Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 32, 2007, 317–340

UNDECIDABLE PROPOSITIONS BY ODE'S

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Abstract. Starting with elementary functions, we generate new functions by multiplication, integration and by solving ODE's so as to obtain a family $\mathscr{M}$ of real holomorphic functions such that: (*) if $E \subseteq \mathbf{N}$ is recursively enumerable then there is $f \in \mathscr{M}$ such that $n \in E$ iff $\int_{-\pi}^{+\pi} f(x)e^{-in\,x}\,dx \neq 0$. Constructive aspects and relations to hypercomputation are discussed.

1. Introduction

It is the aim of this paper to reconsider some questions discussed in [19], which found renewed interest recently ([4], [5], [14]). The question is, whether one can formulate reasonable problems in classical analysis which turn out to be undecidable in the sense of recursive function theory. There is also a relation, mentioned in [19], between this mathematical question and a more philosophical issue, i.e. the subject of hypercomputation. A reader, mainly interested in the latter, may proceed in a first step directly to Sect. 5, where an extended discussion of this point is given.

Now analysis, as encountered in PDE's and ODE's is of a rather different character than the finite combinatorics encountered in mathematical logic. There is however an area which constitutes a bridge between analysis and combinatorics, namely Fourier analysis, which due to its proximity to algebra establishes a connection between analysis and combinatorics. In [19], this observation served as basis for the construction of undecidable propositions in terms of Fourier series. There it was shown that for every recursive enumerable set $E \subseteq \mathbf{N}$, a real analytic function $f(x)$, $x \in [0, 2\pi]$, 2π-periodic in x, can be constructed by restricted means such that

$$E = \left\{ n \Big/ \int_{-\pi}^{+\pi} f(x)e^{-inx}\,dx \neq 0 \right\}. \tag{1.1}$$

The restricted means in [19] were as follows. Starting with a set of basic functions considered as elementary (e.g. e^{ix}, e^{ix+iy}) one can generate new functions by

2000 Mathematics Subject Classification: Primary 03D10, 03F60, 03F65, 42B05, 34A09, 65L80.

Key words: Recursively enumerable sets, theta functions, ordinary differential equations, hypercomputation.

two types of operations: (a) elementary ones such as addition, multiplication, convolution, (b) a nonelementary operation, i.e. solving a Fredholm integral equation

$$f(\zeta,\alpha) = g(\zeta,\alpha) + \frac{1}{2\pi}\int_0^{2\pi} K(\zeta,\eta,\alpha)f(\eta,\alpha)\,d\eta, \quad \zeta,\alpha \in [0,2\pi] \tag{1.2}$$

provided that g, K have already been generated and that the homogeneous equation ($g = 0$ in (1.2)) has no solution for any parameter value of $\alpha \in [0, 2\pi]$. Assuming the result in [13] (not known in 1961) it was shown that a single application of (1.2) suffices to generate (in conjunction with (a)) an $f(x)$ as required by (1.1). This particular application of (1.2) is based on two special functions $g_\lambda(\alpha,\beta)$ $K_\lambda(\alpha,\beta,\gamma)$ constructed from basic functions via elementary operations of type (a) such that the unique solution $\theta_\lambda(\alpha,\beta)$ of

$$\theta_\lambda(\alpha,\beta) = g_\lambda(\alpha,\beta) + \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_\lambda(\alpha,\beta,\gamma)\theta_\lambda(\alpha,\gamma)\,d\gamma \tag{1.3}$$

is given by

$$\theta_\lambda(\alpha,\beta) = 1 + 2\sum_1^\infty \lambda^{n^2} e^{in^2\alpha}\cos(n\beta), \quad \alpha,\beta \in [0,2\pi]. \tag{1.4}$$

Here $\lambda \in (0, 1)$ is a fixed auxiliary parameter which for constructive reasons will be assumed to be rational. With θ_λ at disposal, functions $f(x)$ as stipulated by (1.1) can be constructed, using only the elementary operations from (a). Now a glance at the literature ([10], [11], [12]) shows that $\theta_\lambda(\alpha,\beta)$ is just one of Jacobi's famous theta functions, denoted in most texts by ϑ_3 (by ϑ_2 in [18]). That is, up to a scaling by 2π and a relabeling of variables we have

$$\begin{aligned} \theta_\lambda(2\pi x, 2\pi v) &= \vartheta_3(x + i\gamma, v) = 1 + 2\sum_1 \lambda^{n^2} e^{2i\pi n^2 x}\cos 2n\pi v \\ &\text{with } \lambda = \mathrm{e}^{-2\pi\gamma}, \quad (\gamma > 0). \end{aligned} \tag{1.5}$$

The bridge from our problem to ODE's now arises as follows. As shown by Jacobi [11] it holds that $\vartheta_3(q, 0) = 1+2\sum_1 q^{n^2}$ satisfies an ODE with constant coefficients, i.e. there is a polynomial $P(\zeta_0,\ldots,\zeta_3)$ such that

$$P(y, Dy, D^2y, D^3y) = 0, \quad D = q\partial_q, \quad y = \vartheta_3(q, 0). \tag{1.6}$$

The coefficients of P are rational multiples of powers of π, thus highly constructive. Based on this fact, it was suggested in [19] that one could replace scheme (b), (i.e. solving Fredholms equation) by a scheme (b$'$) which allows the generation of new functions by solving ODE's. This program was carried out in [3]. Here we present a version which differs in various points from [3]; more comments on this will be given later.

We note that there are other approaches to noncomputability in analysis. We refer in particular to [16], [17] where nonrecursive real numbers are constructed via ODE's and PDE's (the linear wave equation in the latter case) by means that differ considerably from ours.

Finally we invoke a field which has recently regained interest and on which our considerations have some bearing. It is the problem of so called hypercomputation, which investigates the possibility of a computer that encompasses the classical Turing machine, i.e. that violates Church's thesis. For an extended discussion of this concept we refer to [4], [5], [14]. We postpone a discussion of this topic to the last section. The main result of this paper is stated in Theorem 1.

2. Description of predicates by Fourier series

We briefly recall a few points from [19] which are important in the sequel. Let P be an s-ary predicate over the integers, i.e. $P \subseteq \mathbf{N}^s$. Let $\phi(x_1, \ldots, x_s)$ be real holomorphic and 2π-periodic in the real variables $x_1, \ldots, x_s$. The function ϕ then has a strongly convergent Fourier series

$$\phi(x_1, \ldots, x_s) = \sum a_{j_1 \ldots j_s} e^{i(j_1 x_1 + \cdots + j_s x_s)}, \quad j_k \in \mathbf{Z}. \tag{2.1}$$

(see e.g. [2]); we note in particular that $\sum |a_{j_1 \ldots j_s}| < \infty$.

Definition 1. $\phi(x_1, \ldots, x_s)$ represents the predicate P if

(a) $a_{j_1 \ldots j_s} \geq 0$ for all $j_1, \ldots, j_s \in \mathbf{Z}$.

(b) $a_{j_1 \ldots j_s} > 0 \leftrightarrow j_k \geq 0,\ k = 1, \ldots, s$ and $P(j_1, \ldots, j_s)$ holds.

Remarks. (1) In connection with (2.1) and Def. 1 we use the notation $[\phi]_{j_1 \cdots j_s} = a_{j_1 \cdots j_s}$ and call $\phi(x_1, \ldots, x_s)$, $\psi(x_1, \ldots, x_s)$ similar if

$$[\phi]_{j_1 \ldots j_s} > 0 \text{ iff } [\psi]_{j_1 \ldots j_s} > 0. \tag{2.2}$$

Thus similar functions represent the same predicate.

(2) We let $\alpha_1, \alpha_2, \ldots, x_1, x_2, \ldots$ denote variables occurring in functions or predicates while α or x often are shorthand for lists of such variables, i.e. $\alpha = \alpha_1, \ldots, \alpha_s$, $x = x_1, \ldots, x_t$.

(3) Let $\alpha = \alpha_1, \ldots, \alpha_s$, $\beta = \beta_1, \ldots, \beta_t$, $\gamma = \gamma_1, \ldots, \gamma_r$ and let $\phi(\alpha, \beta)$, $\psi(\alpha, \gamma)$ be real holomorphic and 2π-periodic in the indicated variables. We call

$$\phi * (\alpha)\psi = (2\pi)^{-s} \int_0^{2\pi} \cdots \int_0^{2\pi} \phi(\alpha_1 - \zeta_1, \ldots, \alpha_s - \zeta_s, \beta)\psi(\zeta_1, \ldots, \zeta_s, \gamma)\, d\zeta^s \tag{2.3}$$

the convolution with respect to $\alpha_1, \ldots, \alpha_s$. Note that $\phi * (\alpha)\psi$ is again real holomorphic and 2π-periodic in the variables α, β, γ.

Proposition 2.1. (a) *For $s \geq 1$, let $\phi(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$, $\psi(\alpha_1, \ldots, \alpha_s, \gamma_1, \ldots, \gamma_r)$ represent $P(x_1, \ldots, x_s, y_1, \ldots, y_t)$ and $Q(x_1, \ldots, x_s, z_1, \ldots, z_r)$ respectively. Then $\phi * (\alpha)\psi$ represents*

$$P(x_1, \ldots, x_s, y_1, \ldots, y_t) \wedge Q(x_1, \ldots, x_s, z_1, \ldots, z_t).$$

(b) *Let $s = 0$ and $\phi(\beta_1, \ldots, \beta_t)$, $\psi(\gamma_1, \ldots, \gamma_r)$ represent $P(y_1, \ldots, y_t)$ and $Q(z_1, \ldots, z_r)$ respectively. Then $\phi(\beta_1, \ldots, \beta_t)\psi(\gamma_1, \ldots, \gamma_r)$ represents $P(y_1, \ldots, y_t) \wedge Q(z_1, \ldots, z_r)$.*

Proof. Consider (a). For simplicity we take $s = t = r = 1$. Thus $\phi(\alpha, \beta) = \sum a_{nm} e^{in\alpha} e^{im\beta}$, $\psi(\alpha, \beta) = \sum b_{pq} e^{ip\alpha} e^{iq\gamma}$ and

$$a_{nm} > 0 \leftrightarrow P(n, m), \quad b_{pq} > 0 \leftrightarrow Q(p, q). \tag{2.4}$$

We then have

$$\begin{aligned} \phi * (\alpha)\psi &= \sum a_{nm} b_{pq} (2\pi)^{-1} \int_0^{2\pi} e^{in(\alpha-\zeta)} e^{ip\zeta}\, d\zeta\, e^{im\beta} e^{iq\gamma} \\ &= \sum a_{nm} b_{pq} \delta_{np} e^{in\alpha} e^{im\beta} e^{iq\gamma} \\ &= \sum a_{nm} b_{nq} e^{in\alpha} e^{im\beta} e^{iq\gamma}. \end{aligned}$$

Moreover, $a_{nm} b_{nq} > 0$ iff $P(n, m) \wedge Q(n, q)$ by (2.4). The general case follows in the same way; likewise case (b). □

Remarks. (a) By Prop. 2.1 we can handle extensions of predicates. Let $\phi(\alpha) = \phi(\alpha_1, \ldots, \alpha_s)$ represent $P(x_1, \ldots, x_s)$; note that for $\lambda \in (0, 1)$,

$$\psi(\beta) = \psi(\beta_1, \ldots, \beta_t) = \prod_1^t (1 - \lambda e^{i\beta_j})^{-1} \tag{2.5}$$

represents $\bigwedge_1^t y_j = y_j$. By Prop. 2.1 we have that $\phi(\alpha)\psi(\beta)$ represents $P(x_1, \ldots, x_s) \wedge \bigwedge_1^t y_j = y_j$, called the natural extension of ϕ from $x_1, \ldots, x_s$ to $x_1, \ldots, x_s, y_1, \ldots, y_t$.

Proposition 2.2. *Let $\phi(\alpha) = \phi(\alpha_1, \ldots, \alpha_s)$, $\psi(\alpha) = \psi(\alpha_1, \ldots, \alpha_s)$ represent $P(x_1, \ldots, x_s)$ and $Q(x_1, \ldots, x_s)$ respectively. Then $\phi(\alpha) + \psi(\alpha)$ represents $P \vee Q$.*

Proof. Take e.g. $s = 2$, set $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and let

$$\phi(\alpha, \beta) = \sum a_{nm} e^{in\alpha} e^{im\beta}, \quad \psi(\alpha, \beta) = \sum b_{nm} e^{in\alpha} e^{im\beta}$$

whence

$$\phi + \psi = \sum (a_{nm} + b_{nm}) e^{in\alpha} e^{im\beta} \quad (n, m \geq 0).$$

Since $a_{nm}, b_{nm} \geq 0$ by Def. 1 and our assumptions we have that

$$a_{nm} + b_{nm} > 0 \leftrightarrow a_{nm} > 0 \vee b_{nm} > 0 \leftrightarrow P(n, m) \vee Q(n, m).$$

The case $s > 2$ is handled likewise. □

Remarks. (1) While P, Q and hence the representing functions in Prop. 2.2 have the same variables we may use Remark (1) above in order to handle the case where P, Q may contain different variables. E.g. if $\phi(\alpha, \beta)$ and $\psi(\beta, \gamma)$ represent $P(x, y)$ and $Q(y, z)$ resp., then

$$\phi(\alpha, \beta)(1 - \lambda e^{i\gamma})^{-1} + \psi(\beta, \gamma)(1 - \lambda e^{i\alpha})^{-1}, \quad \lambda \in (0, 1)$$

represents $(P(x, y) \wedge z = z) \vee (x = x \wedge Q(y, z))$, that is $P(x, y) \vee Q(y, z)$.

(2) If $\phi(\alpha_1, \ldots, \alpha_s)$ represents $P(x_1, \ldots, x_s)$, and if π is a permutation of $x_1, \ldots, x_s$ then $\phi(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(s)})$ represents $P(x_{\pi(1)}, \ldots, x_{\pi(s)})$.

(3) In contrast to conjunction $\wedge$ and disjunction $\vee$ there seems to be no elementary way which, given $\phi(\alpha)$ representing $P(x)$, yields $\psi(\alpha)$ representing $\neg P(x)$.

This follows from our considerations in Sect. 5. There are, however, cases where this *is* possible. With $\alpha = \alpha_1, \ldots, \alpha_s$ and $x = x_1, \ldots, x_s$ let $\phi(\alpha)$ and $f(\alpha)$ represent $P(x)$ and $\bigwedge_1^s x_j = x_j$ respectively. We call f a unity for ϕ if

$$(2.6) \qquad [f]_{n_1 \ldots n_s} = [\phi]_{n_1 \ldots n_s} \quad \text{when} \quad P(n_1, \ldots, n_s) \text{ holds.}$$

Proposition 2.3. (a) *If $\phi(\alpha)$ represents $P(x)$ and if $f(\alpha)$ is a unity for ϕ then $f - \phi$ represents $\neg P(x)$. If $\psi(\alpha)$ represents $Q(x)$ and if g is a unity for ψ then $f * (\alpha) g$ is a unity for $\phi * (\alpha) \psi$.*

Proof. Consider e.g. (b) for $s = 1$, $\alpha_1 = \alpha$, $x_1 = x$. If f, g both represent $x = x$ then $f * (\alpha) g$ represents again $x = x$ by Prop. 2.2. Now if $P(n) \wedge Q(n)$ holds, then $[f]_n = [\phi]_n$ and $[g]_n = [\psi]_n$ by (2.6) and our assumption. In this case

$$[f * (\alpha) g]_n = [f]_n [g]_n = [\phi]_n [\phi]_n = [\phi * (\alpha) \psi]_n$$

i.e. $f * (\alpha) g$ is a unity for $\phi * (\alpha) \psi$. The proof for $s > 1$ and for (a) proceeds in the same way. □

(4) Somewhat surprisingly the existential quantifier admits a simple treatment, i.e. we have

Proposition 2.4. *Let $\phi(\alpha_1, \ldots, \alpha_s, \beta)$ represent $P(x_1, \ldots, x_s, y)$. Then $\phi(\alpha_1, \ldots, \alpha, 0)$ represents $(\exists y) P(x_1, \ldots, x_s, y)$.*

Proof. For simplicity set $s = 1$, $\alpha = \alpha_1$, $x = x_1$. By assumption

$$\phi(\alpha, \beta) = \sum_{n,m} a_{nm} e^{in\alpha} e^{im\beta} \text{ with } a_{nm} \geq 0, \text{ and with } a_{nm} > 0 \text{ iff } P(n, m).$$

Thus $\phi(n, 0) = \sum_{n \geq 0} \left(\sum_{m \geq 0} a_{nm} \right) e^{in\alpha}$, whence $\sum_{m \geq 0} a_{nm} \geq 0$ and therefore

$$\sum_{m \geq 0} a_{nm} > 0 \leftrightarrow (\exists m)(a_{nm} > 0) \leftrightarrow (\exists y) P(n, y). \qquad \square$$

Remarks. (1) By repeated applications of Prop. 2.4 or by extending its proof in an obvious way one shows: if $\phi(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$ represents $P(x_1, \ldots, x_s, y_1, \ldots, y_t)$ then $\phi(\alpha_1, \ldots, \alpha_s, 0, \ldots, 0)$ represents $(\exists y_1 \ldots y_t) P(x_1, \ldots, x_s, y_1, \ldots, y_t)$.

(2) An extended list of predicates and representing functions is given in [19], Sect. 2; frequent use of this list will be made in the sequel.

3. Admissible functions

Our next task is to axiomatise the program outlined in the introduction. We start with some remarks. Below we use the notion of recursive or constructive real number in the sense of [10], pg. 237; however no explicit use of this concept will be made at the moment. A complex number $\alpha + i\beta$ is called constructive if α, β are constructive. We let $\mathscr{F}$ be the set of functions $f(\alpha_1, \ldots, \alpha_s) = g(\alpha_1, \ldots, \alpha_s) + ih(\alpha_1, \ldots, \alpha_s)$, with g, h real and real holomorphic in $\alpha_j \in [a_j, b_j]$, $j \leq s$; a_j, b_j are assumed to be constructive, while $s \geq 0$. As in [3] we could restrict ourselves to real functions; however some arguments simplify when carried out in the complex domain. A function $f \in \mathscr{F}$ is provided with its domain of definition $D = \prod_1^s [a_j, b_j]$ (with $D = \mathbf{R}^s$ admitted); restricting f to $D' = \prod_1^s [a'_j, b'_j]$, $[a'_j, b'_j] \subseteq [a_j, b_j]$ yields basically a new function but for simplicity we do not insist on this distinction. The elementary operations (a) in Sect. 1 are now given by

Definition 2. Let $\mathscr{M} \subseteq \mathscr{F}$; the elementary hull $H_0(\mathscr{M})$ is the smallest set $S \subseteq \mathscr{F}$ satisfying the following conditions (a0)–(a5): (a0) $\mathscr{M} \subseteq S$, (a1) if f, g defined on D are in S, then $f+g$, $f-g$, fg are in S, (a2) if g, defined on D, is in S and if $g \neq 0$ on D then $g^{-1} \in S$, (a3) if $f \in S$ then the complex conjugate $\overline{f}$ is in S, (a4) let $f(\alpha_1, \ldots, \alpha_s)$, $\alpha_j \in [a_j, b_j]$ be in S, let $A_j, B_j, C_j, a'_j, b'_j, a''_j, b''_j$ satisfy

$$(*) \qquad A_j + B_j\beta_j + C_j\gamma_j \in [a_j, b_j] \text{ for } \beta_j \in [a'_j, b'_j],\ \gamma_j \in [a''_j, b''_j],$$

$j \leq s$; then the function

$$f(A_1 + B_1\beta_1 + C_1\gamma_1, \ldots, A_s + B_s\beta_s + C_s\gamma_s), \quad \beta_j \in [a'_j, b'_j],\ \gamma_j \in [a''_j, b''_j]$$

($j \leq s$) is in S, (a5) if $f(\alpha, \ldots, \alpha_s, \beta)$, $\alpha_j \in [a_j, b_j]$, $\beta \in [c, d]$ is in S, then

$$\int_c^d f(\alpha_1, \ldots, \alpha_s, \beta)\, d\beta, \quad \alpha_j \in [a_j, b_j],\ j \leq s$$

is in S.

Remarks. (1) The constants $a_j, a'_j, \ldots$ are tacitly assumed to be constructive.

(2) Clause (a4) is formulated in such a way that it covers all special cases to be met in the sequel, such as the following ones: (a) if $f(\alpha_1, \ldots, \alpha_s)$, $\alpha_j \in [a_j, b_j]$ is in $H_0(\mathscr{M})$, and if π is a permutation of $1, \ldots, s$ then $f(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(s)})$, $\alpha_{\pi(j)} \in [a_{\pi(j)}, b_{\pi(j)}]$ is in $H_0(\mathscr{M})$, (b) if $f(\alpha_1, \ldots, \alpha_s, \beta)$, $\alpha_j \in [a_j, b_j]$, $\beta \in [c, d]$ is in $H_0(\mathscr{M})$ then $f(\alpha_1, \ldots, \alpha_s, e)$, $\alpha_j \in [a_j, b_j]$ is in $H_0(\mathscr{M})$ for any constructive $e \in [c, d]$, (c) if $f(\alpha, \beta)$, $\alpha, \beta \in \mathbf{R}$ is in $H_0(\mathscr{M})$ then $f(\alpha, \beta + \gamma)$, $\alpha, \beta, \gamma \in \mathbf{R}$ is in $H_0(\mathscr{M})$.

(3) Prior to definition 3 below we stipulate:

$$\text{(3.1)} \qquad \text{for } j \leq s,\ e_{js}(\alpha_1, \ldots, \alpha_s) = e^{i\alpha_j}, \quad \alpha_1, \ldots, \alpha_s \in \mathbf{R};$$

but in accord with standard notation we write $e^{i\alpha_j}$ or even $e^{i\alpha}$, $e^{i\beta}$ for $e_{js}(\alpha_1, \ldots, \alpha_s)$.

Definition 3. $\mathscr{M}_0$ is the set of functions $\lambda e_{js}(\alpha_1, \ldots, \alpha_s)$, with $\lambda \in \mathbf{C}$ constructive.

We now form the elementary hull $H_0(\mathscr{M}_0)$. By Def. 2, (a1)–(a5), $H_0(\mathscr{M}_0)$ contains all constructive trigonometric polynomials, i.e. finite sums of monomials

$$\text{(3.2)} \qquad \lambda e^{i(a_1\alpha_1 + \cdots + a_s\alpha_s)}, \quad \text{with } a_j \in \mathbf{R},\ \lambda \in \mathbf{C}$$

constructive constants.

In particular, $\lambda e^{ia\alpha}$, $\alpha \in \mathbf{R}$ is in $H_0(\mathscr{M}_0)$ for $\lambda \in \mathbf{C}$, $a \in \mathbf{R}$ constructive, and thus

$$\text{(3.3)} \qquad (1 - \lambda e^{ia\alpha})^{-1} \in H_0(\mathscr{M}_0) \text{ for } |\lambda| < 1, \quad \lambda, \alpha \in \mathbf{R} \text{ constructive.}$$

One may ask what kind of predicates $P(x_1, \dots, x_s)$ may be represented by some $\phi(\alpha_1, \dots, \alpha_s) \in H_0(\mathscr{M}_0)$. A list of examples is given in [19]. Two examples from this list are

$$\begin{aligned} &\sum_{n,m\geq 0} \lambda^{n+m} e^{in\alpha} e^{im\beta} e^{i(n+m)\gamma} \text{ represents } x+y=z, \\ &\sum_{n,m\geq 0} \lambda^{n+m} e^{in\alpha} e^{i(n+m)\beta} \text{ represents } x \leq y. \end{aligned} \tag{3.4}$$

The lefthand sides of (3.4) are expressed by the functions

$$(1-\lambda e^{i(\alpha+\gamma)})^{-1}(1-\lambda e^{i(\beta+\gamma)})^{-1}, \quad (1-\lambda e^{i(\alpha+\beta)})^{-1}(1-\lambda e^{i\beta})^{-1}$$

respectively, both of which belong obviously to $H_0(\mathscr{M}_0)$. Now while simple predicates such as $x+y=z$, $x \leq y$ are represented by functions $\phi \in H_0(\mathscr{M}_0)$, this is not the case with more complex predicates such as $x \cdot y = z$. In order to force such a representation an additional operation is needed. In [19] it was a rule allowing the generation of new functions by solving Fredholm integral equations, here we generate new functions by solving systems of ODE's. Let $\zeta = \zeta_1, \dots, \zeta_n$, $\xi = \xi_1, \dots, \xi_m$, $\varepsilon = \varepsilon_1, \dots, \varepsilon_n$ be lists of variables. Prior to introduce the definitions below we first stipulate:

$$\begin{aligned} &\text{(a)} \quad D(\zeta), L(\varepsilon) \ \text{ are polynomial } n\times n\text{-matrices,} \\ &\qquad\ \ H(\zeta) \ \text{ is a polynomial } m\times m\text{-matrix,} \\ &\text{(b)} \quad S(\zeta,\xi), R(\varepsilon) \ \text{ are polynomial } n\text{-vectors,} \\ &\qquad\ \ Q(\xi) \ \text{ is a polynomial } m\text{-vector,} \end{aligned} \tag{3.5}$$

("polynomial": depending in a polynomial way on the indicated variables); the coefficients of $D, L, \dots$ may be complex but are tacitly assumed to be constructive.

Definition 4. A vectorfunction $y \in C^1([0,T]\times[a,b];\mathbf{C}^n)$ is "admissible" if there are D, L, H, S, Q, R via (3.5) and $z \in C^1([a,b];\mathbf{C}^m)$, $g \in C^1([a,b];\mathbf{C}^n)$ such that the following holds, where y_t, z_λ, g_λ are the derivatives with respect to the indicated variables:

(a) $D(y(t,\lambda))y_t = S(y(t,\lambda), z(\lambda))$, $H(z(\lambda))z_\lambda = Q(z(\lambda))$,
$L(g(\lambda))g_\lambda = R(g(\lambda))$, $t \in [0,T]$, $\lambda \in [a,b]$,

(b) $\det(D(y(t,\lambda))$, $\det(H(z(\lambda)), \det(L(g(\lambda))$ are $\neq 0$ for $t \in [0,T]$, $\lambda \in [a,b]$,

(c) $y(0,\lambda) = g(\lambda)$, $\lambda \in [a,b]$, and $z(a)$, $g(a)$ are constructive.

$f \in C^1([0,T]\times[a,b];\mathbf{C})$ is admissible if there is an admissible $y = (y_1, \dots, y_n)$ as above such that $f = y_j$, some $j \leq n$. □

Remarks. (1) As before we tacitly assume that a, b, T and the polynomials in def. 4 are constructive.

(2) By standard results on ODE's it follows that the vectorfunctions $y(t,\lambda)$, $z(\lambda)$, $g(\lambda)$ are real holomorphic in $t \in [0,T]$, $\lambda \in [a,b]$; its components are thus in the class $\mathscr{F}$ of functions defined at the beginning of this section.

(3) In order to achieve some formal simplifications, the ODE's in Def. 4 may be complex, but by separating real and imaginary part it would be easy to remain in the real domain.

(4) The variables λ, t are taken from the list $\alpha_1, \alpha_2, \ldots$, i.e. $\lambda = \alpha_j$, $t = \alpha_k$, some $j \neq k$.

Based on Def. 4 we now consider an extension $\mathscr{M}_1 \supseteq \mathscr{M}_0$ of the set of basic functions $\mathscr{M}_0$, given by Def. 3:

Definition 5. A function f is in $\mathscr{M}_1$ iff either $f \in \mathscr{M}_0$ or if f is admissible in the sense of Def. 4. □

Remarks. (1) Since $\mathscr{M}_0$ is the set of trigonometric monomials $\lambda e^{i\alpha_j}$, one infers that any $f \in \mathscr{M}_0$ is admissible, i.e. $\mathscr{M}_1$ is simply the set of functions admissible via Def. 5. (2) Our main result is

Theorem 1. *Given a recursively enumerable predicate $P(x)$ there is $\phi \in H_0(\mathscr{M}_1)$ which represents $P(x)$.*

Remarks. (1) Conditions (b) of Def. 4 complicate the proof of Thm. 1, however they contribute to the constructive aspect of the system of ODE's in Def. 4, (a), as will be seen in Sect. 5.

(2) The class of functions $H_0(\mathscr{M}_1)$ is similar to the class of functions which can be generated by an analogue computer in the sense of [15] (Sect. 2, Def. 10). In both cases, new functions are obtained by solving algebraic ODEs. However, in [15] only functions of a single variable are considered, while here we admit functions of several variables.

Moreover, here we consider ODE's which may depend on a parameter λ via (vector-)functions $z(\lambda), g(\lambda)$ which are themselves solutions of ODEs. A difficult question is whether the use of several variables can be avoided or if (a)–(c) in Def. 4 can be restricted to the case where $z(\lambda), g(\lambda)$, $\lambda \in [a, b]$ are already in $H_0(\mathscr{M}_0)$. This problem will be addressed in Sect. 5.

(3) There are some types of ODEs which are not directly of the form (a)–(c) in Def. 4 but which easily reduce to it. We discuss three typical cases; all other cases to be encountered reduce to (a)–(c) by the same manipulations. To start with, let $P(\zeta_0, \ldots, \zeta_{n-1}, \eta)$ be a constructive polynomial in ζ_j, η and let $y \in C^n([0, T]; \mathbf{C})$ satisfy the following, where $y^{(k)}$ is the k-th derivative:

$$\begin{aligned} &P(y(t), \ldots, y^{(n)}(t)) = 0, \quad (\partial_\eta P)(y(t), \ldots, y^{(n)}(t)) \neq 0 \\ &\text{for } t \in [0, T], \text{ and } y(0), \ldots, y^{(n)}(0) \text{ are constructive.} \end{aligned} \tag{3.6}$$

We claim

$$y^{(j)} \text{ is in } H_0(\mathscr{M}_1) \text{ for } j \leq n. \tag{3.7}$$

To see this we differentiate (3.6) with respect to t and set $y_0 = y, \dots, y_n = y^{(n)}$ so as to get the system:

$$
\begin{aligned}
&y_0' = y_1, \dots, y_{n-1}' = y_n,\\
(3.8)\qquad &(\partial_\eta P)(y_0, \dots, y_n) y_n' = -\sum_0^{n-1} (\partial_{\zeta_j} P)(y_0, \dots, y_n) y_{j+1}
\end{aligned}
$$

which has the form required by Def. 4(a). The functions $g(\lambda), z(\lambda)$ are absent and the determinant in def 4(b) is $(\partial_\eta P)(y(t), \dots, y^{(n)}(t))$, thus $\neq 0$ by (3.6); since Def. 4(c) is implied by clause (3.6), (3.8), claim (3.7) follows from Def. 4, 5.

Next set $\zeta = \zeta_0, \dots, \zeta_n$ resp. $\xi = \xi_1, \dots, \xi_m$ and $\varepsilon = \varepsilon_0, \dots, \varepsilon_n$. Let $P(\zeta, \xi)$ be a constructive polynomial, $H(\xi)$ and $L(\varepsilon)$ an $m \times m$ and an $(n+1) \times (n+1)$-matrix resp., let $Q(\xi)$ be an m-vector and $R(\varepsilon)$ an $n+1$-vector, all polynomial and constructive. Finally, let $y(t, \lambda)$, $z(\lambda) = (z_1, \dots, z_m)$ and $g(\lambda) = (g_0, \dots, g_n)$ be real holomorphic on $t \in [0, T]$, $\lambda \in [a, b]$ such that

$$
\begin{aligned}
&\text{(a) } P(y, \dots, y^{(n)}, z) = 0, H(z) z_\lambda = Q(z), L(g) g_\lambda = R(g), \lambda \in [a, b],\\
&\quad t \in [0, T],\\
(3.9)\qquad &\text{(b) } (\partial_{\zeta_n} P)(y_1, \dots, y^{(n)}, z), \det(H(z)), \det(L(g)) \text{are} \neq 0 \text{ for } t \in [0, T],\\
&\quad \lambda \in [a, b]\\
&\text{(c) } y^{(j)}(0, \lambda) = g_j(\lambda);\ g(0), z(0) \text{ are constructive.}
\end{aligned}
$$

Proposition 3.1. *Under assumptions (3.9), $y^{(j)} \in \mathscr{M}_1$, $j \leq n$.*

The system (3.9) is easily reduced to a system of type (a)–(c) in Def. 4; the details may be omitted.

For our next example, let $\zeta = \zeta_1, \dots, \zeta_n$, $\xi = \xi_1, \dots, \xi_m$, $P(\zeta, \xi)$, $H(\xi)$, $Q(\xi)$ be as in the preceding example, let $y(t), z(t) = (z_1(t), \dots, z_m(t))$, $t \in [0, T]$ be real holomorphic such that

$$
\begin{aligned}
&\text{(a)} P(y, \dots, y^{(n)}, z) = 0, \quad H(z) z_t = Q(z),\ t \in [0, T],\\
(3.10)\qquad &\text{(b)} (\partial_{\zeta_n} P)(y_1, \dots, y^{(n)}, z) \neq 0, \quad \det(H(z)) \neq 0, \quad t \in [0, T],\\
&\text{(c)} y^{(j)}(0), \quad j \leq n, \quad z_k(0), k \leq m \text{ are constructive.}
\end{aligned}
$$

We then have

Proposition 3.2. *Under assumptions (3.10), $y^{(j)} \in \mathscr{M}_1$ for $j \leq n$.*

The proof, which is by the same manipulations as used in the proof of (3.7), is omitted.

Our last case is an instance of the implicit function theorem. Let $P(\eta, \xi_1, \dots, \xi_m)$ be a polynomial, $H(\xi_1, \dots, \xi_m)$ a polynomial $m \times m$-matrix and $Q(\xi_1, \dots, \xi_m)$ a polynomial m-vectorfunction, all constructive, let $y(t), z(t) = (z_1(t), \dots, z_m(t))$, $t \in$

$[0, T]$ be real holomorphic such that

$$\begin{aligned} &P(y(t), z(t)) = 0, \quad H(z(t))z_t = Q(z(t)), \quad (\partial_\eta P)(y(t)z(t)) \neq 0 \\ &\det(H(z(t))) \neq 0, \quad t \in [0, T]; \quad y(0), z(0) \text{ are constructive.} \end{aligned} \tag{3.11}$$

Here the shorthands $P(\eta, \xi) = P(\eta, \xi_1, \ldots, \xi_m)$ etc. have been used.

Proposition 3.3. *Under assumption (3.11) we have that* $y \in \mathscr{M}_1$.

Proof. We set $(z_j)_t = \zeta_j$, i.e. $z_t = \zeta = (\zeta_1, \ldots, \zeta_m)$. We then differentiate the equations in (3.11) so as to get the system

$$\begin{aligned} &(\partial_\eta P)(y, z)y_t = -\sum_1^m (\partial_{\xi_j} P)(y, z)\zeta_j \\ &H(z)\zeta_t = \sum_1^m (\partial_{\xi_j} Q)(z)\zeta_j - \left(\sum_1^m (\partial_{\xi_j} H)(z)\zeta_j\right)\zeta \\ &H(z)z_t = Q(z), \quad t \in [0, T]. \end{aligned} \tag{3.12}$$

This is a system of type Def. 4(a) with parameter λ absent. The determinant as in Def. 4(b) is given by

$$\Delta(t) := (\partial\eta P)(y(t), z(t)) \det(H(z(t))^2, t \in [0, T]. \tag{3.13}$$

By assumption (3.11), $\Delta(t) \neq 0$, $t \in [0, T]$. Finally, $y(0)$, $z(0)$ are constructive by (3.11) whence:

$$\zeta(0) = H(z(0))^{-1}Q(z(0)) \text{ is constructive.} \tag{3.14}$$

To sum up, the vectorfunction

$$\gamma(t) = (y(t), \zeta_1(t), \ldots, \zeta_m(t), z_1(t), \ldots, z_m(t)), \quad t \in [0, T]$$

is real holomorphic, satisfies (3.12), $\gamma(0)$ is constructive and the determinant $\Delta(t)$, associated to (3.12) via Def. 4(b), is $\neq 0$ for $t \in [0, T]$. By Def. 4, 5 all components of γ, hence y, are in $\mathscr{M}_1$. □

Remark. When it comes to an application of (3.11) via Prop. 3.3, one should, in principle, replace (3.11) by its induced system (3.12). However we will refrain from this step and consider (3.11) as the special case of (3.10) which arises for $n = 0$.

The above examples, i.e. (3.6), (3.9), (3.10), (3.11) are typical instances of Def. 4(a)–(c). All cases to be encountered are of the above type or reducible to one of these types along the arguments used in the proof of (3.7) resp. Prop. 3.3.

We conclude with a proposition that leads us to the next section. The proof, given in [19], Sect. 2, is outlined here in order to stress the importance of the Jacobi-Theta function ϑ_3, to which our next section is devoted. We recall $\theta_\lambda(\alpha, \beta)$ in (1.4).

Proposition 3.4. *For every recursively enumerable predicate* $P(x)$ *and any* $\lambda \in (0, 1)$ *there is* $\phi \in H_0(\mathscr{M}_0 \cup \{\theta_\lambda(\alpha, \beta)\})$ *representing* $P(x)$ *(in the sense of Def. 1).*

Proof. Let $S = H_0(\mathscr{M}_0 \cup \{\theta_\lambda(\alpha,\beta)\})$. By Def. 2, 3 the functions $\lambda e^{i(\gamma+\beta)}$, $\lambda e^{i(\gamma-\beta)}$ are in S, where $\lambda \in (0,1)$ is constructive. It follows that $(1-\lambda e^{i(\gamma+\beta)})^{-1}$, $(1-\lambda e^{i(\gamma-\beta)})^{-1}$ and thus $\sum_0^\infty \lambda^m \cos m\beta e^{im\gamma}$ are in S. On the other hand

$$\tfrac{1}{2}(\theta_\lambda(\alpha,\beta)+1) = \sum_0^\infty \lambda^{n^2} e^{in^2\alpha} \cos n\beta \in S.$$

By these facts and by Def. 2 we have that

$$(2\pi)^{-1}\int_0^{2\pi} \left(\sum_0^\infty \lambda^m \cos m\beta e^{im\gamma}\right)\left(\sum_0^\infty \lambda^{n^2} e^{in^2\alpha}\cos n\beta\right) d\beta$$

is in S. By an elementary computation we thus infer

$$\phi_0(\alpha,\beta) := \sum_0^\infty \lambda^{n^2+n} e^{in^2\alpha} e^{in\gamma} \in S \tag{3.15}$$

where λ is kept fixed. This means in terms of Def. 1 that the predicate $P(m,n) \sim m = n^2$ has a representing function $\phi_0 \in S$. Now the predicate $M(m,n,p) \sim p = mn$ is expressed in terms of $m = n^2$ and $q = m+n$ via

$$(\exists uvwst)(s = m+n \wedge m^2 = u \wedge n^2 = v \wedge s^2 = w \wedge u+v = t \wedge t+2p = w). \tag{3.16}$$

On the other hand it follows from Def. 2 and the arguments in Sect. 2: (*) if P, Q are represented by functions in S, then so are $P \wedge Q$, $P \vee Q$, $(\exists y)P$.

By (3.4), (3.15) and remark (*) it follows that there is a function $\phi_1(\alpha,\beta,\gamma) \in S$ which represents the predicate $p = m \cdot n$. By this fact and a repeated application of remark (*) it follows straightforwardly that every diophantine predicate

$$(\exists x_1,\ldots,x_s)(p(x,x_1,\ldots,x_s) = q(x,x_1,\ldots,x_s))$$

(p,q polynomials with integer positive coefficients) admits a representation $\psi(\alpha) \in S$. An inductive proof of this last step is given [3]. The proposition now follows from [13]. □

4. Proof of Theorem 1

We now come to the proof of Thm. 1. By Def. 5 and Prop. 3.4, Thm. 1 follows from

Theorem 1'. *For $\lambda \in (0,1)$ constructive and sufficiently small we have $\theta_\lambda(\alpha,\beta) \in H_0(\mathscr{M}_1)$.*

Remarks. Our proof differs considerably from that in [3], which avoids elliptic function theory but requires heavy computations which are difficult to reproduce. Our proof is based on a device due to Bertrand [1] which relates the ODE for the Jacobi Theta-function $\vartheta_3(q,0)$ to the well known ODE satisfied by the Weierstrass elliptic function $\wp$. In order to be applicable here, the device in [1] has to be modified; however the basic idea in [1] remains unaltered. Our presentation is such that the arguments may be understood without familiarity with elliptic functions.

For anyone interested in the background we refer to [9], [12], [18]. We proceed as follows. We first derive a system of ODEs for $\theta_\lambda(\pi x, 0)$ of type (a)–(c), Def. 4, with x in the role of t and without parameter λ. Then we derive a system of ODEs for $\theta_\lambda(\pi x, 2\pi v)$ with v in the role of t in Def. 4, and with x in the role of λ; we emphasize that the index λ in θ_λ has nothing to do with the parameter λ in Def. 4.

Contrary to expectation, not much is saved if we start with the established ODE for $\theta_\lambda(\pi x, 0)$ (see e.g. [11]). In fact, the steps needed for our derivation of a system of type (a)–(c) for $\theta_\lambda(\pi x, 0)$ are needed again in the derivation of a system of type (a)–(c) for $\theta_\lambda(\pi x, 2\pi v)$. Moreover, the verification of (b) in Def. 4 cannot be avoided even if we take Jacobi's ODE as starting point. And last but not least, our approach may have some interest in itself.

To begin with, we list the objects from elliptic function theory of interest to us; we adopt, with few exceptions, the notation in [18]. Below, $\omega, \omega' \in \mathbf{C}$ are fixed, both $\neq 0$ and such $\omega' = \tau\omega$ with $\tau \in \mathbf{C}$, $im(\tau) > 0$. We call ω, ω' "halfperiods", while τ is a fundamental parameter. We set

$$\tau = x + i\alpha, \quad x \in \mathbf{R}, \quad \alpha > 0; \quad q = e^{i\pi\tau} = e^{i\pi x}e^{-\pi\alpha} = \lambda e^{i\pi x}. \tag{4.1}$$

Here $\lambda \in (0,1)$ (resp. $\alpha > 0$) is an auxiliary fixed constructive constant by assumption, assumed to be small (resp. large) if necessary. The basic functions underlying our analysis are the following (see below)

$$g_j(\tau, \omega), \quad \widetilde{g}_j(\tau), \quad j = 2, 3, \quad \eta(\tau, \omega), \quad \widetilde{\eta}(\tau), \quad \wp(\tau, \omega, z), \quad \vartheta_3(\tau, v) \tag{4.2}$$

where $\tau = x + i\alpha$; z and v may range over $\mathbf{C}$ but in our context v is restricted to $\mathbf{R}$. We first consider the functions g_j, $\widetilde{g}_j$. By definition,

$$\begin{aligned} &\widetilde{g}_2(\tau) = 60 \textstyle\sum{}'(m\tau + n)^{-4} \quad && \widetilde{g}_3(\tau) = 140 \textstyle\sum{}'(m\tau + n)^{-6} \\ &g_2(\tau, \omega) = (2\omega)^{-4}\,\widetilde{g}_2(\tau) && g_3(\tau) = (2\omega)^{-6}\,\widetilde{g}_3(\tau) \\ &\tau = x + i\alpha, \quad \alpha > 0, \end{aligned} \tag{4.3}$$

where the summation is over all $m, n \in \mathbf{Z}$, $(m, n) \neq (0, 0)$. The functions $\widetilde{g}_j$ are modular functions [9], [12], i.e. holomorphic in the upper halfplane ($\alpha > 0$) and characterized by their transformation properties under the modular group. As a function of x, $\widetilde{g}_j(x + i\alpha)$ is 1-periodic. The property of relevance here is the fact that the $\widetilde{g}_j$ are characterized by ODE's as expressed by the lemma below, where

$$\begin{aligned} &y(x) = \widetilde{g}_2(x + i\alpha), \quad z(x) = \widetilde{g}_3(x + i\alpha), \quad y^{(n)} = \partial_x^n y, \\ &z^{(n)} = \partial_x^n z \text{ and } y = y(x), \quad z = z(x). \end{aligned}$$

Lemma 4.1. *There are constructive polynomials $\mathscr{D}(\zeta_0, \ldots, \zeta_3)$, $\mathscr{G}(\xi_0, \xi_1, \zeta_0, \zeta_1)$ such that*

(a) $\mathscr{D}(y, \ldots, y^{(3)}) = 0$, $\mathscr{G}(z, z^{(1)}, y, y^{(1)}) = 0$, $x \in \mathbf{R}$,

(b) $(\partial_{\zeta_3}\mathscr{D})(y, \ldots, y^{(3)}) \cdot (\partial_{\xi_1}\mathscr{G})(z, z^{(1)}, y, y^{(1)}) \neq 0$ *for* $x \in \mathbf{R}$, *and* $\alpha > 0$ *sufficiently large,*

(c) $y(0), \ldots, y^{(3)}(0)$, $z(0)$,$z^{(1)}(0)$ *are constructive.* □

The lemma follows straightforwardly from the theory of modular functions; an outline of proof is given in the appendix. As for η, $\widetilde{\eta}$ we have (e.g. [18], pg. 19)

$$\begin{aligned}\widetilde{\eta}(\tau) &= 4 + \sum{}' \left(\frac{1}{1-2w} + \frac{2}{w} + \frac{1}{w^2}\right)\\ \eta(\tau,\omega) &= \frac{1}{4\omega}\,\widetilde{\eta}(\tau), \quad \tau = x + i\alpha,\ \alpha > 0,\end{aligned} \tag{4.4}$$

where the summation is over all $w = m\tau + n$, $m, n \in \mathbf{Z}$, $(m,n) \neq (0,0)$. The following representations are useful ([9], pg. 212)

$$\begin{aligned}\eta(\tau,\omega) &= \frac{\pi^2}{\omega}\left(\frac{1}{12} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right),\\ g_2(\tau,\omega) &= \left(\frac{\pi}{\omega}\right)^4\left(\frac{1}{12} + 20\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\right),\\ g_3(\tau,\omega) &= \left(\frac{\pi}{\omega}\right)^6\left(\frac{1}{216} - \frac{7}{9}\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}}\right), \quad q = e^{i\pi\tau}.\end{aligned} \tag{4.4)(B}$$

From the series in (4.4) for $\widetilde{\eta}$, one reads off that $\widetilde{\eta}(\tau)$ is holomorphic in $im(\tau) > 0$ and 1-periodic in x. However, $\widetilde{\eta}(\tau)$ is not known to be modular, and a counterpart to Lemma 4.1 does not seem to be available. We come to the Weierstrass elliptic function $\wp(\tau,\omega,z)$ which is meromorphic and doubly periodic in $z \in \mathbf{C}$ with $2\omega, 2\omega' = 2\tau\omega$ as periods. Its poles are situated at $2n\omega + 2m\omega'$, $m, n \in \mathbf{Z}$. It is expressed by an infinite series which involves τ, ω, z ([18] pg. 15, [9] pg. 164). It satisfies an ODE with respect to z:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \quad \wp' = \partial_z\wp, \tag{4.5}$$

([18], pg. 18). We finally recall the Jacobi Theta-function $\vartheta_3(\tau, v)$, (ϑ_2 in [18]), which, with $\tau = x + i\alpha$, $\alpha > 0$ is given by

$$\vartheta_3(x + i\alpha, v) = 1 + 2\sum_{1}^{\infty}\lambda^{n^2}e^{in^2\pi x}\cos 2n\pi v, \quad \lambda = e^{-\pi\alpha}. \tag{4.6}$$

Up to a renaming and rescaling of variables, ϑ_3 coincides with $\theta_\lambda(\alpha,\beta)$ in (1.4): $\theta_\lambda(\pi x, 2\pi v) = \vartheta_3(x + i\alpha, v)$. Theorem 1' thus follows from

Proposition 4.1. *For $\alpha > 0$ sufficiently large, $\vartheta_3(x + i\alpha, v)$, as a function of x, v, is admissible via Def. 4.*

In order to prove Prop. 4.1 we start our analysis with a remark in [1], pg. 4 according to which $\wp$ may be expressed in terms ϑ_3. In fact we have

Proposition 4.2. *Setting $\widetilde{\wp} = -\wp$, $z = 2\omega v$, $\partial = \partial_v$, we have*

$$\widetilde{\wp}(\tau,\omega,z) = \frac{\eta(\tau,\omega)}{\omega} + \frac{1}{4\omega^2}\,\partial\left(\partial\log\vartheta_3(v - \tfrac{1}{2} - \tfrac{\tau}{2})\right). \tag{4.7}$$

Proof. $\wp = -\widetilde{\wp}$ is expressed by the Weierstrass σ-function ([18], Sect. 2.2) via

$$\partial_z(\partial_z \log \sigma(z)) = \widetilde{\wp}(z) = -\wp(z). \tag{4.8}$$

On the other hand, $\sigma(z)$ is related to another Theta-function of Jacobi, i.e. $\vartheta(v)$ ([18], Sect. 6.2) via

$$\sigma(z) = e^{2\omega\eta v^2} C\vartheta(v), \quad z = 2\omega v, \text{ some } C, \tag{4.9}$$

with $\eta = \eta(\tau, \omega)$ as in (4.4). Finally, $\vartheta(v)$ is expressed in terms of our ϑ_3 by

$$\vartheta(v) = q^{-\frac{1}{4}} e^{-i\pi w} \vartheta_3(w), \quad w = v - \tfrac{1}{2} - \tfrac{\tau}{2}, \ q = e^{i\pi\tau} \tag{4.10}$$

according to the table in [18], pg. 42, where ϑ_2 is our ϑ_3. Combination of (4.8)–(4.10) yields the claim. □

In a next step we combine (4.5), (4.7) so as to get an ODE for ϑ_3. We differentiate (4.5) with respect to z and divide by $\wp'$ so as to get

$$2\wp'' + g_2 = 12\wp^2. \tag{4.11}$$

From (4.11), (4.5) we then get

$$3(\wp')^2 = \wp(12\wp^2 - 3g_2) - 3g_3 = 2\wp(\wp'' - g_2) - 3g_3. \tag{4.12}$$

Setting again $\widetilde{\wp} = -\wp$ we get from (4.12) after a rearrangement of terms

$$27((\widetilde{\wp}')^2 + g_3)^2 = (-2\widetilde{\wp}'' + g_2)(\widetilde{\wp}'' + g_2)^2. \tag{4.13}$$

In order to exploit (4.13) we set

$$\psi(v) = \partial_v(\partial_v \log \vartheta_3(v)), \quad \widehat{\wp}(v) = \frac{\eta}{\omega} + \frac{1}{4\omega^2}\,\psi(v), \tag{4.14}$$

where $\vartheta_3(\tau, v) = \vartheta_3(v)$. Based on (4.14) and since $z = 2\omega v$, (4.7) is rewritten as

$$\widetilde{\wp}(2\omega(v + \tfrac{1}{2} + \tfrac{\tau}{2})) = \tfrac{\eta}{\omega} + \tfrac{1}{4\omega^2}\,\psi(v) = \widehat{\wp}(v). \tag{4.15}$$

By combining (4.13) with (4.15) we obtain by a routine computation an ODE for $\psi(v) = \psi(\tau, v)$, i.e.

$$27((\partial_v\psi)^2 + \widetilde{g}_3)^2 = (-2\partial_v^2\psi + \widetilde{g}_2)(\partial_v^2\psi + \widetilde{g}_2)^2 \tag{4.16}$$

with $\widetilde{g}_j$ related to g_j via (4.3); ω does not appear in (4.16). By (4.14), (4.16) is an ODE for $\vartheta_3(\tau, v)$ with respect to v. We exploit this to show that $\varphi(x) = \vartheta_3(x + i\alpha, 0)$ is admissible for α sufficiently large. We first note:

Proposition 4.3. *Let $\partial = \partial_v$; for $n \geq 1$ we have*

$$\partial^n \log \phi = \frac{\partial^n \phi}{\phi} + \frac{\mathscr{D}_n(\phi, \partial\phi, \ldots, \partial^{n-1}\phi)}{\phi^N}, \quad N = N_n > 0,$$

where the integer polynomial $\mathscr{D}_n(\zeta_0, \ldots, \zeta_{n-1})$ is a sum of monomials of the form $c\zeta_0^{\alpha_0}\zeta_1^{\alpha_1} \ldots \zeta_{n-1}^{\alpha_{n-1}}$ with $\alpha_1 + \cdots + \alpha_{n-1} \geq 2$.

The elementary proof by induction is omitted. Recalling Def. 4 we have

Lemma 4.2. *For $\alpha > 0$ constructive, and sufficiently large, $\vartheta_3(x + i\alpha, 0)$, as a function of x, is admissible.*

Proof. We first set: (*) $\phi(\tau, v) = \vartheta_3(\tau, v)$, $\partial = \partial_v$, $\tau = x + i\alpha$, with $\alpha > 0$ a constructive constant, to be choosen suitably. By (4.6) we may take $\alpha > 0$ so large that

$$|\phi(x + i\alpha, v)| \geq \tfrac{1}{2}, \quad x, v \in \mathbf{R}. \tag{4.17}$$

We then recall (4.14) and combine (4.16) with Prop. 4.3. As a result we find polynomials $Q(\zeta_0, \ldots, \zeta_2) = D_3(\zeta_0, \ldots, \zeta_2)$ and $P(\zeta_0, \ldots, \zeta_3) = D_4(\zeta_0, \ldots, \zeta_3)$ having the structure as in Prop. 4.3 and such that:

$$\begin{aligned} &27\left(\left(\frac{\partial^3\phi}{\phi} + \frac{Q(\phi, \partial\phi, \partial^2\phi)}{\phi^M}\right)^2 + \widetilde{g}_3\right)^2 \\ &= \left(-\frac{2\partial^4\phi}{\phi} - \frac{2P(\phi, \ldots, \partial^3\phi)}{\phi^N} + \widetilde{g}_2\right)\left(\frac{\partial^4\phi}{\phi} + \frac{P(\phi, \ldots, \partial^3\phi)}{\phi^N} + \widetilde{g}_2\right)^2 \end{aligned} \tag{4.18}$$

(some M, N). We now apply the device already mentioned in [1], Sect. 2.2. Eq. (4.18) holds for $v, x \in \mathbf{R}$ under assumption (4.17) and thus for $x \in \mathbf{R}$, $v = 0$. Now since $\phi(\tau, 0) = \vartheta_3(\tau, 0)$ and by (4.6) we have (with $\partial = \partial_v$)

$$\begin{aligned} (4\pi i)(\partial_x\phi)(x + i\alpha, 0) &= (\partial^2\phi)(x + i\alpha, 0) \\ (4\pi i)^2(\partial_x^2\phi)(x + i\alpha, 0) &= (\partial^4\phi)(x + i\alpha, 0) \\ (\partial\phi)(x + i\alpha, 0) &= (\partial^3\phi)(x + i\alpha, 0) = 0. \end{aligned} \tag{4.19}$$

Set also

$$\begin{aligned} \widetilde{Q}(\zeta_0, \zeta_2) &= Q(\zeta_0, 0, (4\pi i)\zeta_2) \\ \widetilde{P}(\zeta_0, \zeta_2) &= P(\zeta_0, 0, (4\pi i)\zeta_2, 0). \end{aligned} \tag{4.20}$$

By the structure of P, Q via Prop. 4.3 we have

$$\widetilde{Q}(\zeta_0, \zeta_2) = \zeta_2^2\widehat{Q}(\zeta_0, \zeta_2), \quad \widehat{P}(\zeta_0, \zeta_2) = \zeta_2^2\widehat{P}(\zeta_0, \zeta_2) \tag{4.21}$$

for some polynomials $\widehat{P}, \widehat{Q}$; a closer look shows that actually $\widetilde{Q} = 0$. Setting $\varphi(x) = \phi(x + i\alpha, 0)$ and inserting (4.19), (4.20) into (4.18) yields

$$\begin{aligned} &27\left(\frac{\widetilde{Q}(\varphi, \partial_x\varphi)^2)}{\varphi^{2M}} + \widetilde{g}_3\right)^2 \\ &= \left(-2(4\pi i)^2\frac{\partial_x^2\varphi}{\varphi} - \frac{2\widetilde{P}(\varphi, \partial_x\varphi)}{\varphi^N} + \widetilde{g}_2\right)\left((4\pi i)^2\frac{\partial_x^2\varphi}{\varphi} + \frac{\widetilde{P}(\varphi, \partial_x\varphi)}{\varphi^N} + \widetilde{g}_2\right)^2. \end{aligned} \tag{4.22}$$

This is a polynomial ODE for $\varphi(x) = \phi(x + i\alpha, 0) = \vartheta_3(x + i\alpha, 0)$, i.e. there is a constructive polynomial $E(\zeta_0, \zeta_1, \zeta_2, \eta_0, \xi_0)$ such that after multiplication with φ^k, some $k \in \mathbf{N}$ large, (4.22) transforms into

$$E(\varphi, \partial_x\varphi, \partial_x^2\varphi, \widetilde{g}_2, \widetilde{g}_3) = 0. \tag{4.23}$$

Recalling $\mathscr{D}(\zeta_0, \ldots, \zeta_3)$, $\mathscr{G}(\xi_0, \xi_1, \zeta_0, \zeta_1)$ in Lemma 4.1 we have that

$$\mathscr{D}(\widetilde{g}_2, \ldots, \partial_x^3\widetilde{g}_2) = \mathscr{G}(\widetilde{g}_3, \partial_x\widetilde{g}_3, \widetilde{g}_2, \partial_x\widetilde{g}_2) = 0. \tag{4.24}$$

The system (4.23)+(4.24) is of the type considered in Prop. 3.2. In order to infer that φ is admissible we have to verify (3.10)(b), (c). Now (c) follows straightforwardly from the series expansions (4.6) and [9], pg. 212. It remains to check (3.10)(b). We recall that by Lemma 4.1 we already have

$$(\partial_{\zeta_3}\mathscr{D})(\widetilde{g}_2,\ldots,\partial_x^3\widetilde{g}_2)\neq 0,\quad (\partial_{\xi_1}\mathscr{G})(\widetilde{g}_3,\partial_x\widetilde{g}_3,\widetilde{g}_2,\partial_x\widetilde{g}_2)\neq 0,\quad x\in\mathbf{R}$$

for $\alpha\gg 0$. It thus remains to check

$$(\partial_{\zeta_2}E)(\varphi,\partial_x\varphi,\partial_x^2\varphi,\widetilde{g}_2,\widetilde{g}_3)\neq 0,\quad x\in\mathbf{R}\text{ for }\alpha\gg 0. \tag{4.25}$$

By an elementary computation, based on (4.22) and which we omit, we find $s\in\mathbf{N}$ such that

$$\begin{aligned}&(\partial_{\zeta_2}E)(\varphi,\partial_x\varphi,\partial_x^2\varphi,\widetilde{g}_2,\widetilde{g}_3)=-6a\varphi^s\Pi_1\Pi_2,\quad a=(4\pi i)^2\\ &\Pi_1=\frac{a\partial_x^2\varphi}{\varphi}+\frac{\widetilde{P}(\varphi,\partial_x\varphi)}{\varphi^N},\quad \Pi_2=\Pi_1+\widetilde{g}_2.\end{aligned}\tag{4.26}$$

In order to estimate the Π_j in (4.26) we recall (4.6) according to which we have

$$\partial_x^p\varphi=2\sum_1^\infty(i\pi n^2)^p\lambda^{n^2}e^{in^2\pi x},\quad p>0,\tag{4.27}$$

where $\lambda=e^{-\pi\alpha}$, $\alpha>0$. From (4.6), (4.27) we read off that there are α_0, $k>0$ and remainder terms $R_j(x,\lambda)$, $j=1,2$ such that

$$\begin{aligned}&\partial_x\varphi=\lambda R_1(x,\lambda),\quad \partial_x^2\varphi=2(i\pi)^2\lambda e^{i\pi x}+\lambda^2R_2(x,\lambda),\\ &|R_j(x,\lambda)|\leq k,\quad \tfrac12\leq|\varphi(x)|\leq k\text{ for }x\in\mathbf{R},\alpha\geq\alpha_0.\end{aligned}\tag{4.28}$$

On the other hand it follows from the series expansion for $\widetilde{g}_2$ in (4.4)(B) that there are $k_1>0$, $\alpha_1\leq\alpha_0$ and a remainder term $R_0(x,\lambda)$ such that $\alpha\geq\alpha_1$ implies:

$$\widetilde{g}_2(x,\lambda)=\frac{(2\pi)^4}{12}+\lambda R_0(x,\lambda),|R_0(x,\lambda)|\leq k_1,\ x\in\mathbf{R}.\tag{4.29}$$

From (4.21), (4.28), on the other hand, it follows that there is $C_0>0$ such that

$$|\widetilde{P}(\varphi,\partial_x\varphi)|\leq\lambda^2C_0\text{ for }\alpha\geq\alpha_0,\ x\in\mathbf{R}.\tag{4.30}$$

By combining (4.28), (4.30) we obtain:

$$|\Pi_1|\leq 32\pi^2|2(i\pi)^2\lambda e^{i\pi x}+\lambda^2R_2(x,\lambda)|+2^NC_0\lambda^2\leq C_1\lambda,$$

some C_1, for $\alpha\geq\alpha_0$, $x\in\mathbf{R}$. We then find $\alpha_2\geq\alpha_1$ such that:

$$|\Pi_2|\geq\left|\frac{\pi^4}{12}+\lambda R_0(x,\lambda)\right|-|\Pi_1|\geq\frac{\pi^4}{12}-\lambda k_1-\lambda C_1\geq\frac{\pi^4}{24}\tag{4.31}$$

for $\alpha\geq\alpha_2$, $x\in\mathbf{R}$. Again by (4.28), (4.30) we obtain a lower bound for $|\Pi_1|$:

$$\begin{aligned}|\Pi_1|&\geq\frac{(4\pi)^2}{k}\ |(2\pi i)^2\lambda e^{i\pi x}+\lambda^2R_2(x,\lambda)|-|\widetilde{P}(\varphi,\partial_x\varphi)||\varphi|^{-N}\\ &\geq\lambda\left\{\frac{(4\pi)^2}{k}\ (2\pi^2-\lambda k)-2^N\lambda C_0\right\}\end{aligned}$$

That is, there are $\alpha_3 \geq \alpha_2$ and C_2 such that

$$|\Pi_1| \geq \lambda C_2, \text{ for } x \in \mathbf{R}, \; \alpha \geq \alpha_3. \tag{4.32}$$

To sum up, if $\alpha \geq \alpha_3$ then

$$|\varphi|^s |6a\Pi_1\Pi_2| \geq \frac{6 \cdot 16\pi^2}{2^s} \lambda C_2 \frac{\pi^4}{24} = 2^{2-s}\pi^6 \lambda C_2. \tag{4.33}$$

This proves (4.25) and hence the lemma. □

Remarks. The above arguments show that $\varphi(x)$, $\widetilde{g}_j(x_i\alpha)$, $j = 2, 3$ satisfy for any $\alpha > 0$, a coupled system of polynomial ODEs, i.e. (4.23)+(4.24). Elimination of $\widetilde{g}_j$, $j = 2, 3$ in favour of $\varphi(x)$ yields a single polynomial ODE for Jacobi's $\varphi(x) = \vartheta_3(x + i\alpha, 0)$.

Conditions on $\alpha > 0$ have to be imposed when it comes to the verification of (3.10)(b). We have to assume $\alpha \gg 0$ (i.e. $\lambda = e^{-\pi\alpha} \ll 1$) in order to secure that (4.25) holds. This assumption also appears in [3] in the same context. Next we need a counterpart to Lemma 4.2 for the function $\widetilde{\eta}(x + i\alpha)$, $x \in \mathbf{R}$ ((4.2), (4.4)).

Lemma 4.3. *There is α_0 such that $\widetilde{\eta}(x + i\alpha)$, $x \in \mathbf{R}$ is admissible if $\alpha \geq \alpha_0$.*

Proof. We differentiate (4.5) with respect to z and divide by $\varrho' = \partial_z \varrho$ so as to get

$$g_2 - 2\partial_z^2 \widetilde{\wp} = 12\widetilde{\wp}^2 \text{ where } \widetilde{\wp} = -\wp. \tag{4.34}$$

Setting $z = 2\omega v + \omega + \omega\tau$ in (4.34) yields

$$g_2 - 2\partial_z^2 \widetilde{\wp}(2\omega v + \omega + \omega\tau) = 12\widetilde{\wp}(2\omega v + \omega + \omega\tau)^2, \quad v \in \mathbf{R}. \tag{4.35}$$

We now recall (4.7) where $z = 2\omega v$. We substitute $v + \frac{1}{2} + \frac{\tau}{2}$ for v in (4.7) so as to get (4.15). By combining (4.35) with (4.15) we get by an elementary computation:

$$g_2 - \frac{2}{(2\omega)^4} \partial^4 \log \vartheta_3(v) = 12 \left(\frac{\eta}{\omega} + \frac{1}{(2\omega)^2} \partial^2 \log \vartheta_3(v) \right)^2 \tag{4.36}$$

where $\partial = \partial_v$. We multiply (4.36) by $(2\omega)^4$ and recall (4.3), (4.4) so as to get

$$\widetilde{g}_2 - 2\partial^4 \log \vartheta_3(v) = 12(\widetilde{\eta} + \partial^2 \log \vartheta_3(v))^2. \tag{4.37}$$

We now repeat the arguments in the proof of Lemma 4.2. Recalling the polynomials $\mathscr{D}_n$ in Prop. 4.3 we stipulate

$$\begin{aligned} &P(\zeta_0, \ldots, \zeta_3) = \mathscr{D}_4(\zeta_0, \ldots, \zeta_3), \; H(\zeta_0, \zeta_1) = \mathscr{D}_2(\zeta_0, \zeta_1), \\ &\widetilde{P}(\zeta_0, \zeta_2) = \mathscr{D}_4(\zeta_0, 0, (4\pi i)\zeta_2, 0), \quad \phi = \vartheta_3, \; \varphi(x) = \phi(x + i\alpha, 0) \end{aligned} \tag{4.38}$$

and recall (4.21) according to which

$$\widetilde{P}(\zeta_0, \zeta_2) = \zeta_2^2 \widehat{P}(\zeta_0, \zeta_2), \quad H(\zeta_0, 0) = 0 \tag{4.39}$$

by virtue of Prop. 4.3. By these stipulations and Prop. 4.3 we infer from (4.37):

$$\widetilde{g}_2 - 2\left\{ \frac{\partial^4 \phi}{\phi} + \frac{P(\phi, \ldots, \partial^3 \phi)}{\phi^N} \right\} = 12 \left\{ \widetilde{\eta} + \frac{\partial^2 \phi}{\phi} + \frac{H(\phi, \partial\phi)}{\phi^M} \right\}^2, \tag{4.40}$$

(some $M, N > 0$). In (4.40) we set $v = 0$. Due to relations (4.19) and by (4.38) we get

$$\widetilde{g}_2 - 2\left\{(4\pi i)^2\,\frac{\partial_x^2\varphi}{\varphi} + \frac{\widetilde{P}(\varphi,\partial_x\varphi)}{\varphi^N}\right\} = 12\left(\widetilde{\eta} + \frac{(4\pi i)\partial_x\varphi}{\varphi}\right)^2. \tag{4.41}$$

Now (4.41) is an instance of the implicit function theorem as addressed by (3.11) and Prop. 3.3, with $\widetilde{\eta}$ as “unknown”, and $\varphi, \widetilde{g}_2$ in the role of $z(\ )$. In fact there is a constructive polynomial $E_1(\zeta, \zeta_0, \zeta_1, \zeta_2, \xi)$ such that (4.41) transforms into a polynomial equation

$$E_1(\widetilde{\eta}, \varphi, \partial_x\varphi, \partial_x^2\varphi, \widetilde{g}_2) = 0 \tag{4.42}$$

after multiplication of (4.41) with φ^{N+1}. With $E, \mathscr{G}, \mathscr{D}$ as in (4.23), (4.24) and (4.42) we get a constructive polynomial system:

$$\begin{aligned} &E_1(\widetilde{\eta}, \varphi, \partial_x\varphi, \partial_x^2\varphi, \widetilde{g}_2) = E(\varphi, \partial_x\varphi, \partial_x^2\varphi, \widetilde{g}_2, \widetilde{g}_3) = 0\\ &\mathscr{D}(\widetilde{g}_2, \ldots, \partial_x^3\widetilde{g}_2) = \mathscr{G}(\widetilde{g}_3, \partial_x\widetilde{g}_3, \widetilde{g}_2, \partial_x\widetilde{g}_2) = 0. \end{aligned} \tag{4.43}$$

The system (4.43) is of the form (or rather can be put into the form) of clause (3.11). In order to recognize $\widetilde{\eta}(x + i\alpha)$, $x \in \mathbf{R}$ for $\alpha \gg 0$ via Prop. 3.3 one has to check the two conditions in (3.11). That $\widetilde{\eta}(i\alpha)$, $(\partial_x^j\varphi)(0)$, $(\partial_x^j\widetilde{g}_2)(i\alpha)$ and $(\partial_x^j\widetilde{g}_3)(i\alpha)$ are all constructive for constructive α follows again from their series expansions (4.4)(B), (4.6) for $v = 0$.

In view of Lemma 4.1 and of (4.25) it is easily seen that it only remains to show that

$$(\partial_\zeta E_1)(\widetilde{\eta}, \varphi, \partial_x\varphi, \partial_x^2\varphi, \widetilde{g}_2) \neq 0,\ x \in \mathbf{R} \text{ for } \alpha \gg 0. \tag{4.44}$$

This is easily seen to be equivalent to the condition

$$\widetilde{\eta} + (4\pi i)\frac{\partial_x\varphi}{\varphi} \neq 0,\ x \in \mathbf{R},\ \text{for } \alpha \gg 0. \tag{4.45}$$

Since $\widetilde{\eta} = 4\omega\eta$ and by the series expansion for η in (4.4)(B) there are α', k' and a remainder term $R_3(x, \lambda)$ such that $\alpha \geq \alpha'$ implies

$$\widetilde{\eta} = \frac{\pi^2}{3} + R_3(x,\lambda)\lambda,\ |R_3(x,\lambda)| \leq k',\ \lambda = e^{-\pi\alpha}. \tag{4.46}$$

Recalling (4.28) one infers from (4.46) that there is $\alpha'' \geq \alpha'$ such that

$$\left|\widetilde{\eta} + (4\pi i)\,\frac{\partial_x\varphi}{\varphi}\right| \geq \frac{\pi^2}{3} - k'\lambda - 2k\lambda > 0,\ x \in \mathbf{R}, \tag{4.47}$$

concluding the proof. □

Proof of Proposition 4.1. We take (4.40) as a starting point and put it into polynomial form by multiplication with ϕ^k, some $k \in \mathbf{N}$ sufficiently large. That is there is a constructive polynomial $E_2(\zeta_0, \ldots, \zeta_4, \xi_0, \xi_2)$ such that (4.40) is rewritten as

$$E_2(\phi, \ldots, \partial^4\phi, \widetilde{g}_2, \widetilde{\eta}) = 0, \quad x, v \in \mathbf{R}, \quad (\partial = \partial_v) \tag{4.48}$$

where, with $\alpha > 0$ fixed, $\phi(x+i\alpha, v)$, $x, v \in \mathbf{R}$ is a function of x, v while $\widetilde{g}_j(x+i\alpha)$, $\widetilde{\eta}(x+i\alpha)$, $\varphi(x) = \phi(x+i\alpha, 0)$, $x \in \mathbf{R}$ are functions of x alone. We now extend (4.48) into a system of type Def. 4(a) and then verify conditions (b), (c). This extension is routine and will only be outlined. The variable x corresponds to λ in Def. 4 while the variable v corresponds to t. The vector $y(t, \lambda)$ in Def. 4 is represented by $\phi, \partial\phi, \ldots, \partial^4\phi$, while $z(\lambda)$ is represented by $\widetilde{g}_j$, $j = 2, 3$, $\widetilde{\eta}$, φ and by their derivatives $\partial_x \widetilde{g}_j, \ldots$, properly ordered. The vector $g(\lambda)$ in Def. 4 finally is represented by $\widetilde{g}_2, \widetilde{g}_3, \varphi$ and their derivatives. The system associated with $g(\lambda)$ is now given by (4.23)+(4.24), while the system associated with $z(\lambda)$ is given by (4.43), i.e. by (4.23)+(4.24)+(4.42). The system associated with $y(t, \lambda)$ is now given by (4.19)+(4.43)+(4.48), i.e. by (4.19)+(4.23)+(4.24)+(4.42)+(4.48). The constructivity condition in Def. 4(c) is again handled by the series expansions (4.4)(B) and by (4.6). Finally, an inspection shows that in view of Lemma 4.1(b) and by (4.25), (4.44), condition (b) in Def. 4 holds if we can prove

$$(\partial_{\zeta_4} E_2)(\phi, \ldots, \partial^4\phi, \widetilde{g}_2, \widetilde{\eta}) \neq 0, \quad x, v \in \mathbf{R} \text{ for } \alpha \gg 0. \tag{4.49}$$

However, due to the structure of (4.40) it follows that (4.49) is a consequence of the assertion

$$|\phi(x+i\alpha, v)| \geq \tfrac{1}{2}, \quad x, v \in \mathbf{R}, \text{ for } \alpha \gg 0, \tag{4.50}$$

which is obvious by (4.6). With all conditions in Def. 4 satisfied by our system, the admissibility of $\phi(x+i\alpha, v) = \vartheta_3(x+i\alpha, v)$, $x, v \in \mathbf{R}$ follows. □

Remarks. We mention some open problems. The first is whether one can prove Prop. 4.1 by using a variant of Def. 4, (a)–(c) which allows only to solve ODE's of type (3.6), i.e. without auxiliary parameter λ. Another question is wether the following stronger form of Prop. 3.4 can be proved:

(P*) for any recursively enumerable predicate $P(x)$ there is $\phi(\gamma)$ in $H_0(\mathscr{M}_0 \cup \{\theta_\lambda(\gamma, 0)\})$ which represents $P(x)$. If $(P*)$ holds, then the only ODE to be solved in order to obtain undecidable predicates of type (1.1) would be Jacobis ODE for $\vartheta_3(x+i\alpha, 0) = \varphi(x)$. A further question is the following. Our proof of Thm. $1'$ is based on the properties of Jacobis Theta-function $\vartheta_3(x, v)$ which (up to a minor transformation) represents the predicate $m^2 = n$. The problem is whether there are other types of functions and ODEs which lead to a representation of $m^2 = n$. We do not know of any such possibility.

Remarks on constructive aspects. Theorem 1 has a constructive counterpart which for reasons of space can be touched only briefly. Let $f(\alpha) \in \mathscr{F}$ represent a predicate $P(x)$ in the sense of Def. 1. As stressed subsequently to Prop. 2.2, it does not seem possible to construct by simple means a function $\widetilde{f}(\alpha) \in \mathscr{F}$ which represents $\neg P(x)$. This is a consequence of

Conjecture 1. If $\phi(\alpha) \in H_0(\mathscr{M}_1)$ represents the predicate $P(x)$ then $P(x)$ is recursively enumerable.

This conjecture is mentioned in [3]. If it is valid then Thm. 1 is in a certain sense optimal. At the moment, the conjecture is not yet proven, but preliminary work strongly indicates that it is valid. It is in the proof of conjecture 1 where full use of condition (b) in Def. 4 has to be made. This condition guarantees on the one hand the uniqueness of the solutions $y(t,\lambda), g(\lambda), z(\lambda)$ of the ODEs in Def. 4(a). That is if $\widetilde{y}(t,\lambda)$, $\widetilde{g}(\lambda)$, $\widetilde{z}(\lambda)$ is another triple of solutions such that $\widetilde{y}(0,\lambda) = \widetilde{g}(\lambda)$, $\widetilde{g}(a) = g(a)$, $\widetilde{z}(a) = z(a)$, then $y = \widetilde{y}$, $g = \widetilde{g}$, $z = \widetilde{z}$. This follows from classical theorems of global existence and uniqueness of solutions of ODEs (see e.g. [8]). On the other hand, Def. 4(b) also guarantees that efficient approximation methods exist for the solutions y, z, g in Def. 4(a), which can be refined so as to yield approximation methods in the sense of recursive analysis. A detailed discussion of these methods will be given elsewhere.

5. Relations to hypercomputation

We briefly digress on the relation between our considerations in sec. 2–4 and the concept of hypercomputation which has recently gained interest. Here it is not possible to discuss the various aspects of this notion; we have to focus on a single aspect. For an extended review we refer to Copeland [4] and to the vast literature cited therein; see also [5] and the many papers in [14] which we do not list individually. By "hypercomputation" we mean two different but related circles of problems, i.e.: (a) do there exist processes in nature which give rise to nonrecursive phenomena? (b) if so, can these processes be organized into an analogue computer which violates Church's thesis in one way or the other? We note that (a) does not necessarily imply (b), but that (b) presupposes (a).

Here we consider (b). In order to interpret (b) properly, we proceed to a thought experiment. We assume that we are given a configuration of n pendulums, interacting nonlinearly; concrete examples can easily be given. This configuration is described by a Hamiltonian system, which consists of $6n$ coupled ODE's denoted in abbreviated form by (i) $x_t = f(x)$, $x = (x_1, \ldots, x_{6n})$. Let $S \subseteq \mathbf{R}^{6n}$ be the phase space of (i); we assume that S is well behaved, i.e. that for any rational vector $\zeta \in \mathbf{Q}^{6N}$ it can be decided whether $\zeta \in S$ or $\zeta \notin S$.

Thus given $\zeta \in S$, (i) admits a solution $x(\zeta, t)$ defined on a maximal interval of existence $[0, T_\zeta)$, $T_\zeta \leq \infty$ such that $x(\zeta, 0) = \zeta$. Let $\mathscr{M} \subseteq S$ be defined as follows: (ii) $\zeta \in \mathscr{M}$ if (α) $\zeta \in \mathbf{Q}^{6n} \cap S$, ($\beta$) $1 < T_\zeta$, (γ) $x_1(\zeta, 1) > 0$. We assume as part of our though experiment: (iii) $\mathscr{M}$ is nonrecursive. I.e. we assume that there is no program P, which when started with the input $\zeta \in \mathbf{Q}^{6n} \cap S$, computes during a finite time τ_ζ, stops and prints "yes" if $\zeta \in \mathscr{M}$ and "no" if $\zeta \notin \mathscr{M}$. The pendulum configuration is thus a process of nature which admits (or exhibits) a nonrecursive phenomenon. But in addition one can, from the point of view of classical physics, perform experiments and measurements on it with arbitrary precision. One can fix an initial condition $\zeta \in \mathbf{Q}^{6n} \cap S$, determine the trajectory $x(\zeta, t)$ and decide by measurement if $x_1(\zeta, 1) > 0$, i.e. if $\zeta \in \mathscr{M}$. Under our assumption (iii), the pendulum configuration may thus be viewed as a (classical) analogue machine, by which undecidable propositions such as "$\zeta \in \mathscr{M}$" may be decided by measurement. In order to set Thm. 1 and the underlying theory into relation with this concept of "hypercomputation" given by (b), we refer to [15], where an abstract generalisation (termed G.P.A.C) of Bush's differential analyser is introduced. The construction of G.P.A.C is based on existing hardware, and can in principle at least be realised by a physical apparatus. Whether G.P.A.C admits nonrecursive phenomena in our sense is not discussed in [15]; thus G.P.A.C does not subsume under (b) unless proved otherwise. Nevertheless, G.P.A.C is based on a definition which is very similar to our Def. 4 of admissible functions in that both allow for the solution of algebraic ODE's as the principal tool for the construction of new functions. There is however a major difference between [15] and our setting. In [15], only functions of a single variable "x", representing time, are considered, and the ODE's to be solved act only on x. Here however we consider functions of several variables, $f(x_1, \ldots, x_s)$, and the ODE's to be solved may act on any of the x_j's.

The situation is thus as follows. In order to construct an analogue machine which simulates the operations allowed by Def. 2, 4, 5 at least one of two conditions must be satisfied. Either the hardware underlying the construction of G.P.A.C in [15] is modified such that the machine can solve successive ODE's, acting thereby on different variables. In this case the distinguished role of time "t" in analogue computing has to be overcome. Or else one has to improve the mathematical side of our analysis, leading to Thm. 1 and prove e.g. conjecture (P^*) at the end of Sect. 4, so that only a single ODE, acting on a distinguished variable, say x_1, has to be solved. To sum up, if our construction of undecidable propositions via Thm. 1 can indeed be simulated by hardware, the actual procedure is as follows. Given a recursively enumerable set $E \subseteq N$, one relies on [13] and seeks an integer polynomial $p(x, y_1, \ldots, y_n)$ $(= p(x, y))$ such that $n \in E$ iff $(\exists y)\ p(n, y) = 0$. With $p(x, y)$ at disposal, one can assemble the hardware in accordance with the prescriptions of Def. 2, 4, 5 and the proof of Thm. 1 so as to produce a function $f(x)$ related to E via (1.1).

In this connection we have to mention a thesis by Penrose ([7]) which, in a loose way, states: (Pth) significant nonrecursive phenomena do not occur in classical physics. In this form, (Pth) defies a precise mathematical analysis. But we may look at an implication which would emerge from a truth of (Pth). It would follow that no refinement of Thm. 1 would produce a structure, based on elementary functions, operations and ODE's which would on the one hand admit undecidable phenomena of the sort put forward by Thm. 1, and which on the other hand could reasonably be implemented into physical hardware. A proof of this implication does not seem to be within reach of available techniques.

For further work, whose aim is to reconcile the combinatorial nature of computation with classical analysis, we refer to [6], [7], [16], [17]; see also [20] which points into the same direction, although in a different way.

Appendix: Proof of Lemma 4.1

Lemma 4.1 is essentially a corollary to the theory of modular forms ([12], Chpt. 3). The only part calling for some attention is (b) of Lemma 4.1. In order to remain in accord up to a constant factor with [12], I.§3 we set

$$\widetilde{g}_2(\tau) = G_4(\tau), \quad \widetilde{g}_3(\tau) = G_6(\tau), \quad \tau = x + i\alpha, \quad \alpha > 0. \tag{A.1}$$

G_4, G_6 give rise to a complex vector space M_k via

$$M_k = \mathrm{span}\{G_4^\beta G_6^\gamma \,/\, 4\beta + 6\gamma = k\}. \tag{A.2}$$

There are two operations [ ] ([12], pg. 157) and # ([1], Prop. 1.1, pg. 2) which map elements of M_k into elements of M_p, some p. According to [1], Prop. 1.1 we have: if $f = f(x + i\alpha) \in M_k$, then

$$f^\# := kfD^2f - (k+1)(Df)^2 \in M_{2k+4}, \text{ where } D = (2\pi i)^{-1}\partial_x. \tag{A.3}$$

By [12], pg. 157, (5) on the other hand one has: if $f \in M_k$, $g \in M_\ell$ then

$$[f, g] := \ell(Df)g - kf(Dg) \in M_{k+\ell+2}. \tag{A.4}$$

We apply this to $f = G_4$ and note that

$$M_{12} = \mathrm{span}(G_4^3, G_6^2), \quad M_{18} = \mathrm{span}(G_4^3 G_6, G_6^3), \tag{A.5}$$

by virtue of (A.2). By (A.3), (A.5) we have

$$G_4^\# = aG_4^3 + bG_6^2 \in M_{12}, \text{ some } a, b \in \mathbf{C}. \tag{A.6}$$

The argument below simplifies if $b = 0$; we thus assume $b \neq 0$. Likewise we infer from (A.4), (A.5):

$$[G_4, G_4^\#] = cG_4^3 G_6 + dG_6^3, \text{ some } c, d \in \mathbf{C} \tag{A.7}$$

whence

$$[G_4, G_4^\#]^2 = G_6^2(cG_4^3 + dG_6^2)^2. \tag{A.8}$$

By combining (A.6), (A.8) we find a polynomial $P(\zeta_0, \zeta_1)$ such that

$$[G_4, G_4^\#]^2 - P(G_4, G_4^\#) = \mathscr{D}(G_4, \ldots, G_4^{(3)}) = 0, \tag{A.9}$$

with $\mathscr{D}(\xi_0, \ldots, \xi_3)$ a polynomial and $G_4^{(j)} = \partial_x^j G_4$. By a straightforward computation one may rewrite (A.9) in the form

$$(4^2 G_4^2 D^3 G_4 + B(G_4, DG_4, D^2 G_4))^2 - P(G_4, G_4^\#) = 0, \tag{A.10}$$

where $B(\zeta_0, \zeta_1, \zeta_2)$ is a polynomial of the form

$$B(\zeta_0, \zeta_1, \zeta_2) = \sum A(\zeta_0)\zeta_1^\delta \zeta_2^\varepsilon, \quad \varepsilon + \delta \geq 2. \tag{A.11}$$

Now half of the condition in Lemma 4.1(b) amounts to verify that $(\partial_{\xi_3}\mathscr{D})(G_4, \ldots, G_4^{(3)}) \neq 0$ for $x \in [0, 1]$, $\alpha \gg 0$.

To see this we observe using (A.10) that up to a constant factor this condition assumes the form

$$G_4^2(4G_4^2D^3G_4 + B(G_4, DG_4, D^2G_4)) \neq 0, \quad x \in [0,1] \tag{A.12}$$

for $\alpha \gg 0$. To verify (A.12) we recall the series expansion of G_4 in [9], pg. 212 according to which

$$G_4(x + i\alpha) = a_0 + a_1 q + O(q^2), \quad a_0 a_1 \neq 0, \quad q = e^{2\pi i x - 2\pi\alpha}.$$

Thus for small q resp. $\alpha \gg 0$ large we have

$$G_4(x + i\alpha) = a_0 + O(q), \quad (D^j G_4) = a_1 q + O(q^2), \quad 1 \leq j \leq 3. \tag{A.13}$$

By virtue of (A.11), (A.13) we have

$$B(G_4, DG_4, D^2G_4) = O(q^2). \tag{A.14}$$

Based on (A.13), (A.14) it then follows that there are $k > 0$, $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ implies

$$|G_4^2(4^2G_4^2D^3G_4 + B(G_4, DG_4, D^2G_4))| \geq ke^{-2\pi\alpha} > 0, \quad x \in [0,1]. \tag{A.15}$$

This proves (A.12) and hence one half of Lemma 4.1(b).

In order to conclude the proof, we note that by virtue of (A.4), (A.5) we get $[G_4, G_6] \in M_{12}$ that is

$$6(DG_4)G_6 - 4G_4(DG_6) - eG_4^3 - fG_6^2 = 0, \quad \text{some } e, f \in \mathbf{C}. \tag{A.16}$$

That is there is a polynomial $G(\xi_0, \xi_1, \zeta_0, \zeta_1)$ such that (A.16) may be rewritten as $\mathscr{G}(G_6, G_6^{(1)}, G_4, G_4^{(1)}) = 0$. By (A.15), already proved, Lemma 4.1(b) is proved if we can show:

$$(\partial_{\xi_1}\mathscr{G})(G_6, G_6^{(1)}, G_4, G_4^{(1)}) \neq 0, \quad x \in [0,1], \quad \alpha \gg 0. \tag{A.17}$$

Now a glance at (A.16) shows that (A.17) follows from

$$G_4(x + i\alpha) \neq 0, \quad x \in [0,1], \text{ for } \alpha \gg 0. \tag{A.18}$$

By (A.13), clause (A.18) evidently holds if $\alpha \geq \alpha_0$, for some $\alpha_0 > 0$. This concludes the proof of Lemma 4.1(b).

Remark. There is a constructive part in Lemma 4.1 whose proof has been omitted for reasons of space, but also because it is routine.The first part is the assertion that the polynomials $\mathscr{D}$, $\mathscr{G}$ have constructive coefficients, the second part that the initial conditions $G_4(i\alpha)$, $(\partial_x G_4)(i\alpha), \ldots$ are all constructive. The first claim follows from a comparison of coefficients based on the series expansions in [9], pg. 212, while the second follows directly from these series expansions.

Acknowledgements. We are indebted to D. Masser and V. Bosser for valuable discussions on modular and elliptic functions and to D. Bertrand for his help concerning the work in [1].

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Received 15 December 2005