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ON THE MEAN SQUARE OF THE ZETA-FUNCTION AND THE DIVISOR PROBLEM

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Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac12+it)|$. If $E^*(t)=E(t)-2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x)=-\Delta(x)+2\Delta(2x)-\frac12\Delta(4x)$, then we obtain the asymptotic formula

$$\int_0^T (E^*(t))^2\,dt \;=\; T^{4/3}P_3(\log T)+O_\varepsilon(T^{7/6+\varepsilon}),$$

where P_3 is a polynomial of degree three in $\log T$ with positive leading coefficient. The exponent $7/6$ in the error term is the limit of the method.

1. Introduction

As usual, let the error term in the classical Dirichlet divisor problem be

$$\Delta(x)=\sum_{n\le x}d(n)-x(\log x+2\gamma-1), \tag{1.1}$$

and

$$E(T)=\int_0^T|\zeta(\tfrac12+it)|^2\,dt-T\left(\log\big(\frac{T}{2\pi}\big)+2\gamma-1\right), \tag{1.2}$$

where $d(n)$ is the number of divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma=-\Gamma'(1)=0.577215\ldots$ is Euler's constant. In view of Atkinson's classical explicit formula for $E(T)$ (see [1] and [2, Chapter 15]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see Jutila [5], [6] and Meurman [9]), where

$$\Delta^*(x):=-\Delta(x)+2\Delta(2x)-\tfrac12\Delta(4x)=\tfrac12\sum_{n\le 4x}(-1)^nd(n)-x(\log x+2\gamma-1), \tag{1.3}$$

which is a better analogue of $E(T)$ than $\Delta(x)$. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t):=E(t)-2\pi\Delta^*\big(\frac{t}{2\pi}\big),$$

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and in particular in [6] he proved that

$$\int_0^T (E^*(t))^2\,dt \ll T^{4/3}\log^3 T. \tag{1.4}$$

On the other hand, we have the asymptotic formulas

$$\begin{gathered}\int_0^x \Delta^2(y)\,dy = Ax^{3/2} + F(x),\\ A = \frac{1}{6\pi^2}\sum_{n=1}^\infty d^2(n)n^{-3/2},\quad F(x) \ll x\log^4 x,\end{gathered} \tag{1.5}$$

and

$$\begin{gathered}\int_0^T E^2(t)\,dt = BT^{3/2} + R(T),\\ B = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^\infty d^2(n)n^{-3/2},\quad R(T) \ll T\log^4 T,\end{gathered} \tag{1.6}$$

and an analogous formula to (1.5) holds if $\Delta(x)$ is replaced by $\Delta^*(x)$, since Lemma 2 below is the analogue of the truncated Voronoï formula for $\Delta(x)$, and a full Voronoï formula for $\Delta^*(x)$ follows from the corresponding Voronoï formula for $\Delta(x)$ (see e.g., [2, Chapter 3]) and (1.3). Note that we have

$$\sum_{n=1}^\infty d^2(n)n^{-3/2} = \frac{\zeta^4(3/2)}{\zeta(2)}.$$

The bounds in (1.5) and (1.6) are at present the sharpest ones known, and the history is given in [3, Chapter 2]. The bounds with $\log^5$ instead of $\log^4$ were independently obtained by Motohashi and Meurman [10]. The actual bound in (1.5) is due to Preissmann [11], and the bound in (1.6) was noticed independently by Preissmann (loc. cit.) and the author [3]. This follows by using a variant of the so-called Hilbert's inequality (see [3], eq. (2.101)]). Further results on $F(x)$ were obtained by Lau and Tsang (see [7] and [13]). In particular, in [13] Tsang has shown that, for almost all x and a suitable constant κ,

$$F(x) \;=\; -\frac{1}{4\pi^2}x\log^2 x + \kappa x\log x + O(x), \tag{1.7}$$

and he conjectured that (1.7) holds unconditionally. Thus (1.5) seems to be close to best possible, and very likely the same holds for (1.6).

For an extensive discussion of $E(T)$ see the author's monographs [2], [3]. The significance of (1.4) is that it shows that $E^*(t)$ is in the mean square sense much smaller than either $E(t)$ or $\Delta^*(t)$. It is expected that $E(t)$ and $\Delta^*(t)$ are 'close' to one another in order, but this has never been satisfactorily established, although Jutila [5] obtained significant results in this direction. It is also conjectured that $E^*(T) \ll_\varepsilon T^{1/4+\varepsilon}, \Delta^*(x) \ll_\varepsilon x^{1/4+\varepsilon}$, the latter being a consequence of the classical conjecture $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$ in the Dirichlet divisor problem. In fact, Lau and Tsang [8] recently proved that $\alpha = \alpha^*$, where

$$\alpha = \inf\Big\{a \ : \ \Delta(x) \ll x^a\Big\}, \quad \alpha^* = \inf\Big\{a^* \ : \ \Delta^*(x) \ll x^{a*}\Big\}.$$

In the first part of the author's work [4] the bound in (1.4) was complemented with the new bound

$$\int_0^T (E^*(t))^4\,dt \ \ll_\varepsilon \ T^{16/9+\varepsilon}; \tag{1.8}$$

neither (1.4) or (1.8) seem to imply each other. Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ means that the $\ll$-constant depends on ε. In the second part of the same work (op. cit.) it was proved that

$$\int_0^T |E^*(t)|^5\,dt \ \ll_\varepsilon \ T^{2+\varepsilon}, \tag{1.9}$$

and some further results on higher moments of $|E^*(t)|$ were obtained as well. The aim of this note is to show that the bound in (1.4) is of the correct order of magnitude. The result is the asymptotic formula, contained in

Theorem 1. *We have*

$$\int_0^T (E^*(t))^2\,dt \ = \ T^{4/3}P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{1.10}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be evaluated explicitly.

From (1.10) and Hölder's inequality for integrals we obtain the following

Corollary 1. *If $A \ge 2$ is fixed, then*

$$\int_0^T |E^*(t)|^A\,dt \ \gg \ T^{1+A/6}(\log T)^{3A/2}. \tag{1.11}$$

Note that there is a discrepancy between the bound in (1.11) (for $A = 4$ and $A = 5$), and the upper bounds in (1.8) and (1.9). The true order of the integrals in (1.8) and (1.9) is difficult to determine. If, as usual, $f(x) = \Omega(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) \ne 0$, then from Theorem 1 we immediately obtain

Corollary 2. *We have*

$$E(T) = 2\pi\Delta^*\Big(\frac{T}{2\pi}\Big) + \Omega(T^{1/6}\log^{3/2} T). \tag{1.12}$$

Thus equation (1.12) shows that $E(T)$ and $\Delta^*\big(\frac{T}{2\pi}\big)$ cannot be too 'close' to one another in order. A reasonable conjecture, compatible with the classical conjectures $E^*(T) \ll_\varepsilon T^{1/4+\varepsilon}$, $\Delta^*(x) \ll_\varepsilon x^{1/4+\varepsilon}$, is that one has

$$E(T) = 2\pi\Delta^*\Big(\frac{T}{2\pi}\Big) + O_\varepsilon(T^{1/4+\varepsilon}). \tag{1.13}$$

2. The necessary lemmas

In this section we shall state three lemmas which are necessary for the proof of Theorem 1. The first two are Atkinson's classical explicit formula for $E(T)$ (see [1] or e.g., [2] or [3]) and the Voronoï-type formula for $\Delta^*(x)$, which is the analogue of the classical truncated Voronoï formula for $\Delta(x)$. The third is an asymptotic formula involving $d^2(n)$.

Lemma 1. *Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then*

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{2.1}$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4} \sum_{n\le N} (-1)^n d(n) n^{-3/4} e(T,n) \cos(f(T,n)), \tag{2.2}$$

$$\Sigma_2(T) = -2 \sum_{n\le N'} d(n) n^{-1/2} (\log T/(2\pi n))^{-1} \cos\left(T \log\Big(\frac{T}{2\pi n}\Big) - T + \tfrac14\pi\right), \tag{2.3}$$

with

$$\begin{aligned} f(T,n) &= 2T \operatorname{arsinh}\big(\sqrt{\pi n/(2T)}\,\big) + \sqrt{2\pi n T + \pi^2 n^2} - \tfrac14\pi \\ &= -\tfrac14\pi + 2\sqrt{2\pi n T} + \tfrac16\sqrt{2\pi^3} n^{3/2} T^{-1/2} \\ &\quad + a_5 n^{5/2} T^{-3/2} + a_7 n^{7/2} T^{-5/2} + \dots , \end{aligned} \tag{2.4}$$

$$\begin{aligned} e(T,n) &= (1 + \pi n/(2T))^{-1/4} \Big\{ (2T/\pi n)^{1/2} \operatorname{arsinh}\Big(\sqrt{\pi n/(2T)}\,\Big)\Big\}^{-1} \\ &= 1 + O(n/T) \qquad (1 \le n < T), \end{aligned} \tag{2.5}$$

and $\operatorname{arsinh} x = \log(x + \sqrt{1+x^2}\,)$.

Lemma 2. [2, Chapter 15] *We have, for $1 \ll N \ll x$,*

$$\Delta^*(x) = \frac{1}{\pi\sqrt2} x^{\frac14} \sum_{n\le N} (-1)^n d(n) n^{-\frac34} \cos(4\pi\sqrt{nx} - \tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon} N^{-\frac12}). \tag{2.6}$$

Lemma 3. *For $a > -\frac12$ a constant we have*

$$\sum_{n\le x} d^2(n) n^a = x^{a+1} P_3(\log x; a) + O_\varepsilon(x^{a+1/2+\varepsilon}), \tag{2.7}$$

where $P_3(y;a)$ is a polynomial of degree three in y whose coefficients depend on a, and whose leading coefficient equals $1/(\pi^2(a+1))$. All the coefficients of $P_3(y;a)$ may be explicitly evaluated.

Proof. One can obtain (2.7) by partial summation from [2, eq. (14.30)], but a direct proof seems in order. From the Perron inversion formula (see e.g., the

Appendix of [2]) we have, for $1 \ll T \ll x$,

$$\sum_{n\le x} d^2(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{\zeta^4(s)}{\zeta(2s)s} x^s\,ds + O_\varepsilon(x^{1+\varepsilon}T^{-1}), \tag{2.8}$$

since $d^2(n)$ is generated by $\zeta^4(s)/\zeta(2s)$ (op. cit., eq. (1.106)). The segment of integration in (2.8) is replaced by segments joining the points $1+\varepsilon \pm iT, \frac{1}{2}+\varepsilon\pm iT$. There is a pole of the integrand at $s=1$ of degree four, making a contribution of $xP_3(\log x)$ by the residue theorem, where $P_3(y)$ is a generic polynomial of degree three in y. The leading coefficient of P_3 is found to be $1/\pi^2$ (see [2, eq. (5.24)]). From $\zeta(\frac{1}{2}+it) \ll t^{1/6}$ the contribution of the vertical segments $[\frac{1}{2}+\varepsilon\pm iT, 1+\varepsilon\pm iT]$ is found to be $\ll_\varepsilon x^{1+\varepsilon}T^{-1} + x^{1/2+\varepsilon}T^{-1/3}$. By using the elementary bound

$$\int_0^T |\zeta(\tfrac{1}{2}+\varepsilon+it)|^4\,dt \ll_\varepsilon T$$

for the integral over the segment joining the points $\frac{1}{2}+\varepsilon\pm iT$, one arrives at

$$\sum_{n\le x} d^2(n) = xP_3(\log x) + O_\varepsilon(x^{1/2+\varepsilon}) \tag{2.9}$$

on taking $T=\sqrt{x}$ in (2.8), from which (2.7) follows by partial summation. Note that we cannot rule out the existence of zeros of $\zeta(s)$ in the region $\frac{1}{2} < \Re\mathrm{e}\, s < 1$. Therefore the error terms in (2.9) and (2.7) are essentially the best ones that we can obtain without assuming some hypotheses on the zeros of $\zeta(s)$, like e.g., the famous Riemann Hypothesis (that all complex zeros of $\zeta(s)$ are on $\Re\mathrm{e}\, s = \frac{1}{2}$). □

3. The proof of Theorem 1

It is sufficient to prove the formula (1.10) with the integral over $[T, 2T]$ and then to replace T by $T2^{-j}$ and to sum all the results over $j=1,2,\dots$. We use Lemma 1 and Lemma 2 with $N=T$ to deduce that, for $T\le t\le 2T$,

$$E^*(t) = S_1(t)+S_2(t)+S_3(t), \tag{3.1}$$

where

$$\begin{aligned} S_1(t) &= \sqrt{2}\left(\frac{t}{2\pi}\right)^{1/4}\sum_{n\le T}(-1)^n d(n)n^{-3/4}\Bigg\{e(t,n)\cos f(t,n) \\ &\qquad - \cos(\sqrt{8\pi nt}-\tfrac{1}{4}\pi)\Bigg\}\,dt \\ S_2(t) &= -2\sum_{n\le c_1T} d(n)n^{-1/2}\left(\log\frac{t}{2\pi n}\right)^{-1}\cos\left(t\log\frac{t}{2\pi n}-t+\frac{\pi}{4}\right), \\ S_3(t) &= O_\varepsilon(T^\varepsilon), \end{aligned} \tag{3.2}$$

where c_1 (>0) is to be found from the definition of N' in Lemma 1. We have

$$\int_T^{2T} S_1^2(t)\,dt \ll T^{4/3}\log^3 T, \qquad \int_T^{2T} \{S_2^2(t)+S_3^2(t)\}\,dt \ll_\varepsilon T^{1+\varepsilon}. \tag{3.3}$$

The first bound is in fact embodied in the proof of (1.4), since it is $S_1(t)$ that makes the major contribution to the integral in question. This can be seen if one follows the proof of (1.4) in [6] or in [2], and the same holds for the estimate in (3.3) containing $S_2(t)$, while the estimate containing $S_3(t)$ is trivial. From (3.1)–(3.3) and the Cauchy–Schwarz inequality for integrals it transpires that

$$\int_T^{2T} (E^*(t))^2\,dt = \int_T^{2T} S_1^2(t)\,dt + O_\varepsilon(T^{7/6+\varepsilon}). \tag{3.4}$$

By using the method of Meurman [10] the error term in (3.4) could be improved to $O_\varepsilon(T^{1+\varepsilon})$, but it is the exponent $7/6$ in (3.11) that cannot be improved at present. Squaring out $S_1(t)$ it is seen that the integral on the right-hand side of (3.4) equals

$$\begin{aligned}&\sqrt{\frac{2}{\pi}}\sum_{n\le T} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}\Big\{e(t,n)\cos f(t,n) - \cos(\sqrt{8\pi nt}-\tfrac14\pi)\Big\}^2\,dt\\ &+\sqrt{\frac{2}{\pi}}\sum_{m\ne n\le T}(-1)^{m+n}d(m)d(n)(mn)^{-3/4}\int_T^{2T} t^{1/2}\cdots,\end{aligned} \tag{3.5}$$

where the main contribution comes from the 'diagonal' terms $m=n$, while the terms with $m\ne n$ will contribute to the error term in (3.4). The terms in the first sum in (3.5) for which $n>T^{2/3}$ trivially contribute $\ll_\varepsilon T^{7/6+\varepsilon}$. The contribution of the second sum in (3.5) (with $m\ne n$) is $\ll_\varepsilon T^{1+\varepsilon}$. This follows e.g., by the method of [2, Chapter 15] or [3, Chapter 2] for the mean square of $E(t)$. Therefore (3.5) leads to

$$\begin{aligned}\int_T^{2T} (E^*(t))^2\,dt = \sqrt{\frac{2}{\pi}}\sum_{n\le T^{2/3}} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}\Big\{&e(t,n)\cos f(t,n)\\ &-\cos(\sqrt{8\pi nt}-\tfrac14\pi)\Big\}^2\,dt + O_\varepsilon(T^{7/6+\varepsilon}),\end{aligned} \tag{3.6}$$

and furthermore we use (see (2.5)) $e(t,n)=1+O(n/T)$ in (3.6) to see that we may replace $e(t,n)$ by unity, with the ensuing error term absorbed by the error term in (3.6). To manage the cosines in (3.6) we use the elementary identity

$$(\cos\alpha-\cos\beta)^2 = 1-\cos(\alpha-\beta)+\{\tfrac12\cos 2\alpha+\tfrac12\cos 2\beta-\cos(\alpha+\beta)\}$$

with $\alpha=f(t,n), \beta=\sqrt{8\pi nt}-\frac14\pi$. By the first derivative test (cf. [2, Lemma 2.1]) it is seen that the terms coming from curly braces contribute $\ll T$ to (3.6). Likewise the values of n in $\cos(\alpha-\beta)$ for which $T^{1/2} < n \le T^{2/3}$ contribute

$$\ll \sum_{n>T^{1/2}} d^2(n)n^{-3/2}T^{1/2}T^{3/2}n^{-3/2} = \sum_{n>T^{1/2}} d^2(n)n^{-3}T^2 \ll T\log^3 T.$$

For the terms $n \le T^{1/2}$ in $\cos(\alpha-\beta)$ we use the series expansion

$$\cos(\alpha-\beta) = \cos(x_0+h) = \cos x_0 + \sum_{m=1}^{\infty} \frac{h^m}{m!}\cos(x_0 + \tfrac{1}{2}\pi m) \tag{3.7}$$

with ($a_1 = \frac{1}{6}\sqrt{2\pi^3}$, see (2.4))

$$x_0 = a_1 n^{3/2} t^{-1/2},\ h = a_3 n^{5/2} t^{-3/2} + a_5 n^{7/2} t^{-5/2} + \dots .$$

We truncate the series over m so that the tail, by trivial estimation, makes a total contribution which is $\ll T$. This is possible because

$$h \ll n^{5/2}T^{-3/2} \ll T^{-1/4} \qquad (T \le t \le 2T,\ 1 \le n \le T^{1/2}).$$

The contribution of the finitely many remaining terms in the sum over m in (3.7) is estimated by the first derivative test, and it is

$$\ll \sum_{n\le T^{1/2}} d^2(n)n^{-3/2}T^{1/2}n^{5/2}T^{-3/2}n^{-3/2}T^{3/2} \ll T^{3/4}\log^3 T.$$

Finally it follows from (3.6) that we obtain, since $1-\cos\gamma = 2\sin^2(\frac{1}{2}\gamma)$,

$$\int_T^{2T} (E^*(t))^2\,dt = \sqrt{\frac{8}{\pi}} \sum_{n\le T^{2/3}} d^2(n)n^{-3/2} \int_T^{2T} t^{1/2}\sin^2(\tfrac{1}{2}a_1 n^{3/2}t^{-1/2})\,dt + O_\varepsilon(T^{7/6+\varepsilon}). \tag{3.8}$$

If we use the bound

$$\sin^2(\tfrac{1}{2}a_1 n^{3/2}t^{-1/2}) \ \ll\ n^3 T^{-1} \qquad (T \le t \le 2T),$$

then the contribution of the terms with $n < T^{4/15}$ in (3.8) is seen to be

$$\ll \sum_{n\le T^{4/15}} d^2(n)n^{-3/2}n^3T^{-1}\int_T^{2T} t^{1/2}\,dt \ll T^{1/2}\sum_{n\le T^{4/15}} d^2(n)n^{3/2}$$
$$\ll T^{1/2}(T^{4/15})^{5/2}\log^3 T = T^{7/6}\log^3 T.$$

Hence (3.8) reduces to

$$\int_T^{2T} (E^*(t))^2\,dt = \sqrt{\frac{8}{\pi}} \sum_{T^{4/15}\le n\le T^{2/3}} d^2(n)n^{-3/2} \int_T^{2T} t^{1/2}\sin^2(\tfrac{1}{2}a_1 n^{3/2}t^{-1/2})\,dt + O_\varepsilon(T^{7/6+\varepsilon}). \tag{3.9}$$

In the integral above we make the change of variable

$$\tfrac{1}{2}a_1 n^{3/2}t^{-1/2} = y, \quad dt = -\tfrac{1}{2}a_1^2 n^3 y^{-3}\,dy.$$

We set $z = (4(y/a_1)^2T)^{1/3}$ so that, after changing the order of integration and summation, the main term on the right-hand side of (3.9) becomes

$$
\begin{aligned}
&\sqrt{\frac{2}{\pi}}\sum_{T^{4/15}\le n\le T^{2/3}} d^2(n)n^{-3/2}\int_{\frac12 a_1n^{3/2}(2T)^{-1/2}}^{\frac12 a_1n^{3/2}T^{-1/2}}\frac{a_1n^{3/2}}{y}\cdot\frac{a_1^2n^3}{2y^3}\cdot\sin^2 y\,dy \\
&= \sqrt{\frac{1}{2\pi}}a_1^3\int_{2^{-3/2}a_1T^{-1/10}}^{\frac12 a_1T^{1/2}}\sum_{\max(T^{4/15},z)\le n\le\min(T^{2/3},2^{1/3}z)} d^2(n)n^3\cdot\frac{\sin^2 y}{y^4}\,dy.
\end{aligned}
\tag{3.10}
$$

The range of summation for n is the interval $[z,\,2^{1/3}z]$ if

$$
y\,\in\,J,\quad J := \left[\frac12 a_1T^{-1/10},\,\frac{1}{2\sqrt2}a_1T^{1/2}\right].
$$

If we replace the interval of integration in the second integral in (3.10) by J, then by using $|\sin y|\le\min(1,|y|)$ it follows that the error that is made is $\ll_\varepsilon T^{7/6+\varepsilon}$. Now we use Lemma 3 with $a=3$ to obtain that the integral over J equals, with suitable constants d_j,

$$
\begin{aligned}
&\int_{\frac12 a_1T^{-1/10}}^{\frac{1}{2\sqrt2}a_1T^{1/2}}\left(T^{4/3}y^{8/3}\sum_{j=0}^{3}d_j\log^j(y^{2/3}T^{1/3}) + O_\varepsilon(T^{7/6+\varepsilon}y^{7/3})\right)\frac{\sin^2 y}{y^4}\,dy \\
&= T^{4/3}\int_{\frac12 a_1T^{-1/10}}^{\frac{1}{2\sqrt2}a_1T^{1/2}}\left(\sum_{j=0}^{3}d_j3^{-j}\log^j(y^2T)\right)\frac{\sin^2 y}{y^{4/3}}\,dy + O_\varepsilon(T^{7/6+\varepsilon}).
\end{aligned}
\tag{3.11}
$$

Replacing the segment of integration in the integral on the right-hand side of (3.11) by $(0,\,\infty)$, we make an error which is $\ll_\varepsilon T^{7/6+\varepsilon}$. Namely, for $0<\alpha<1<\beta,\ j = 0,1,\ldots$ fixed, we have

$$
\int_\alpha^\beta\frac{\sin^2 y}{y^{4/3}}\log^j\,dy = \int_0^\infty\frac{\sin^2 y}{y^{4/3}}\log^j\,dy + O(\alpha^{5/3}) + O(\beta^{-1/3}\log^j\beta),
\tag{3.12}
$$

where we used again $|\sin y|\le\min(1,|y|)$. Hence the expression in (3.11) becomes, on using (3.12) with $\alpha = \frac12 a_1T^{-1/10}$, $\beta = 2^{-3/2}a_1T^{1/2}$,

$$
T^{4/3}\sum_{j=0}^{3}b_j\log^j T + O_\varepsilon(T^{7/6+\varepsilon})
$$

with some constants $b_j\,(b_3>0)$ which may be explicitly evaluated, and (1.10) follows from (3.9) and (3.11). With some more effort one could improve the error term in (1.10) to $O(T^{7/6}\log^C T)$, by using the method of Meurman [10] (see also [3, Chapter 2]) and improving the error term in Lemma 3 (see Ramachandra and Sankaranarayanan [12]) to $O(x^{a+1/2}\log^5 x\log\log x)$, but the exponent 7/6 is the limit of the method.

References

[1] Atkinson, F. V.: The mean value of the Riemann zeta-function. - Acta Math. 81, 1949, 353–376.

[2] Ivić, A.: The Riemann zeta-function. - John Wiley & Sons, New York, 1985, 2nd edition, Dover, Mineola, New York, 2003.

[3] Ivić, A.: The mean values of the Riemann zeta-function. - LNs 82, Tata Inst. of Fundamental Research, Bombay, distributed by Springer Verlag, Berlin etc., 1991.

[4] Ivić, A.: On the Riemann zeta-function and the divisor problem. - Cent. Eur. J. Math. 2:4, 2004, 1–15, and II, ibid. 3:2, 2005, 203–214.

[5] Jutila, M.: Riemann's zeta-function and the divisor problem. - Ark. Mat. 21, 1983, 75–96, and II, ibid. 31, 1993, 61–70.

[6] Jutila, M.: On a formula of Atkinson. - In: Colloq. Math. Soc. János Bolyai 34, Topics in classical Number Theory, Budapest, 1981, North-Holland, Amsterdam, 1984, 807–823.

[7] Lau, Y.-K., and K.-M. Tsang: Mean square of the remainder term in the Dirichlet divisor problem. - J. Théor. Nombres Bordeaux 7, 1995, 75–92.

[8] Lau, Y.-K., and K.-M. Tsang: Omega result for the mean square of the Riemann zeta-function. - Manuscripta Math. 117, 2005, 373–381.

[9] Meurman, T.: A generalization of Atkinson's formula to L-functions. - Acta Arith. 47, 1986, 351–370.

[10] Meurman, T.: On the mean square of the Riemann zeta-function. - Quart. J. Math. Oxford Ser. (2) 38, 1987, 337–343.

[11] Preissmann, E.: Sur la moyenne quadratique du terme de reste du problème du cercle. - C. R. Acad. Sci. Paris Sér. I Math. 306, 1988, 151–154.

[12] Ramachandra, K., and A. Sankaranarayanan: On an asymptotic formula of Srinivasa Ramanujan. - Acta Arith. 109, 2003, 349–357.

[13] Tsang, K.-M.: Mean square of the remainder term in the Dirichlet divisor problem II. - Acta Arith. 71, 1995, 279-299.

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