, n_2, \dots)$ accumulates at infinity. In this construction, there will be countably many different choices for each n_j, so that R is in fact uncountable, as claimed.

Suppose that $j \geq 1$ such that n_k has been chosen for all $k < j$. Let $\underline{r}^k$ denote the address

$$\underline{r}^j := \underline{r}(n_1, \dots, n_{j-1}) := T_1 s_1 s_2 \dots s_{n_1 - 1} T_{n_1} s_1 \dots s_{n_{j-1}-1} T_{n_{j-1}} \underline{s}.$$

Then $\underline{r}^j$ is a preimage of $\underline{s}$, and thus lands at ∞. Let $t_j \in (t^{\kappa}_{\underline{r}^j}, T_0]$ be some value satisfying $|g_{\underline{r}^j}(t_j)| > j$. By Lemma 3.2, there is some $N_j \in \mathbf{N}$ such that

$$|g_{\underline{r}}(t_j)| \geq j$$

for all $\underline{r} \in R_1(n_1, \dots, n_{j-1}, n)$ with $n \geq N_j$. We choose n_j to be any such n.

Let $\underline{r} = \underline{r}(n_1, n_2, \dots)$ be an address constructed in this way. Then

$$g_{\underline{r}}(t_j) \to \infty.$$

Since $t_j \leq T_0$ for all j, we must have $t_j \to t^{\kappa}_{\underline{r}}$. I.e., $g_{\underline{r}}$ accumulates at ∞ and thus $\underline{r} \in R$, as required. ■

Proof of Theorem 1.2. This theorem is a direct corollary of the previous theorem together with Lemma 4.2 and Theorem 4.1. ■

Remark. Our construction yields nonseparating indecomposable continua, each containing exactly one dynamic ray. For many addresses $\mathrm{addr}(\kappa)$, it is possible to modify the above construction to obtain two addresses $\underline{r}^1, \underline{r}^2$ with $\mathrm{Addr}^-(\underline{r}^1) = \mathrm{Addr}^-(\underline{r}^2) = \{\underline{r}^1, \underline{r}^2\}$. This leads to an indecomposable limit set which separates the plane into two components and contains the two dynamic rays $g_{\underline{r}^1}$ and $g_{\underline{r}^2}$; see also [DJM], where this is carried out for the case of the Misiurewicz parameter $\kappa = \log(2\pi) + \pi i/2$ with $\mathrm{addr}(\kappa) = 0111\dots$. (For this particular parameter, [DJM] also constructs addresses $\underline{r}^1$ and $\underline{r}^2$ with $\mathrm{Addr}^-(\underline{r}^1) = \mathrm{Addr}^-(\underline{r}^2) = \{\underline{r}^2\}$, so that $g_{\underline{r}^1}$ accumulates on an indecomposable continuum but not on itself.)

Proof of Theorem 1.1. For attracting or parabolic parameters, every dynamic ray lands and the Julia set is the union of these rays and their landing points [BDD, Theorem 5.7] (compare also [R4, Corollary 9.3] for the stronger statement that the dynamics on $J(E_\kappa)$ is semiconjugate to that of an exponential map with an attracting fixed point). For the converse direction, let E_κ be an exponential map with $\kappa \in J(E_\kappa)$. Then one of the following holds:

(a) The singular value κ is not accessible from $I(E_\kappa)$; i.e., κ is neither on a dynamic ray nor the endpoint of a ray, or
(b) The singular value κ is accessible from $I(E_\kappa)$, in which case there exists a nonlanding dynamic ray of E_κ by Theorem 1.2. ■

Proof of Corollary 1.3. Since the indecomposable continua constructed in Theorem 5.1 all have different itineraries, they only have the point at ∞ in common. As indicated in Section 2, it is not difficult to compactify $\mathbf{C}$ to a space $\widetilde{\mathbf{C}}$ in such a way that each external address $\underline{s} \in \overline{\mathscr{S}}$ corresponds to (exactly) one point at ∞. So if, for $\underline{r} \in R$, we take the closures $\widetilde{g_{\underline{r}}}$ in $\widetilde{\mathbf{C}}$, the resulting indecomposable continua are pairwise disjoint. The space $\widetilde{\mathbf{C}}$ is clearly homeomorphic to the closed unit disk $\overline{\mathbf{D}} \subset \mathbf{C}$, concluding the proof. ■

Let us conclude the section by indicating how the above construction can be modified to yield a proof of Theorem 1.4 (concerning Siegel disks). Suppose that E_κ has a Siegel disk U whose boundary contains the singular value, and suppose furthermore that κ is accessible from the escaping set; say $\operatorname{addr}(\kappa) = \underline{s}$. Then clearly $\mathtt{u} := \mathbb{K}(\underline{s})$ is periodic, but $\underline{s}$ is not. The period of $\mathtt{u}$ is at most the period of the Siegel disk U. It is less obvious, but also true, that these periods must be equal. (This follows, for example, from the combinatorial results of [RS]; we omit the details here.)

5.2. Theorem. *Let $\underline{s} \in \mathscr{S}_0$ be a non-periodic external address for which the kneading sequence $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$ is periodic. Suppose that κ is a parameter with* $\operatorname{addr}(\kappa) = \underline{s}$*. Then there exist uncountably many addresses $\underline{r} \in \mathscr{S}_0$ with* $\operatorname{itin}_{\underline{s}}(\underline{r}) = \underline{\mathtt{u}}$ *and* $\underline{r} \in \operatorname{Addr}^-(\underline{r})$.

Suppose furthermore that E_κ has an unbounded Siegel disk U. Then, for each of these addresses, $g_{\underline{r}} \subset \partial U$. If ∞ is accessible from U, then $\operatorname{Addr}^-(\underline{r}) = \{\underline{r}\}$*; in particular, $L_{\underline{r}}$ is an indecomposable continuum.*

Sketch of proof. For simplicity, we will restrict to the case where U is a fixed Siegel disk (the general case is analogous, but requires slightly more bookkeeping). Shifting κ by an integer multiple of $2\pi i$, we may then furthermore assume that $\underline{\mathtt{u}} = \mathtt{000}\ldots$.

Let $X \subset \mathscr{S}_0$ be the compact set consisting of all infinite external addresses $\underline{r}$ whose itineraries are adjacent to $\underline{\mathtt{u}}$; i.e. $\underline{\mathtt{u}} \in \{\operatorname{itin}^+(\underline{r}), \operatorname{itin}^-(\underline{r})\}$. We claim that every interval of $\overline{\mathscr{S}} \setminus X$ is bounded by two iterated preimages of $\underline{s}$ under the shift, and contains an intermediate external address $\underline{t}$ with $\operatorname{itin}(\underline{t}) = \mathtt{00}\ldots\mathtt{00}*$. Indeed, this follows easily from Lemma 2.3 and the fact that $\underline{s} \in X$. In particular, iterated preimages of $\underline{s}$ are dense in X. (In fact, the map which collapses the two addresses $0\underline{s}$ and $1\underline{s}$, and every pair of their preimages, semi-conjugates $\sigma|_X$ to an irrational rotation of the circle. In particular, *every* backward (and forward) orbit is dense in X.)

Now we can inductively construct uncountably many addresses $\underline{r} \in X$ with $\operatorname{Addr}^-(\underline{r}) \neq \emptyset$: as in the proof of Theorem 5.1, $\underline{r}$ is the limit of iterated preimages $\underline{r}^j \in X$ of $\underline{s}$, where $\underline{r}^{j+1}$ is chosen sufficiently close to $\underline{r}^j$.

To obtain the stronger claim, stated in the theorem, that actually $\underline{r} \in \operatorname{Addr}^-(\underline{r})$, let $\underline{r}^{j-}$ be the intermediate address which is the unique element of $\operatorname{Addr}^-(\underline{r}^j)$. By choosing $\underline{r}^{j+1}$ sufficiently close to $\underline{r}^j$ in each step, we can easily ensure that $\operatorname{Addr}^-(\underline{r})$ contains a limit address of the sequence $\underline{r}^{j-}$. However, it is not difficult to show (using Lemma 2.3) that $\underline{r}^{j-} \to \underline{r}$. Hence $\underline{r} \in \operatorname{Addr}^-(\underline{r})$, which completes the proof of the first claim.

Now suppose that E_κ has an unbounded, fixed Siegel disk U, and consider the set $\operatorname{Addr}(U)$. By [R2], we have $\kappa \in \partial U$. In particular, U is unbounded to the left, so $\infty \in \operatorname{Addr}(U)$. It follows that any intermediate external address $\underline{t}$ with $\operatorname{itin}(\underline{t}) = \mathtt{00}\ldots\mathtt{00}*$ also belongs to $\operatorname{Addr}(U)$. By the above observation (and since $\operatorname{Addr}(U)$ is compact), this implies that $X \subset \operatorname{Addr}(U)$, and hence $g_{\underline{r}} \subset \overline{U}$ for every $\underline{r} \in X$.

Finally, suppose that κ is accessible from U, say by a curve γ connecting it to the indifferent fixed point. Then, for any $\underline{r} \in X$ which is not an iterated preimage of $\underline{s}$, we can use preimages of γ to separate $g_{\underline{r}}$ from all other dynamic rays. This shows that $\operatorname{Addr}^-(\underline{r}) \subset \{\underline{r}\}$, as desired. If ∞ is accessible from U by a curve γ, but κ is not, then $\underline{t} := \operatorname{addr}(\gamma) \in X$. The iterated preimages of $\underline{t}$ are dense in X, and the claim follows analogously. ■

6. Exponential maps with rays accumulating at infinity

In the previous sections, we dealt with the case where κ is accessible. In this section, we will briefly explore the situation where κ is merely an accumulation point of some dynamic ray, or even more generally when we merely know that some dynamic ray accumulates at ∞. The latter case is equivalent to saying that there is some $\underline{s} \in \mathscr{S}_0$ with $\operatorname{Addr}^-(\underline{s}) \neq \emptyset$; let us begin by showing that there is a similar criterion for the former case.

6.1. Lemma. (Rays accumulating at the singular value) *Let $\kappa \in \mathbf{C}$ and $\underline{s} \in \mathscr{S}_0$. Then $g_{\sigma(\underline{s})}$ accumulates on the singular value κ if and only if $\infty \in \operatorname{Addr}^-(\underline{s})$.*

Proof. (Compare also [R2].) We may assume that $\kappa \in J(E_\kappa)$, since otherwise all dynamic rays land in $\mathbf{C}$. The “only if” part of the statement is trivial. So let $\underline{s} \in \mathscr{S}_0$ such that $g_{\sigma(\underline{s})}$ does not accumulate on κ, and let U be the component of $\mathbf{C} \setminus L_{\sigma(\underline{s})}$ containing κ. Since $\kappa \in J(f)$, the set U contains some escaping point z_0, say $z_0 = g_{\underline{r}}(t_0)$. Consequently $\gamma_0 := g_{\underline{r}}\big([t_0, \infty)\big) \subset U$. Extend the curve γ_0 to a curve $\gamma \subset U$ by connecting κ and z_0. Then there is a branch of E_κ^{-1} defined on

$\mathbf{C} \setminus \gamma$ taking $g_{\sigma(\underline{s})}$ to $g_{\underline{s}}$. It follows easily that $\mathrm{Addr}^-(\underline{s})$ is contained in the interval $(m\underline{r}, (m+1)\underline{r})$ of $\mathscr{S}$ for some $m \in \mathbf{Z}$; in particular, $\infty \notin \mathrm{Addr}^-(\underline{s})$. ■

Let us now investigate what happens when we attempt to adapt the proof of Theorem 5.1 to our more general situation. Again, there are two cases to consider. If there is some address $\underline{s} \in \mathscr{S}_0$ for which $\kappa \in L_{\underline{s}}$ and $L_{\underline{s}}$ is bounded, then we still have the tool of itineraries at our disposal, and the combinatorial part of the proof will go through just like before. Otherwise, we will still be able to construct many rays $g_{\underline{r}}$ which accumulate at infinity, but without much control over $\mathrm{Addr}^-(\underline{r})$.

6.2. Theorem. (Rays accumulating at ∞) *Let $\kappa \in \mathbf{C}$.*

(a) *The set of addresses $\underline{r} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{r}) \neq \emptyset$ is either empty or uncountable.*
(b) *If there is an address $\underline{s} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{s}) = \emptyset$ such that $\kappa \in L_{\underline{s}}$, then there are uncountably many addresses $\underline{r} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{r}) = \{\underline{r}\}$.*

Proof. If there is an address $\underline{s} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{s}) \neq \emptyset$, then we can construct uncountably many addresses with $\mathrm{Addr}^-(\underline{r}) \neq \emptyset$ as in (the second part of) the proof of Theorem 5.1, as a limit of preimages of $\underline{s}$. If furthermore (b) holds, then the first part of the proof of Theorem 5.1 will also go through, allowing us to construct addresses with $\mathrm{Addr}^-(\underline{r}) = \{\underline{r}\}$. ■

It is not difficult to modify the proof of (a) to directly yield the existence of uncountably many addresses $\underline{r}$ for which $\mathrm{Addr}^-(\underline{r}) \cap \mathscr{S}_0 \neq \emptyset$, proving Theorem 1.5. Instead, we will use a little more combinatorics to obtain a slightly stronger statement: if there is no address $\underline{s}$ as in Theorem 6.2 (b), then every $\underline{r} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{r}) \neq \emptyset$ satisfies $\mathrm{Addr}^-(\underline{r}) \cap \mathscr{S}_0 \neq \emptyset$.

6.3. Lemma. (Close accumulation addresses) *Let $\kappa \in \mathbf{C}$, and let $\underline{s} \in \mathscr{S}_0$. Then for every $\underline{r} \in \mathrm{Addr}^-(\sigma(\underline{s}))$, there is $\widetilde{\underline{r}} \in \mathrm{Addr}^-(\underline{s})$ with $\sigma(\widetilde{\underline{r}}) = \underline{r}$ and $|\widetilde{r}_1 - s_1| \leq 1$.*
In particular, $\mathrm{Addr}^-(\sigma(\underline{s})) = \sigma(\mathrm{Addr}^-(\underline{s}) \setminus \{\infty\})$.

Proof. The first statement is a consequence of the fact that $g_{\underline{s}}$ cannot intersect its $2\pi i\mathbf{Z}$-translates. To prove the second statement, note that $\mathrm{Addr}^-(\sigma(\underline{s})) \subset \sigma(\mathrm{Addr}^-(\underline{s}) \setminus \{\infty\})$ follows from the first claim, and the converse inclusion follows from the definitions. ■

6.4. Corollary. *Let E_κ be an exponential map and let $\underline{s} \in \mathscr{S}_0$ and $m \geq 1$ such that $\mathrm{Addr}^-(\sigma^m(\underline{s})) \neq \emptyset$. Then there is $\widetilde{\underline{s}} \in \mathrm{Addr}^-(\underline{s})$ satisfying $|s_j - \widetilde{s}_j| \leq 1$ for $j \in \{1, \dots, m\}$.*

Proof. This follows from the previous lemma by a simple induction. ■

6.5. Lemma. (Exponentially bounded accumulation addresses) *Let E_κ be an exponential map and let $\underline{s} \in \mathscr{S}_0$. Then one of the following holds:*

(a) $\mathrm{Addr}^-(\underline{s}) \cap \mathscr{S}_0 \neq \emptyset$, *or*
(b) $\mathrm{Addr}^-(\sigma^m(\underline{s})) = \emptyset$ *for some $m \geq 0$.*

Proof. Suppose that $\mathrm{Addr}^-(\sigma^m(\underline{s})) \neq \emptyset$ for all $m \geq 0$. Then we can find a (possibly constant) sequence $\underline{r}^n \in \mathrm{Addr}^-(\underline{s})$ with the property that each $\underline{r}^n$ is not an intermediate external address of length $\leq n+1$. By Corollary 6.4, we may assume that each $\underline{r}^n$ furthermore satisfies

$$|r_j^n - s_j| \leq 1$$

for $j \in \{1, \dots, n\}$. It follows that every accumulation point of the sequence $\underline{r}^n$ is exponentially bounded. Since $\mathrm{Addr}^-(\underline{s})$ is compact, this concludes the proof. ■

6.6. Corollary. (Rays accumulating on escaping points) *Let E_κ be an exponential map. Then one of the following holds:*

(a) *There is some $\underline{s} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{s}) = \{\infty\}$; in particular, there are uncountably many addresses $\underline{r} \in \mathscr{S}_0$ satisfying $\mathrm{Addr}^-(\underline{r}) = \{\underline{r}\}$; or*
(b) *every $\underline{r} \in \mathscr{S}_0$ satisfies either $\mathrm{Addr}^-(\underline{r}) = \emptyset$ or $\mathrm{Addr}^-(\underline{r}) \cap \mathscr{S}_0 \neq \emptyset$.*

Proof. By the previous lemma, we see that (b) holds unless there is some $\underline{s} \in \mathscr{S}_0$ for which $\mathrm{Addr}^-(\underline{s}) \neq \emptyset$ but $\mathrm{Addr}^-(\sigma(\underline{s})) = \emptyset$. This means that $\mathrm{Addr}^-(\underline{s}) = \{\infty\}$, as required. ■

Proof of Theorem 1.5. Using Lemma 3.3, the Theorem is an immediate consequence of the previous Corollary and Theorem 6.2. ■

We conclude this section by mentioning a few further questions suggested by this line of investigation.

(a) If some dynamic ray accumulates at ∞, is it true that some ray must accumulate on the singular value?
(b) Can we replace $\mathrm{Addr}^-(\underline{r}) \cap \mathscr{S}_0 \neq \emptyset$ by $\underline{r} \in \mathrm{Addr}^-(\underline{r})$ in Corollary 6.6 (b)?
(c) Can we replace $\mathrm{Addr}^-(\underline{r}) \cap \mathscr{S}_0 \neq \emptyset$ by $\mathrm{Addr}^-(\underline{r}) = \{\underline{r}\}$ in Corollary 6.6 (b)?
(d) If some dynamic ray accumulates on the singular value, can we ensure that there is some $\underline{r}$ for which $L_{\underline{r}}$ is an indecomposable continuum?
(e) If $\kappa \in J(E_\kappa)$, is there always some $\underline{s} \in \mathscr{S}_0$ with $\mathrm{Addr}^-(\underline{s}) \neq \emptyset$?

It seems reasonable to expect that the first two questions could be answered using a further development of the methods in this section, while the remaining problems appear more difficult.

Appendix A. Remarks on higher-dimensional parameter spaces

For many entire functions with a bounded set of singular values, the escaping set consists of dynamic rays, just as in the exponential family. In fact, this is now known to be true for all finite-order entire functions with a bounded set of singular values [Ro, R^{3}S]. Let us shortly discuss how our main result generalizes to these cases.

Schleicher [S] showed that, for postcritically pre-periodic maps in the cosine family $z \mapsto a\exp(z) + b\exp(-z)$, every dynamic ray lands and every point of $\mathbf{C}$ is either on a ray or the landing point of a ray. Thus our main result is not true for the cosine family.

The reason for this is that cosine maps do not have any asymptotic values, and the presence of a dynamic ray landing at an asymptotic value was the driving factor in our proof. Indeed, this is the only obstruction to carrying our proof over to the general case. In particular, one might expect the following dichotomy: if $f\colon \mathbf{C} \to \mathbf{C}$ is a postcritically finite entire function of finite order, then

- if f has an asymptotic value, some dynamic ray of f accumulates on an entire dynamic ray, and conversely,
- if f has only critical values, then every dynamic ray of f lands and every point in $J(f)$ is on a dynamic ray or the landing point of such a ray.

We should note that the condition of "accessible singular values" is also of interest in the study of these more general families of entire functions. In particular, under such a hypothesis it is possible to prove that all periodic dynamic rays land. (The only known proof that periodic rays of *all* exponential maps land [R3], regardless of whether the singular value is accessible, relies strongly on the one-dimensionality of exponential parameter space and does not generalize to the higher-dimensional case.)

References

[BDD] Bhattacharjee, R., R. L. Devaney, R. E. L. Deville, K. Josić, and M. Moreno Rocha: Accessible points in the Julia sets of stable exponentials. - Discrete Contin. Dyn. Syst. Ser. B 1:3, 2001, 299–318.

[BDG] Bodelón, C., R. L. Devaney, M. Hayes, G. Roberts, L. R. Goldberg, and J. H. Hubbard: Hairs for the complex exponential family. - Internat. J. Bifur. Chaos Appl. Sci. Engrg. 9:8, 1999, 1517–1534.

[CP] Carmona, J. J., and Ch. Pommerenke: Decomposition of continua and prime ends. - Comput. Methods Funct. Theory 3:1-2, 2003, 253–272.

[C] Curry, S. B.: One-dimensional nonseparating plane continua with disjoint ε-dense subcontinua. - Topology Appl. 39:2, 1991, 145–151.

[De] Devaney, R. L.: Knaster-like continua and complex dynamics. - Ergodic Theory Dynam. Systems 13:4, 1993, 627–634.

[DJ] Devaney, R. L., and X. Jarque: Indecomposable continua in exponential dynamics. - Conform. Geom. Dyn. 6, 2002, 1–12.

[DJM] Devaney, R. L., X. Jarque, and M. Moreno Rocha: Indecomposable continua and Misiurewicz points in exponential dynamics. - Internat. J. Bifur. Chaos Appl. Sci. Engrg. 15:10, 2005, 3281–3293.

[Do] Douady, A.: Descriptions of compact sets in $\mathbf{C}$. - In: Topological methods in modern mathematics (Stony Brooks, NY, 1991), Publish or Perish, Houston, TX, 1993, 429–465.

[M] Milnor, J.: Dynamics in one complex variable. - Ann. of Math. Stud. 160, third ed., Princeton University Press, Princeton, NJ, 2006.

[P] Pommerenke, Ch.: Boundary behaviour of conformal maps. - Grundlehren Math. Wiss. 229, Springer-Verlag, Berlin, 1992.

[R1] Rempe, L.: Dynamics of exponential maps. - Doctoral thesis, Christian-Albrechts-Universität Kiel, 2003, http://e-diss.uni-kiel.de/diss_781/.

[R2] Rempe, L.: On a question of Herman, Baker and Rippon concerning Siegel disks. - Bull. London Math. Soc. 36:4, 2004, 516–518, arXiv:math.DS/0307174.

[R3] Rempe, L.: A landing theorem for periodic rays of exponential maps. - Proc. Amer. Math. Soc. 134:9, 2006, 2639–2648, arXiv:math.DS/0307371.

[R4] Rempe, L.: Topological dynamics of exponential maps on their escaping sets. - Ergodic Theory Dynam. Systems 26:6, 2006, 1939–1975, arXiv:math.DS/0309107.

[RS] Rempe, L., and D. Schleicher: Combinatorics of bifurcations in exponential parameter space. - Transcendental Dynamics and Complex Analysis (edited by P. Rippon and G. Stallard), Cambridge Univ. Press, to appear, arXiv:math.DS/0408011.

[Ro] Rottenfußer, G.: Dynamical fine structure of entire transcendental functions. - Doctoral thesis, International University Bremen, 2005.

[R^{3}S] Rottenfußer, G., J. Rückert, L. Rempe, and D. Schleicher: Dynamic rays of bounded-type entire functions. - Preprint, 2006, arXiv:0704.3213.

[S] Schleicher, D.: The dynamical fine structure of iterated cosine maps and a dimension paradox. - Duke Math. J. 136:2, 2007, 343–356, arXiv:math.DS/0406255.

[SZ1] Schleicher, D., and J. Zimmer: Escaping points of exponential maps. - J. London Math. Soc. (2) 67:2, 2003, 380–400.

[SZ2] Schleicher, D., and J. Zimmer: Periodic points and dynamic rays of exponential maps. - Ann. Acad. Sci. Fenn. Math. 28, 2003, 327–354.

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