Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 32, 2007, 179–198

ON THE SHAPE OF BERS–MASKIT SLICES

Yohei Komori and Jouni Parkkonen

Osaka City University, Department of Mathematics
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585 Japan; komori@sci.osaka-cu.ac.jp

University of Jyväskylä, Department of Mathematics and Statistics
P.O. Box 35, 40014 University of Jyväskylä, Finland; parkkone@maths.jyu.fi

Abstract. We consider complex one-dimensional Bers–Maskit slices through the deformation space of quasifuchsian groups which uniformize a pair of punctured tori. In these slices, the pleating locus on one of the components of the convex hull boundary of the quotient three-manifold has constant rational pleating and constant hyperbolic length. We show that the boundary of such a slice is a Jordan curve which is cusped at a countable dense set of points. We will also show that the slices are not vertically convex, proving the phenomenon observed numerically by Epstein, Marden and Markovic.

1. Introduction

In recent years, there has been considerable interest in the topology of the deformation spaces of Kleinian groups. In particular, the space of punctured torus groups $\mathscr{D}$, being the most basic example of such spaces, has been widely studied, and great progress has been made. Here $\mathscr{D} = \mathscr{D}(\pi_1 S)$ the set of $\mathrm{PSL}(2, \mathbf{C})$-conjugacy classes of discrete and faithful type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representations of the fundamental group $\pi_1 S$ of a once-punctured torus S. The space of punctured torus groups contains the classical and well-studied subspace $\mathscr{QF}$, called the quasifuchsian space of punctured tori, the set of $\rho \in \mathscr{D}$ for which $G_\rho = \rho(\pi_1 S)$ is a quasifuchsian group whose conformal boundary uniformizes a pair of punctured tori. By Bers' simultaneous uniformization theorem [14], $\mathscr{QF}$ is biholomorphic to $\mathbf{H} \times \mathbf{H}$, hence we know the topology of $\mathscr{QF}$ very well. Here we use the standard identification of the Teichmüller space $T(S)$ of punctured tori with the upper half plane $\mathbf{H}$.

The identification of $\mathscr{QF}$ with $\mathbf{H} \times \mathbf{H}$ can be extended to the end invariant map

$$\nu\colon \mathscr{D} \to \overline{\mathbf{H}} \times \overline{\mathbf{H}} \setminus \Delta \tag{1.1}$$

where $\overline{\mathbf{H}} = \mathbf{H} \cup \overline{\mathbf{R}}$, $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, and $\Delta = \{(r, r) : r \in \overline{\mathbf{R}}\}$ is the boundary diagonal. The map ν associates to each representation $\rho \in \mathscr{D}$ a pair of end invariants (ν_+, ν_-) which describe the asymptotic geometry of the two noncompact ends of the truncated manifold obtained from $\mathbf{H}^3/G_\rho$ by removing a neighbourhood of the main

2000 Mathematics Subject Classification: Primary 30F40, 30F60, 57M50.

Key words: Kleinian groups, punctured torus groups, Teichmüller space, pleating coordinates, end invariants.

cusp of the associated hyperbolic 3-manifold $\mathbf{H}^3/G_\rho$. See [17] and the references therein for more details.

Minsky [17] proved that the map ν is bijective, but not continuous, while the inverse map ν^{-1} is continuous. Thus, the topology of $\mathscr{D}$, in particular that of the boundary of $\mathscr{D}$, seems to be highly non-trivial. On the other hand, Minsky also showed that the boundaries of the Bers slices and the Maskit slices, which are holomorphic slices of $\mathscr{D}$, are Jordan curves, which implies that the total space $\mathscr{D}$ might be complicated, but if we restrict to its holomorphic sections, they are not so wild. For more details on the topology of deformation spaces see e.g. [1], [2], [4], [10], [15].

In this paper we consider another holomorphic slice $\mathscr{BM}_c$ called the Bers–Maskit slice defined as follows: For $\rho \in \mathscr{QF}$, the boundary of the hyperbolic convex hull $\partial\mathscr{C}_\rho$ of the limit set $\Lambda(G_\rho)$ in $\mathbf{H}^3$ consists of two components $\partial\mathscr{C}_\rho^\pm$ facing the ordinary set $\Omega(G_\rho)^\pm$. The two boundary components $\partial\mathscr{C}^\pm/G_\rho$ are pleated surfaces whose pleating loci we denote by $pl^\pm(\rho)$. Let α be a free generator of $\pi_1 S$ and suppose that α is the bending locus of $\partial\mathscr{C}^\pm/G_\rho$. We denote the hyperbolic length of $pl^\pm(\rho) = \alpha$ in $\partial\mathscr{C}_\rho^\pm/G_\rho$ by $l_\alpha(\partial\mathscr{C}^\pm/G_\rho)$. Then we define

$$\mathscr{BM}_c^\pm = \{\rho \in \mathscr{QF} : pl^\pm(\rho) = \alpha \text{ and } l_\alpha(\partial\mathscr{C}^\pm/G_\rho) = c\}.$$

In practice, $\mathscr{BM}_c$ consists of the closure of $\mathscr{BM}_c^\pm$ in $\mathscr{QF}$. Figure 1 shows a computer-generated image of one such slice, $\mathscr{BM}_c$ is the region bounded by the two cusped curves, extended periodically to an infinite strip. The real line corresponds to Fuchsian groups, $\mathscr{BM}_c^-$ is the part of $\mathscr{BM}_c$ contained in the upper half plane.

The Bers–Maskit slices have been investigated by Keen and Series [7] and McMullen [15] to prove the existence of pleating coordinates for $\mathscr{QF}$ and for (limit) Bers slices for punctured tori. Epstein, Marden and Markovic [6] also considered these slices to give a counter-example for the equivariant $K = 2$ conjecture.

We will show that

(i) The boundary of the Bers–Maskit slice is the disjoint union of two Jordan arcs.

(ii) At the p/q-cusp boundary point of the Bers–Maskit slice, the complex length function of the p/q-word is conformal.

(iii) Any cusp boundary point of the Bers–Maskit slice is an inward-pointing cusp.

(iv) The Maskit slice and the Bers–Maskit slice are not vertically convex.

For the case of the Maskit slice, (i) was proved by Minsky [17], (ii) was shown by Miyachi [19] and Parkkonen [23], and (iii) was proved by Miyachi [18, 19]. The fourth claim was observed numerically for the Maskit slice in [25] and for the Bers–Maskit slices in [6].

We follow the idea of Miyachi [18, 19] to show our results: After preparing basic notions in section 2, we prove (iii) assuming (i) and (ii) in section 3. We prove (i) in section 4 and (ii) in section 5. As a corollary of the third claim, we prove (iv) in section 6.

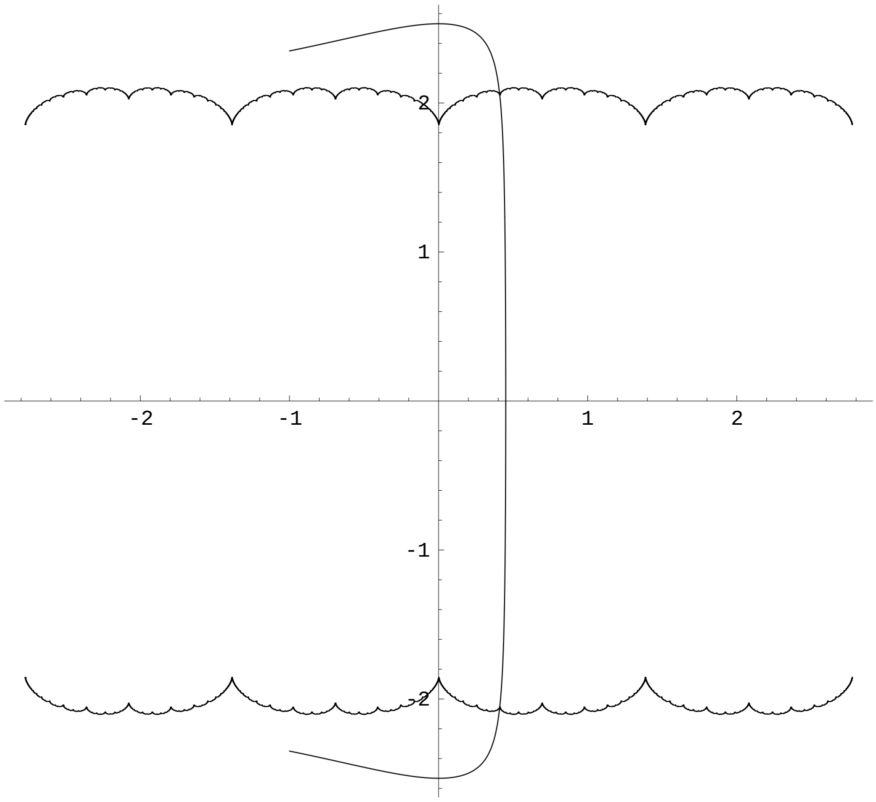

Figure 1. The Bers–Maskit slice $\mathscr{BM}_c$ for $c = 2\,\mathrm{arcosh}(5/4)$ ($\mathrm{tr}\,B = 5/2$) and the real locus of the trace of the word $W_{1/3}$.

Acknowledgments. We would like to thank Hideki Miyachi, Caroline Series and David Epstein for helpful discussions. In particular, the idea to consider the action of the Dehn twist on $\overline{\nu(\mathscr{BM}_c^-)}$ in the last part of the proof of Theorem 3.2 is due to Epstein.

The work presented in this article was started at the University of Warwick workshop "Kleinian Groups and Hyperbolic 3-manifolds". After collaborations at Osaka City University and University of Jyväskylä, it was completed at Centre Interfacultaire Bernoulli, EPFL, Lausanne during the research program "Spaces of Negative Curvature". The authors were supported by the Center of Excellence "Geometric analysis and mathematical physics" of the Academy of Finland, the Ministry of Education of Japan, and EPFL.

Figures 1 and 2 were produced with `Mathematica` using data produced with the program `kleinian` of D. J. Wright. Figure 3 was produced with the program `lim` of C. T. McMullen, and modified using `Adobe Illustrator`. We would like to thank Ari Lehtonen for his help with this figure.

2. Background and definitions

In this section we define and review briefly the basic objects which are treated in this paper. We refer to [13, 14] for the basic definitions and the theory of Kleinian groups, to [9, 15, 20] for more details on the Bers–Maskit slices, to [17] for a treatment of end invariants and punctured torus groups, and to [7] for a more extensive treatment of the words W_r, $r \in \mathbf{Q}$ which are defined in this section.

2.1. The space of punctured torus groups. Let S be a once-punctured torus, and let $\pi_1 S$ be the fundamental group of S. A representation of $\pi_1 S$ into $\mathrm{PSL}(2, \mathbf{C})$ is called *type-preserving* if the image of the commutator of (any) free generators of $\pi_1 S$ is parabolic. Let $\mathscr{R} = \mathscr{R}(\pi_1 S)$ denote the space of type-preserving representations of $\pi_1 S$ into $\mathrm{PSL}(2, \mathbf{C})$ up to conjugation by Möbius transformations. Note that we do not require that the representations in $\mathscr{R}$ are faithful (i.e. injective). Denote by $\mathscr{D} = \mathscr{D}(\pi_1 S)$ the subset of $\mathscr{R}$ which consists of discrete and faithful representations, and by $\mathscr{QF}$ the set of $[\rho] \in \mathscr{D}$ for which $\rho(\pi_1 S)$ is a quasifuchsian group which uniformizes a pair of punctured tori. It is well known that $\mathscr{D}$ is closed and $\mathscr{QF}$ is open in $\mathscr{R}$, see e.g. [14].

An element $[\rho] \in \mathscr{D}$ and the corresponding Kleinian group $G_\rho = \rho(\pi_1 S)$ are both referred to as *a punctured torus group*. In order to simplify notation, we will write $\rho \in \mathscr{D}$ instead of $[\rho] \in \mathscr{D}$. To each punctured torus group ρ we associate the pair of end invariants (ν_+, ν_-) which describe the asymptotic geometry of the two noncompact ends of the associated hyperbolic 3-manifold $\mathbf{H}^3/G_\rho$ as follows: If $\rho \in \mathscr{QF}$, the end invariants are the Teichmüller parameters of the pair of punctured tori which correspond to ρ. Using the standard identification of the Teichmüller space $T(S)$ of punctured tori with $\mathbf{H}$, the pair of end invariants in this case is a point in $\mathbf{H} \times \mathbf{H}$. In general, we have a map

$$\nu \colon \mathscr{D} \to \overline{\mathbf{H}} \times \overline{\mathbf{H}} \setminus \Delta \tag{2.1}$$

where $\mathbf{H}$ is the upper half plane, $\overline{\mathbf{H}} = \mathbf{H} \cup \overline{\mathbf{R}}$, $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ and Δ is the boundary diagonal $\{(r, r) : r \in \overline{\mathbf{R}}\}$. The restriction of ν to $\mathscr{QF}$ is homeomorphic onto $\mathbf{H} \times \mathbf{H}$: ν_+ is holomorphic whereas ν_- is anti-holomorphic on $\mathscr{QF}$. Minsky [17] proved that the map ν is bijective, but not continuous, (see [1, 17]). On the other hand the inverse map ν^{-1} is continuous. By means of this, he showed that $\mathscr{D}$ is the closure of $\mathscr{QF}$ in $\mathscr{R}$, which gives a positive answer to a conjecture of Bers [3] in the case of punctured torus groups.

Quasifuchsian space $\mathscr{QF}$ contains a real 2-dimensional manifold $\mathscr{F}$ which consists of $\rho \in \mathscr{QF}$ for which G_ρ is a Fuchsian group. The image of $\mathscr{F}$ under the end invariant map is equal to the diagonal $\{(\tau, \tau) : \tau \in \mathbf{H}\}$.

2.2. Complex Fenchel–Nielsen coordinates. Fix free generators g and h of $\pi_1 S$. They represent simple closed curves α and β on S whose intersection number is one. Let $[\rho] \in \mathscr{R}$. We can choose a representative $\rho = \rho_{\lambda,\mu}$ for this class such that

$$A = \rho_{\lambda,\mu}(g) = \begin{pmatrix} \cosh(\lambda/2) & \cosh(\lambda/2) + 1 \\ \cosh(\lambda/2) - 1 & \cosh(\lambda/2) \end{pmatrix}$$

$$B = \rho_{\lambda,\mu}(h) = \begin{pmatrix} \cosh(\mu/2)\coth(\lambda/4) & -\sinh(\mu/2) \\ -\sinh(\mu/2) & \cosh(\mu/2)\tanh(\lambda/4) \end{pmatrix}.$$

The parameters λ and μ are related with the geometry of the quotient 3-manifold: λ is the complex translation length of the geodesic corresponding to A and μ is the complex shear parameter with respect to A and B. For more details and the definition of complex shear we refer to [21].

Kourouniotis [12] and Tan [24] showed that the map $FN\colon \mathscr{QF} \to \mathbf{C}^2$ given by $FN(\rho) = (\lambda(\rho), \mu(\rho))$ is a complex analytic embedding. This map is referred to as the *complex Fenchel–Nielsen parametrization* of $\mathscr{QF}$. On $\mathscr{F}$, FN has real values and it gives the classical Fenchel–Nielsen coordinates of Teichmüller space, with λ the hyperbolic translation length of A and μ the twist parameter with respect to A and B, see e.g. [5]. The image of FN is also studied in [20].

We will denote the restriction of λ on $\mathscr{F}$ (the hyperbolic length function) by l_α. For any $c > 0$ we can also define *the earthquake path*

$$\mathscr{E}_c := \{\rho \in \mathscr{F} : l_\alpha(\rho) = c\}$$

which is the locus of punctured tori in $\mathscr{F}$ on which l_α is constant. This curve is parametrized by the twist parameter μ.

2.3. Enumeration of simple closed curves. The set of free homotopy classes of unoriented and non-boundary parallel simple closed curves on S can be naturally identified with $\overline{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$ satisfying the following condition: The boundary point $p/q \in \overline{\mathbf{R}} = \partial T(S)$ of the Teichmüller space $T(S) = \mathbf{H}$ is the point where the hyperbolic length of the unique geodesic in the homotopy class corresponding to p/q has shrunk to zero. We denote the unique geodesic in the homotopy class corresponding to p/q in $\mathbf{H}^3/G_\rho$ by $\gamma_{p/q}$. For each $p/q \in \overline{\mathbf{Q}}$, we can find an explicit word $W_{p/q}$ in the marked generators $\langle g, h\rangle$ of $\pi_1(S)$ representing $\gamma_{p/q}$ recursively as follows:

$$W_{1/0} = W_\infty = g^{-1} \quad \text{and} \quad W_{0/1} = W_0 = h.$$

If $a/b < c/d$ (with the convention $1/0 = \infty > r$ for all $r \in \mathbf{Q}$) satisfy $ad - bc = -1$, we set

$$W_{(a+c)/(b+d)} = W_{c/d} W_{a/b}.$$

The construction of the words implies that for $p \geq 0$, p is the number of g^{-1}'s and q is the number of h's in $W_{p/q}$. For $p < 0$, $-p$ is the number of g's in the word $W_{p/q}$.

2.4. Pleating locus. We will discuss the convex hull boundary and the pleating locus. Let $\Omega(G)$ be the ordinary set (or the set of discontinuity), and $\Lambda(G)$ be the limit set of a Kleinian group G. If $\rho \in \mathscr{QF}$, then the regular set $\Omega(G_\rho)$ consists of two invariant components $\Omega(G_\rho)^\pm$. Let $\partial\mathscr{C}_\rho$ be the boundary of the hyperbolic convex hull of $\Lambda(G_\rho)$ in $\mathbf{H}^3$. Then $\partial\mathscr{C}_\rho$ consists of two components $\partial\mathscr{C}_\rho^\pm$ facing $\Omega(G_\rho)^\pm$.

The two connected components of $\partial\mathscr{C}_\rho/G_\rho$ are pleated surfaces in $\mathbf{H}^3/G_\rho$. For G_ρ non-Fuchsian, the pleating loci with bending angles are measured geodesic laminations on S. We denote these laminations by $pl^\pm(\rho)$, and their projective classes by $|pl^\pm(\rho)|$. Let $PML(S)$ be the set of projective classes of measured geodesic laminations on S. Then $PML(S)$ can be naturally identified with $\overline{\mathbf{R}}$ and $\overline{\mathbf{H}} = T(S) \cup PML(S)$ is the Thurston compactification of the Teichmüller space of once-punctured tori. Since measured geodesic laminations on S are uniquely ergodic, we can identify $GL(S)$, the set of geodesic laminations on S, with $PML(S)$ (see [9] p. 459, Section 2.2).

For $\rho \in \mathscr{D}$ in general, $\Omega(G_\rho)^+$ or $\Omega(G_\rho)^-$ may not be a simply connected domain. In this case we can consider the end invariants $\nu_\pm \in PML(S)$ as follows: If $\Omega(G_\rho)^+$ is an infinite union of round disks, then $\Omega(G_\rho)^+/G_\rho$ is a thrice-punctured sphere, obtained from S by deleting a simple closed geodesic $\gamma_{p/q}$. In this case $W_{p/q} \in G_\rho$ is parabolic and we define the ending invariant $\nu_+ = p/q$. If $\Omega(G_\rho)^+$ is empty, then we can find a sequence of simple closed geodesics $\{\gamma_{p_n/q_n}\}$ in S whose geodesic representatives in $\mathbf{H}^3/G_\rho$ are eventually contained in any neighborhoods of the end e_+, and p_n/q_n converges in $\mathbf{R}$ to a unique irrational number r. In this case we define the ending invariant $\nu_+ = r$.

2.5. BM-slices. For any $c > 0$, let V_c be the subspace of $\mathbf{C}^2$ in complex Fenchel–Nielsen coordinates defined by the condition $\lambda = c$ (we will usually identify V_c with $\mathbf{C}$). Let

$$\mathscr{QF}_c = \{\zeta \in V_c : (c,\zeta) \in FN(\mathscr{QF})\}.$$

Thus, $\mathscr{QF}_c$ corresponds to the intersection of $\mathscr{QF}$ with V_c. It should be remarked that the real line of V_c is equal to the earthquake path $\mathscr{E}_c$. We denote by $\mathscr{BM}_c$ the component of $\mathscr{QF}_c$ which contains Fuchsian groups, and call it *the Bers–Maskit-slice or BM-slice* (see [9]). Let us consider the set of $\rho \in \mathscr{QF}$ such that the pleating locus of the convex hull boundary $\partial\mathscr{C}_\rho^\pm$ is equal to α and the hyperbolic length of α in $\mathbf{H}^3/G_\rho$ is equal to c. Then $A = \rho(g)$ must be purely hyperbolic (see [7] Lemma 4.6), hence the image of this set in complex Fenchel–Nielsen coordinates is also in V_c. We denote it by $\mathscr{BM}_c^\pm$. The basic properties of $\mathscr{BM}_c^\pm$ are given by the following theorem:

Theorem 2.1. ([9, 15]) *The complement of the Fuchsian locus $\mathscr{F}$ in $\mathscr{BM}_c$ consists of two connected components $\mathscr{BM}_c^+$ and $\mathscr{BM}_c^-$ meeting $\mathscr{F}$ along the real line corresponding to the earthquake path $\mathscr{E}_c$. The slice $\mathscr{BM}_c$ is simply connected and invariant under the action of the Dehn twist along α. The component $\mathscr{BM}_c^-$ is in $\mathbf{H}$ while $\mathscr{BM}_c^+$ is in the lower half plane: They are interchanged by complex conjugation.*

Parker and Parkkonen also studied $\mathscr{BM}_c$ by the name λ-*slice* and proved a similar result, Theorem 4.2 in [20]. The outside of $\mathscr{BM}_c$ in $\mathscr{QF}_c$ was studied by Komori and Yamashita (see [11]). They also draw exotic pictures of $\mathscr{QF}_c$ by means of Jorgensen's algorithm for discreteness of $\rho \in \mathscr{R}$.

2.6. End invariants and BM-slices. In sections 3 and 4 we want to control the image of a BM-slice under the end invariant maps $\nu_\pm$. This can be achieved by means of the following result of McMullen which compares the various lengths associated with the free homotopy class of a curve α in a quasifuchsian 3-manifold:

Theorem 2.2. ([15] Corollary 3.5) *Let G be a marked quasifuchsian and not Fuchsian group whose bending locus of $\partial\mathscr{C}^-/G$ is α. Then*

$$l_\alpha(\Omega^-/G) < l_\alpha(\partial\mathscr{C}^-/G) = l_\alpha(\mathbf{H}^3/G) < l_\alpha(\Omega^+/G).$$

Corollary 2.3. *The restriction of the end invariant map* (2.1) *to $\mathscr{BM}_c^-$ satisfies*

$$\nu_+(\mathscr{BM}_c^-) \subset \{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) > c\}$$

and

$$\nu_-(\mathscr{BM}_c^-) \subset \{\tau_- \in \mathbf{H} : l_\alpha(\tau_-) < c\}.$$

2.7. Complex earthquakes. We also need McMullen's results on complex earthquakes which are essential for our results in section 3 and 4. For more details we refer to [15] and the references therein.

Our definition of a complex earthquake is slightly different from the original one: Let X be a once-punctured torus, and let $\overline{\mathbf{L}}$ denote the closure of the lower half plane $\mathbf{L}$. When $\lambda \in \mathbf{L}$ the complex earthquake $\mathrm{eq}_\lambda(X)$ of X along α is defined as the composition of twisting of distance $\operatorname{Re}\lambda$ and grafting of height $-\operatorname{Im}\lambda$. Recall that grafting means that the surface is cut along α and a Euclidean right cylinder of height $-\operatorname{Im}\lambda$ is inserted.

Let $D(X,\alpha)$ be the union of $\overline{\mathbf{L}}$ and $\mathscr{BM}^-_{l_\alpha(X)}$. The complex earthquake map $f\colon D(X,\alpha) \to T(S)$ is defined by

$$f(\lambda) = \begin{cases} \mathrm{eq}_\lambda(X), & \text{if } \lambda \in \overline{\mathbf{L}}, \\ \Omega^+/G, & \text{if } \lambda \in \mathscr{BM}^-_{l_\alpha(X)}, \end{cases}$$

where G is the quasifuchsian group corresponding to λ.

McMullen [15] proved that the complex earthquake map f is conformal, and that it maps $\overline{\mathbf{L}}$ onto $\{Y \in T(S) : l_\alpha(Y) \le l_\alpha(X)\}$. Identifying $T(S)$ with $\mathbf{H}$, this implies that f is a conformal map from $\mathscr{BM}^-_{l_\alpha(X)}$ onto $\{\tau \in \mathbf{H} : l_\alpha(\tau) > l_\alpha(X)\}$. Since $f = \nu_+$ on $\mathscr{BM}^-_{l_\alpha(X)}$, we have the following theorem:

Theorem 2.4. ($\mathscr{BM}_c^-$ as a parameter space of once-punctured tori) *The end invariant map $\nu_+\colon \mathscr{BM}_c^- \to \{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) > c\}$ is a conformal surjective map.*

We should remark that the surjectivity of the above map is a key point to define the curve $\sigma(s)$ in the proof of Theorem 3.1.

3. Cusps are inward-pointing cusps

The word cusp in the title of this section refers to two a priori different objects: On the one hand, it is customary to call a geometrically finite boundary point of a deformation space a cusp. In a BM-slice these points coincide with the endpoints of rational pleating rays (see [9, 15, 20] and Section 5). These boundary points of the deformation space correspond to Riemann surfaces which are obtained from the surfaces represented by the interior points of the deformation space by pinching, which produces Riemann surfaces with a number of cusps. On the other hand, if $U \subset \mathbf{C}$ is an open set, a boundary point $\zeta \in \partial U$ is called an inward-pointing cusp of U if there is a disk $D = \{z \in \mathbf{C} : |z - z_0| = |z_0|\} \subset \mathbf{C}$ (tangent to the origin in $\mathbf{C}$), such that the map

$$f_{\zeta,\theta}(z) = \zeta + e^{i\theta} z^2$$

is an embedding of D into U for some $0 \geq \theta \geq 2\pi$. The image $f_{\zeta,\theta}(D)$ is a cardioid, and its cusp is at ζ.

In this section we prove that in the boundary of a Bers–Maskit slice the first meaning of the word cusp actually implies the second one. We follow the outline of Miyachi's proof of the analogous statement for the Maskit embedding in [18]. Figure 1 illustrates the fractal-like structure of the boundary of a Bers–Maskit slice.

Let $r \in \mathit{PML} = \mathbf{R} \cup \{\infty\}$. The r-pleating ray in $\mathscr{BM}_c^-$ is

$$\mathscr{P}_r = \{\mu \in \mathscr{BM}_c^- : pl^+(\rho_{c,\mu}) = r\}.$$

If r corresponds to a simple closed curve (and to a rational number in the identification $\mathit{PML} = \mathbf{R} \cup \{\infty\}$), we say that $\mathscr{P}_r$ is a rational pleating ray. On the boundary of either half of the BM-slice, say $\mathscr{BM}_c^-$, for any $p/q \in \mathbf{Q}$ there is a unique non-Fuchsian, geometrically finite boundary point $\mu(p/q)$ called *the p/q-cusp*, which is the endpoint of the p/q-pleating ray.

Our first theorem is about the shape of the BM-slice at the cusp points:

Theorem 3.1. *For any $p/q \in \mathbf{Q}$, the cusp $\mu(p/q)$ at the boundary of $\mathscr{BM}_c^-$ is an inward-pointing cusp.*

Proof. Consider the map $\Pi\colon \mathbf{C} \to V_c$ defined by $\Pi(t) = \mu(p/q) + t^2$. Fix a component of $\Pi^{-1}(\mathscr{BM}_c^-)$ and denote it by $\widetilde{\mathscr{BM}}_c^-$. Note that $\Pi(0) = \mu(p/q)$, and that $0 \in \partial\widetilde{\mathscr{BM}}_c^-$. To prove the existence of a cardioid stated in the theorem, we will show that there is a round disk B_1 in $\widetilde{\mathscr{BM}}_c^-$ whose boundary contains 0. In order to establish this, we will find a curve $\sigma(s)$ ($s \in \mathbf{R}$) in $\widetilde{\mathscr{BM}}_c^-$ such that $\lim_{s\to\pm\infty} \sigma(s) = 0$ and which is sufficiently flat at 0 so that it separates a round disk in $\widetilde{\mathscr{BM}}_c^-$ containing 0 on the boundary, from the boundary of $\widetilde{\mathscr{BM}}_c^-$, see Figure 2.

To define $\sigma(s)$ and check its properties, we will use the following coordinate change of the end invariants related to p/q: Take $h \in \mathrm{PSL}(2,\mathbf{Z})$ satisfying $h(p/q) = \infty$, and let $\nu_\pm[p/q] = h \circ \nu_\pm$. Note that h is not uniquely determined (it is determined only up to postcomposition by a horizontal translation), but the quantity $\nu_+[p/q] - \overline{\nu_-[p/q]}$ is well-defined.

Now we define $\sigma(s)$ as follows:

$$\sigma(s) = \Pi^{-1} \circ \nu_+[p/q]^{-1}(s + ri) \quad (s \in \mathbf{R}) \tag{3.1}$$

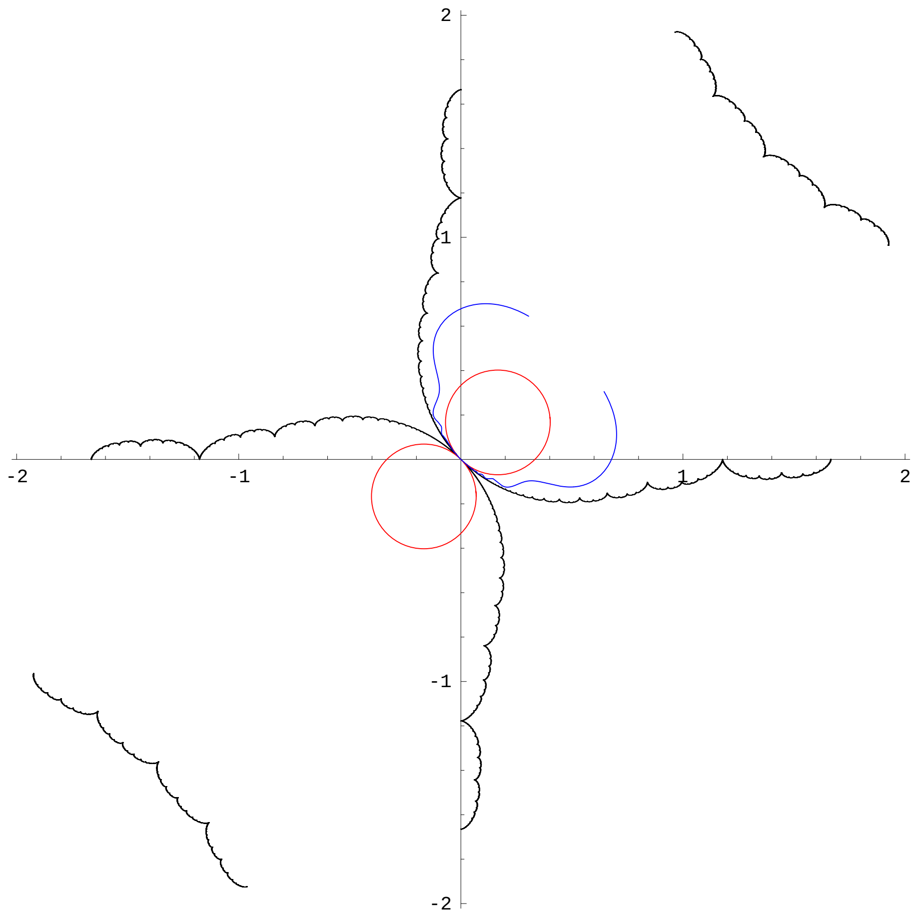

Figure 2. The preimage $\Pi^{-1}\mathscr{BM}_c$ of $\mathscr{BM}_c$ in $\mathbf{C}$ with the curve σ defined by (3.1), and the disks B_1 and $-B_1$. In this figure we have used the slice of Figure 1 and the 'main cusp' $p/q = 0/1$ on the imaginary axis. The preimage of this cusp in Π is 0.

where $\nu_+[p/q]$ and Π are only considered on $\mathscr{BM}_c^-$ and $\widetilde{\mathscr{BM}}_c^-$ respectively, and $r = r(c)$ is a function of c satisfying the condition that $l_\alpha(s+ri) > c$ for all $s \in \mathbf{R}$. We can find such r since the earthquake path $\mathscr{E}_c \subset \mathbf{H}$ is a periodic topological horocycle tangent to $\mathbf{R}$ at $h(1/0)$. In particular, it is bounded in the $\nu_+[p/q]$-coordinate, see Corollary 2.3. Theorem 2.4 implies that $\Pi \circ \sigma(s)$ is contained in $\mathscr{BM}_c^-$.

To prove that $\sigma(s)$ converges to 0 when $s \to \pm\infty$, it is enough to show that $\Pi \circ \sigma(s)$ converges to $\mu(p/q)$ when $s \to \pm\infty$. This follows from the next theorem which we will prove in section 4. See also Figure 1:

Theorem 3.2. *The boundary of $\mathscr{BM}_c$ consists of two Jordan arcs.*

The horizontal line $s + ri \quad (s \in \mathbf{R})$ in $\mathbf{H}$ arrives at ∞ on the boundary of $\mathbf{H}$ when $s \to \pm\infty$. Theorem 3.2 implies that the map $\nu_+^{-1}[p/q]$ extends to the boundary. Thus, $\Pi \circ \sigma(s)$ converges to $\mu(p/q)$ when $s \to \pm\infty$.

It remains to consider the regularity of $\sigma(s)$ at $0 \in \partial\widetilde{\mathscr{B}\mathscr{M}}_c^-$. Let $\lambda_{p/q}(t)$ denote the complex translation length of $W_{p/q}(\Pi(t))$. Note that $\lambda_{p/q}(0) = 0$, and that $\lambda_{p/q}$ extends as a holomorphic function in a neighborhood of 0 satisfying $l_{p/q} := \operatorname{Re}\lambda_{p/q} > 0$ on $\widetilde{\mathscr{B}\mathscr{M}}_c^-$. From the equality

$$\operatorname{tr} W_{p/q}(\Pi(t)) = 2\cosh(\lambda_{p/q}(t)/2),$$

we get that the derivative of $\operatorname{tr} W_{p/q}$ vanishes at $\mu(p/q)$ if and only if the derivative of $\lambda_{p/q}$ vanishes at 0. Furthermore, in section 5 we prove the following result:

Theorem 3.3. *The derivative of* $\operatorname{tr} W_{p/q}$ *is nonzero for any* $p/q \in \mathbf{Q}$.

Thus, $\lambda_{p/q}$ is conformal at 0, and the shape of the curve $\lambda_{p/q}(\sigma(s))$ is conformally the same as that of $\sigma(s)$. We get the smoothness of $\lambda_{p/q}(\sigma(s))$ (and that of $\sigma(s)$ by the above reasoning) from the following result of Minsky:

Theorem 3.4. (Pivot theorem, [17]) *There are universal constants* $\varepsilon, c_1 > 0$ *such that if* $l_{p/q} < \varepsilon$, *then*

$$d_{\mathbf{H}}\Big(\frac{2\pi i}{\lambda_{p/q}}, \nu_+[p/q] - \overline{\nu_-[p/q]} + i\Big) < c_1,$$

where $d_{\mathbf{H}}$ *is the hyperbolic metric on* $\mathbf{H}$.

Note that $l_{p/q}(\sigma(s)) < \varepsilon$ for sufficiently large $|s|$, since $\sigma(s)$ arrives at 0 when $s \to \pm\infty$ and $\lambda_{p/q}(0) = 0$. Hence, we can apply Theorem 3.4 to our curve $\sigma(s)$. Moreover, by (3.1), the curve in $\mathbf{H}$ defined by

$$\nu_+[p/q](\Pi \circ \sigma(s)) - \overline{\nu_-[p/q](\Pi \circ \sigma(s))} + i \tag{3.2}$$

can be written as

$$s + (r+1)i - \overline{\nu_-[p/q](\Pi \circ \sigma(s))}.$$

Because $\nu_-[p/q](\mathscr{B}\mathscr{M}_c^-)$ is a bounded domain in $\mathbf{H}$, we can find a horizontal line in $\mathbf{H}$ lying above $\lambda_{p/q}(\sigma(s))$ which guarantees the existence of a round ball B_1 in $\widetilde{\mathscr{B}\mathscr{M}}_c^-$ whose boundary contains 0 by Minsky's pivot theorem. This concludes the proof of Theorem 3.1. □

4. The boundary of $\mathscr{B}\mathscr{M}_c$

In this section we will prove theorem 3.2. Since $\mathscr{B}\mathscr{M}_c^+$ is the image of $\mathscr{B}\mathscr{M}_c^-$ under the complex conjugation, we restrict our attention only to $\mathscr{B}\mathscr{M}_c^-$. See 2.5 and 2.6 for background material for this section.

4.1. The boundary in $\mathscr{D}$ and in V_c. The boundary of $\mathscr{B}\mathscr{M}_c^-$ in $V_c \subset \mathbf{C}^2$ is naturally identified with that of $FN^{-1}(\mathscr{B}\mathscr{M}_c^-)$ in $\mathscr{D}$: The complex Fenchel–Nielsen coordinate map FN is a complex analytic embedding of the set

$$\{\rho \in \mathscr{D} : A = \rho(g) \text{ is not parabolic}\}$$

into $\mathbf{C}^2$ (see the proof of proposition 2.1 in [21]), and

$$FN^{-1}(\mathscr{BM}_c^-) \subset \{\rho \in \mathscr{D} : \operatorname{tr}\rho(g) = \operatorname{tr} A = 2\cosh(c/2)\}$$

with $c \neq 0$. Hence, to prove Theorem 3.2, it is enough to show that the boundary of $FN^{-1}(\mathscr{BM}_c^-)$ in $\mathscr{D}$ consists of the earthquake path $\mathscr{E}_c$ and a Jordan arc. In order to simplify notation, we will write $\mathscr{BM}_c^-$ for $FN^{-1}(\mathscr{BM}_c^-)$ in the rest of this section.

4.2. The image $\nu(\mathscr{BM}_c^-)$ as a graph in $\mathbf{H} \times \mathbf{H}$. Theorem 2.4 implies that there is a continuous function

$$h\colon \{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) > c\} \to \{\tau_- \in \mathbf{H} : l_\alpha(\tau_-) < c\}$$

such that $\nu(\mathscr{BM}_c^-)$ can be written as the graph of h:

$$\nu(\mathscr{BM}_c^-) = \{(\tau_+, h(\tau_+)) \in \mathbf{H} \times \mathbf{H} : \tau_+ \in \nu_+(\mathscr{BM}_c^-)\}.$$

To prove the next proposition, we will use the following result about bending coordinates of the limit Bers slice due to McMullen [15]: Let $[\omega] \in PML(S) = \mathbf{R} \cup \{\infty\}$. The subset $B^{[\omega]} := \nu^{-1}(\{[\omega]\} \times \mathbf{H})$ is called the limit Bers slice corresponding to ω.

Theorem 4.1. (Coordinates of a limit Bers slice, [15]) *There is a homeomorphism $B^{[\omega]} \to (\mathbf{R} \cup \{\infty\} - \{[\omega]\}) \times \mathbf{R}^+$, given by*

$$G \mapsto \left([\beta],\ \frac{l_\beta(\partial\mathscr{C}^-/G)}{i(\beta,\omega)}\right),$$

where β is the bending lamination of $\partial\mathscr{C}^-/G$, l_β is the length function of β, and $i(\beta,\omega)$ is the intersection number of measured laminations β and ω.

Proposition 4.2. *The function h has a unique continuous extension*

$$h\colon \{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) \geq c\} \cup \mathbf{R} \to \{\tau_- \in \mathbf{H} : l_\alpha(\tau_-) \leq c\}.$$

Proof. Let $\tau_+^\infty \in \mathbf{R}$. There is a sequence $\{\tau_{+,n}\}$ in $\{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) > c\}$ converging to τ_+^∞. Since $\{\tau_- \in \mathbf{H} : l_\alpha(\tau_-) \leq c\} \cup \{1/0\}$ is compact, we can take a convergent subsequence $\{h(\tau_{+,n_j})\}$ from $\{h(\tau_{+,n})\}$. Let τ_-^∞ be the limit of this sequence.

First we show that τ_-^∞ is not equal to $1/0$. Suppose $\tau_-^\infty = 1/0$. Then the sequence $\{\nu^{-1}((\tau_{+,n_j}, h(\tau_{+,n_j}))\}$ in $\mathscr{BM}_c^-$ converges to $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ which must be in the closure of $\mathscr{BM}_c^-$ because of the continuity of ν^{-1}. Since the negative end invariant of $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ is equal to $1/0$, A is parabolic for $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$. On the other hand, $\operatorname{tr} A$ is continuous on $\mathscr{R}$, and from the definition of $\mathscr{BM}_c^-$, $\operatorname{tr} A$ is constant on the closure of $\mathscr{BM}_c^-$, which is a contradiction.

Next we show that τ_-^∞ does not depend on the choice of convergent subsequences. This implies that $\{h(\tau_{+,n})\}$ itself converges to τ_-^∞ and we can define $h(\tau_+^\infty) = \tau_-^\infty$. There are two cases to be considered: First suppose that τ_+^∞ is on the earthquake path $\mathscr{E}_c$. Then the end invariants of $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ are both in $\mathbf{H}$, hence it must be a quasifuchsian group. On the other hand, the condition $l_\alpha(\Omega^+/G) = l_\alpha(\partial\mathscr{C}^-/G) = c$ implies that $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ is Fuchsian from Theorem 2.2 hence $\tau_-^\infty = \tau_+^\infty$. Next suppose that τ_+^∞ is on the boundary $\mathbf{R}$. Then $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ must be a boundary group of $\mathscr{BM}_c^-$ since $\tau_+^\infty \in \mathbf{R}$. Consider $\nu^{-1}(\tau_+^\infty, \tau_-^\infty)$ as a point of the limit Bers slice $B^{\tau_+^\infty}$ and apply Theorem 4.1, then τ_-^∞ is uniquely determined by τ_+^∞.

It is clear from the construction that the extension is a continuous function. □

This proposition implies that the closure $\overline{\nu(\mathscr{BM}_c^-)}$ of $\nu(\mathscr{BM}_c^-)$ in $\overline{\mathbf{H}} \times \overline{\mathbf{H}} \setminus \Delta$ is the graph of f over $\{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) \geq c\} \cup \mathbf{R}$, and we have the following result:

Corollary 4.3. *The projection pr_+ is a homeomorphism from $\overline{\nu(\mathscr{BM}_c^-)}$ onto $\{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) \geq c\} \cup \mathbf{R}$.*

Proof of Theorem 3.2. From Corollary 4.3, we get that the boundary of $\overline{\nu(\mathscr{BM}_c^-)}$ corresponds to two Jordan arcs $\mathscr{E}_c$ and $\mathbf{R}$ under the homeomorphism pr_+. The map ν^{-1} is a continuous bijection from $\overline{\nu(\mathscr{BM}_c^-)}$ onto $\nu^{-1}(\overline{\nu(\mathscr{BM}_c^-)})$ containing $\mathscr{BM}_c^-$. To show that the restriction of ν^{-1} on $\overline{\nu(\mathscr{BM}_c^-)}$ is homeomorphic, we consider the action of the Dehn twist along α : $\overline{\nu(\mathscr{BM}_c^-)}$ is invariant and ν^{-1} is equivariant under the Dehn twist along α. The quotient space of $\overline{\nu(\mathscr{BM}_c^-)}$ by this action is homeomorphic to the quotient space of $\{\tau_+ \in \mathbf{H} : l_\alpha(\tau_+) \geq c\} \cup \mathbf{R}$ by the action of the translation $\tau_+ \mapsto \tau_+ + c$. Thus, it is topologically a closed annulus, hence in particular a compact set. Therefore ν^{-1} on $\overline{\nu(\mathscr{BM}_c^-)}$ is homeomorphic. □

5. Nonfaithful representations close to the cusps

In this section we study the representations close to the cusp for which the element, which is hyperbolic in the interior of $\mathscr{BM}_c$ and parabolic at the cusp is a primitive elliptic transformation. Recall that an elliptic element is primitive if it has minimal rotation around its axis in $\mathbf{H}^3$ among all elements in the group it generates.

As in section 4, we will consider representations in $\mathscr{BM}_c^-$, and the corresponding results for $\mathscr{BM}_c^+$ follow by symmetry. Let us fix $c > 0$, and let $\mu \in \mathbf{H}$.

We will use the following notation: The parameter $\mu \in \mathbf{H}$ corresponds to a representation ρ, and $\mu_\infty, \mu_n \in \mathbf{H}$ ($n \in \mathbf{N}$) correspond to specific representations ρ_∞, ρ_n defined later in this section. Furthermore, $G_\infty = G_{\rho_\infty}$, and $G_n = G_{\rho_n}$. A similar convention is used for subgroups of G_n defined in the course of the proof of Theorem 5.1.

5.1. Fuchsian subgroups. If $c > 0$ and $\mu \in \mathbf{H}$, then the subgroup

$$\Gamma_\rho = \rho\langle g, hg^{-1}h^{-1}\rangle = \langle A, BA^{-1}B^{-1}\rangle$$

is a Fuchsian group of the second kind, and $\mathbf{H}/\Gamma_\rho$ is a "punctured cylinder" with boundary geodesics of equal lengths corresponding to the generators A and $BA^{-1}B^{-1}$. For fixed c all groups $\Gamma_\rho = \Gamma_{\rho_{c,\mu}}$ are conjugates of the same group by Möbius transformations. Recall that if $\rho \in \mathscr{BM}_c^-$, then the pleating locus on $\partial\mathscr{C}^-/G_\rho$ is α, which corresponds to the generator A of Γ_ρ.

5.2. Koebe groups and b-groups. In the proof of Theorem 5.1, we will use the results of [7, 22, 23] on the combinatorial structure and the deformation spaces of terminal regular b-groups and Koebe groups of type (1,1). A Kleinian group H is a Koebe group of type $(1,1)$ if its regular set consists of an invariant component Δ_0 and a collection of disks Δ_i, $i \in \mathbf{N}$, such that Δ_0/H is a punctured torus and the stabilizers of the disks are Fuchsian triangle groups. Such a group H is a terminal regular b-group if the invariant component is simply connected. See Figure 3 for examples of limit sets of Koebe groups and b-groups.

For any cusp $\mu_\infty \in \partial\mathscr{B}\mathscr{M}_c$ the corresponding group G_∞ is a terminal regular b-group. There is some word W_r, $r \in \mathbf{Q}$ (see 2.3 for the definition of W_r) for which $\rho_\infty W_r$ is parabolic. In Theorem 5.1, we will show that there are unique points μ_n close to μ_∞ in the complement of $\mathscr{B}\mathscr{M}_c$ such that, for the corresponding representation ρ_n, $\rho_n W_r$ is primitive elliptic and the corresponding Kleinian group G_n is a Koebe group.

5.3. Circle chains. In Theorem 5.1, we use circle chains in a manner similar to [23]: Let A and B be Möbius transformations such that A is either hyperbolic, parabolic or a primitive elliptic of order n, and $F = \langle A, BA^{-1}B^{-1}\rangle$ is a Fuchsian group which uniformizes a punctured cylinder, a thrice punctured sphere or a punctured sphere with two cone points of order n on its invariant disk, according to the type of A. Note that F is a Fuchsian group of the second kind (i.e. its limit set is not a circle) in the first case, while in the other cases it is of the first kind. Let $W_{p/q} = W_{p/q}(A,B)$ be the p/q-word in the generators . A collection $\{\delta_i\}$, $i \in \mathbf{Z}$, of closed, round disks $\delta_i \subset \widehat{\mathbf{C}}$ is a (*combinatorial*) *p/q-chain for the group* $\langle A, B\rangle$ (*with generators A and B*) if it satisfies the following conditions:

(i) δ_0 is tangent to the invariant circle of F which contains $\Lambda(F)$ at the fixed point of the parabolic element
$$K = W_{r/s}^{-1} W_{p/q}^{-1} W_{r/s} W_{p/q},$$
where r/s is a Farey neighbour of p/q,

(ii) $W_{p/q}\delta_0 = \delta_0$,

(iii) $B(\delta_j) = \delta_{j+p}$ for all $j = 0, \dots, q$, and

(iv) $A(\delta_j) = \delta_{j+q}$ for all $j \in \mathbf{Z}$.

The chain is *proper* if

(v) the interiors of the disks δ_i are contained in $\Omega(G)$ for all i,

(vi) the interiors of adjacent disks δ_i and δ_{i+1} intersect for all i, and

(vii) $\operatorname{int}\delta_i \cap \operatorname{int}\delta_j = \emptyset$ for $|i-j| > 1$.

Note that this definition enables us to work with circle chains in quasifuchsian groups, terminal b-groups and Koebe groups depending on whether the generator A is hyperbolic, parabolic or primitive elliptic. The circle chain does not close up for the quasifuchsian group, it is an infinite chain with one accumulation point for the terminal b-group, and a finite chain consisting of n copies of the "basic piece" for the Koebe group. See Figure 3.

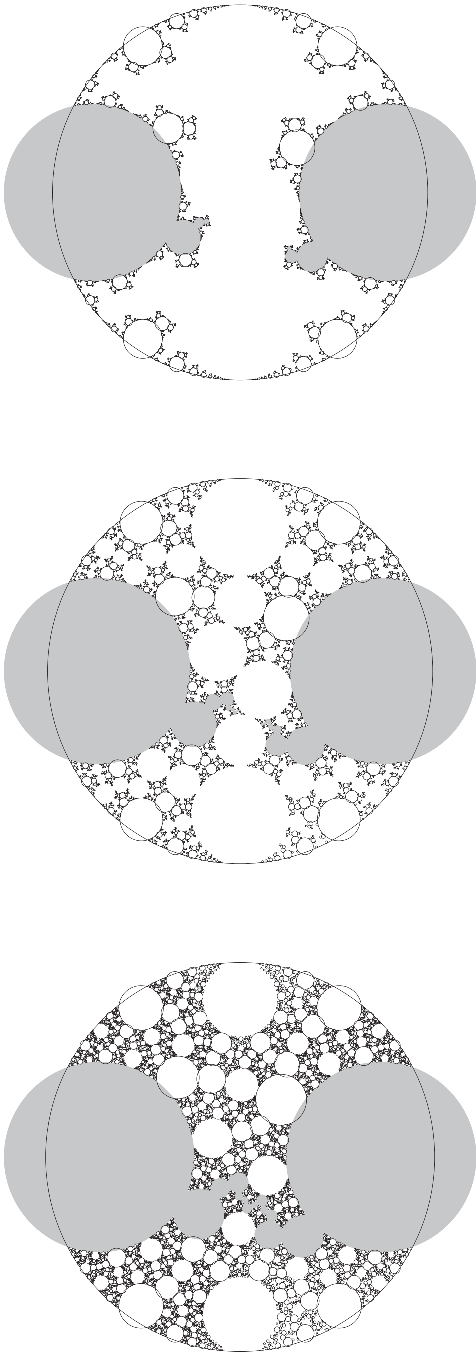

Figure 3. Limit sets of a quasifuchsian group, a terminal b-group and a Koebe group with $c = 1/2$ and $r = 2/5$. The circle chain $\{\delta_i\}$ consists of the shaded disks and the exterior of the biggest disk in each case.

The existence of circle chains is closely related with the pleating structure of $\partial\mathscr{C}$: As in [7] for the Maskit slice, $\mu \in \mathscr{P}_r$, $r \in \mathbf{Q}$ if and only if G_ρ has an r-chain for the generators A and B. For more details of pleating rays and related material we refer to [7, 8, 9, 23].

Theorem 5.1. *Let $c > 0$, $r \in \mathbf{Q}$. Let μ_∞ be the cusp at the end of the rational pleating ray $\mathscr{P}_r \cap \mathscr{BM}_c^-$. Then there is a sequence of parameters $\mu_n \in \mathbf{C} \setminus \overline{\mathscr{BM}_c^-}$ and corresponding representations ρ_n such that*

(i) *$\rho_n(W_r)$ is primitive elliptic of order n, and*
(ii) *$\lim_{n\to\infty} \mu_n = \mu_\infty$,*
(iii) *$G_n = \rho_n(\pi_1 S)$ is discrete, and $\Omega(\rho_n)/G_n$ is the disjoint union of a punctured torus and a 2-orbifold of signature $(0, 3; n, n, \infty)$.*

Furthermore, the sequence μ_n is uniquely determined for n big enough.

Proof. Let us consider $\pi_1 S$ with generators $W_r = W_r(g, h)$ and U_r, where $U_r = W_t(g, h)$ is a (non-uniquely determined) element which corresponds to any Farey neighbour t of r. There is some $s \in \mathbf{Q}$ such that $W_s(W_r, U_r) = g^{\pm 1}$, the s-word in the generators W_r and U_r. See [16] for details on how s depends on r.

By the analyticity of the trace of W_r, there are parameters μ_n in any neighbourhood of the cusp μ_∞ for which

$$\operatorname{tr} \rho_n W_r = \pm 2 \cos \pi/n,$$

where the sign is the same as for the cusp parameter μ_∞. In other words, $\rho_n W_r$ is a primitive elliptic transformation of order n. For parameters $\mu \in \mathscr{P}_r$, the subgroup

$$\Phi_\rho = \rho\langle W_r, U_r^{-1} W_r^{-1} U_r \rangle$$

is a Fuchsian group of the second kind, $\Phi_\infty = \Phi_{\rho_\infty}$ is a torsion-free triangle group, and $\Phi_n = \Phi_{\rho_n}$ is a triangle group of signature (n, n, ∞) for $n \in \mathbf{Z}$, $n \geq 3$.

Let

$$\mathscr{V}_r = \{\mu \in V_c : \operatorname{tr} \rho W_r \in \mathbf{R}\}$$

denote the *real locus* of W_r in $\mathbf{C}$. Recall that $\mathscr{P}_r \subset \mathscr{V}_r \cap \mathscr{BM}_c^-$. Assume $\mu \in \mathscr{V}_r$. The fact that $\partial\mathscr{C}_\rho^-$ is pleated along the orbit of the axis of A in $\mathbf{H}^3$ for $\mu \in \overline{\mathscr{BM}_c^-}$ implies that there is a proper combinatorial s-chain $\{\delta_{j,\mu}\}_{j\in\mathbf{Z}}$ in $\Omega(G_\rho)$ with respect to the generators $\rho_{c,\mu} W_r$ and $\rho_{c,\mu} U_r$, where the disks $\delta_{j,\mu}$ are stabilized by conjugates of the Fuchsian group Γ_ρ of the second kind, see 5.1. Recall that all such groups with equal values of c are conjugate in $\mathrm{PSL}(2, \mathbf{C})$.

The cusp group G_∞ is a terminal regular b-group of the type treated in [7]. The circle chain $\{\delta_i\}$ in the invariant component $\Omega_0(G_\infty)$ is proper, and the closures of any two disks the chain are disjoint unless they are adjacent in the chain. If $s = N/M$, with $N, M \in \mathbf{Z}$, then

$$\delta_{i+kN} = W_r^k(\delta_i)$$

for all $k \in \mathbf{Z}$. Thus, it is enough to consider the perturbation of a finite collection of circles to understand the behaviour of the chain $\{\delta_{i,\mu}\}$ when the parameter μ varies.

The continuity of the circles in the parameter μ, along with the fact that $\rho_n W_r$ is a primitive elliptic transformation, implies that for big enough n the group G_n has a finite proper combinatorial s-chain $\{\delta_{j,n}\}_{j\in\mathbf{Z}}$ with respect to $\rho_n W_r$ and $\rho_n U_r$. This chain consists of n copies of the "basic chain" $\delta_{1,n}, \delta_{2,n}, \dots, \delta_{N,n}$. See [23] Section 9 for more details. In [23] the same situation is treated in the case of the Maskit embedding, which corresponds to $c = 0$, and the subgroup corresponding to Γ_ρ is the triangle group of signature (∞, ∞, ∞).

The facts that

$$G_n = \Phi_n *_{\rho_n(U_r)} \tag{5.1}$$

i.e. that G_n is the HNN extension of of the triangle group Φ_n of signature (n, n, ∞) by the element $\rho_n(U_r)$, and that G_n is a discrete group follow using Maskit's second combination theorem [13, VII.E.5]: Let us begin by constructing a topological disc D_0 for the cusp group G_∞, which is precisely invariant under $\rho_\infty W_r$ in Φ_n and such that the disc $\rho_\infty U_r^{-1}(\widehat{\mathbf{C}} \setminus D_0)$ is precisely invariant under $\rho_\infty(U_r^{-1} W_r^{-1} U_r)$, and the closures of D_0 and $\rho_\infty U_r^{-1}(\widehat{\mathbf{C}} \setminus D_0)$ are disjoint. This can be done by a modification of Wright's method for groups where ρg is parabolic as in [25], or by the following non-constructive argument: Let γ be a simple closed geodesic on the punctured torus $\Omega(G_\infty)/G_\infty$ in the free homotopy class determined by W_r. Let $\widetilde{\gamma}$ be the closure of the lift of γ to the invariant component of G_∞ which is invariant under $\rho_\infty W_r$. Let D_0 be the component of $\widehat{\mathbf{C}} \setminus \widetilde{\gamma}$ which does not contain any points of the limit set of the triangle group Γ_{r,μ_∞}. By construction, $\widetilde{\gamma} \cap g(\widetilde{\gamma}) = \emptyset$ for all $g \in G_\infty \setminus \langle \rho_\infty W_r \rangle$.

Let n be big enough so that G_n has a proper finite s-chain as constructed above. By continuity of the circles $\delta_{i,\mu}$ in μ, the circle $\delta_{i,n}$ is very close to δ_i for all $i \in \{1, 2, \dots, N\}$. Thus, we can assume

$$\widetilde{\gamma} \cap \big(\delta_1 \cup \delta_2 \cup \cdots \cup \delta_N\big) \subset \widetilde{\gamma} \cap \big(\delta_{0,n} \cup \delta_{1,n} \cup \cdots \cup \delta_{N,n} \cup \delta_{N+1,n}\big)$$

Pasting together n copies of this arc with small adjustments in the $\rho_n W_r$ translates of δ_{0,μ_n} produces a loop $\widetilde{\gamma}_n$ which is precisely invariant under $\langle \rho_n W_r \rangle$ in Φ_n, and satisfies $\rho_n U_r(\widetilde{\gamma}_n) \cap \widetilde{\gamma}_n = \emptyset$. The loop $\widetilde{\gamma}_n$ bounds a topological disk $D_{0,n}$ which satisfies the conditions of Maskit's second combination theorem. This implies discreteness and the group theoretical structure (5.1). The groups G_n are Koebe groups, which have an infinitely connected invariant component, see [22].

It remains to prove the uniqueness of the "elliptic values". This follows from the existence of pleating coordinates for the deformation spaces of Koebe groups proved in [22]: Assume there are two parameters t_n and t'_n arbitrarily close to the cusp which satisfy the condition (i) in the statement of this theorem. We can assume the parameters are in the neighbourhood where (i) and (iii) hold for t_n and t'_n. The convex hulls are pleated along the image of g in the groups. Thus, ρ_{c,t_n} and $\rho_{c,t'_n})$ are both determined by c, which is the translation length of of both $\rho_{c,t_n}(g)$ and $\rho_{c,t'(n)}(g)$. This implies $t_n = t'_n$, see [22] Section 3. □

Proof of Theorem 3.3. Theorem 5.1 implies that close to the cusp there is only one parameter for which $\operatorname{tr} W_r = \pm 2\cos(\pi/n)$ for n big enough. The holomorphicity of $\mu \mapsto \operatorname{tr} W_r$ implies there is no critical point. □

6. The slices are not vertically convex

In this section we show that the Maskit slice and the Bers–Maskit slices are not vertically convex, i.e. that there are vertical lines in parameter space such that the intersection of such a line with $\mathscr{M}$ or $\mathscr{BM}_c$ is not connected. This phenomenon was observed experimentally for the Maskit slice by Wright [25], and for the Bers–Maskit slices it was pointed out by Epstein, Marden and Markovic [6].

It is easy to check that the intersection of a vertical line through a cusp of $\mathscr{M}$ with integer or half-integer index is connected. We will show that this is not true in general for other indices. To be more precise, we show that there are points close to the cusps at the ends of the $-1/3$-rays in $\mathscr{M}$ and $\mathscr{BM}_c$, $c > 0$, which make it impossible for these slices to be vertically convex.

The proof is based on the results of section 3 and the following completely elementary observation: Let C be the interior of the standard cardioid defined in polar coordinates by the equation

$$r = 2(1 + \cos\phi).$$

One sees from the equation that any line through the origin, except the horizontal one, intersects ∂C in three points. Thus, the intersection of this line, and of nearby parallel lines to one side of it, with C is not connected. If a scaled and rotated copy of C is embedded in one of our slices such that the origin is mapped to a cusp and the axis of symmetry of the image is not vertical, then this observation implies that the slice is not vertically convex.

The case of the Maskit slice can be treated by a relatively simple calculation, whereas the Bers–Maskit slices require the manipulation of hyperbolic functions. However, even these expressions simplify sufficiently to be manageable explicitly. We will study the real locus of the transformation

$$W_{-1/3} = B^2AB.$$

For computations it is convenient to replace $W_{-1/3}$ by a conjugate transformation B^3A whose real locus is identical. Note that the notation here differs slightly from section 5 where the words W_r were elements of $\pi_1 S$.

The trace of $W_{-1/3}$ can be computed by finding the expression of $W_{-1/3}$ from the generators, or by using the relation

$$\operatorname{tr} MN + \operatorname{tr} MN^{-1} = \operatorname{tr} M \operatorname{tr} N \quad (\forall M, N \in \mathrm{SL}(2, \mathbf{C})) \tag{6.1}$$

which implies

$$\operatorname{tr} W_{-1/3} = (\operatorname{tr}^2 B - 2)\operatorname{tr} AB - \operatorname{tr} AB^{-1}. \tag{6.2}$$

Proposition 6.1. *The Maskit slice is not vertically convex.*

Proof. It follows from Theorem 2 of [18] (this is the analog of our Theorem 3.1) that there is an embedded cardioid in $\mathscr{M}$ at the cusp which is symmetric with respect to reflection in the tangent direction of the pleating ray at the cusp. If the tangent direction of the pleating ray at the cusp is not vertical, there is a point z in the lower half of the cardioid such that the intersection of the vertical line through z and $\mathscr{M}$ has at least two components. We will show that the pleating ray $\mathscr{P}_{-1/3}$ does not have a vertical tangent at the cusp $\mu_{-1/3}$.

The trace of the transformation which defines this cusp is

$$\operatorname{tr} W_{-1/3} = i(\mu^3 + 2\mu^2 + 3\mu + 2).$$

Let $\mu = x + iy$. The real locus of $W_{-1/3}$ is determined by the equation

$$f(x, y) = 2 + 3x + 2x^2 + x^3 - 2y^2 - 3xy^2 = 0.$$

This curve has vertical tangent only at points where

$$\partial_y f(x, y) = -2y(2 + 3x) = 0.$$

It is easy to check that this happens only at the point $\mu = -1$ which is outside $\mathscr{M}$. Thus, the cusp cardioid at $\mu_{-1/3}$ is tilted. The claim now follows from the above observations on cardioids. □

A perturbation argument using the convergence of the Bers–Maskit slices to the Maskit slice and the continuity of pleating rays would give the nonconvexity of the Bers–Maskit slices for small c. However, the following result shows that the result holds for all $c > 0$.

Proposition 6.2. *The Bers–Maskit slice $\mathscr{BM}_c$ is not vertically convex for any $c > 0$.*

Proof. We will follow the same method of proof as in Proposition 6.1. Let us consider the cusp $\mu_{-1/3} \in \partial\mathscr{BM}_c^-$. Using (6.2) or a direct computation (with the aid of symbolic computation software such as Mathematica) we get an expression for the trace in terms of the parameters μ and c (in the following formulas we denote hyperbolic cosine, sine and tangent by ch and sh and th to make the formulas more compact):

$$\frac{2\operatorname{sh}^3\frac{c}{2}}{\operatorname{ch}\frac{c}{2}}\operatorname{tr} W_{-1/3} =$$
$$2\operatorname{ch}\frac{c-3\mu}{2} + 8\operatorname{ch}\frac{c-\mu}{2} + \operatorname{ch}\frac{3(c-\mu)}{2} + 4\operatorname{ch}\frac{c+\mu}{2} + \operatorname{ch}\frac{c+3\mu}{2}.$$

Let $\mu = x + iy$. A calculation (again using appropriate software) shows that the imaginary part of $\operatorname{tr} W_{-1/3}$ is equal to

$$\frac{2\sin\frac{y}{2}}{\operatorname{th}^3\frac{c}{2}}\left(\operatorname{sh}\frac{3x-c}{2} + \frac{\left(3\operatorname{sh}\frac{x}{2} - \operatorname{ch}\frac{x}{2}\operatorname{th}\frac{c}{2}\right)}{\operatorname{ch}\frac{c}{2}} - 2\cos y\operatorname{sh}\frac{c-3x}{2}\right).$$

The real locus of $W_{-1/3}$ outside the locus $y = 0$ is therefore determined by the equation

$$\operatorname{sh}\frac{3x-c}{2} + \frac{\left(3\operatorname{sh}\frac{x}{2} - \operatorname{ch}\frac{x}{2}\operatorname{th}\frac{c}{2}\right)}{\operatorname{ch}\frac{c}{2}} - 2\cos y\operatorname{sh}\frac{c-3x}{2} = 0. \tag{6.3}$$

The dependence of the real locus on the parameter y is surprisingly simple! As in Proposition 6.1 it is enough to show that the common zeros of (6.3) and its partial derivative with respect to y,

$$2\sin y\operatorname{sh}\frac{c-3x}{2} = 0 \tag{6.4}$$

do not contain a point of the pleating ray $P_{-1/3}$ such that $y \neq 0$.

Suppose that $P_{-1/3}$ has a vertical tangent vector at a point (x, y) with $y \neq 0$. Then (6.4) implies $x = c/3$. Now, the value of the left hand side of (6.3) is

$$\frac{8\operatorname{sh}^3(c/6)}{\operatorname{ch}(c/6)(2\operatorname{ch}(c/3)-1)^2},$$

which is readily seen to be positive for all $c > 0$. Thus, the pleating ray does not have a vertical tangent outside the Fuchsian locus $\{y = 0\}$. This implies the claim. □

References

[1] Anderson, J. W., and R. D. Canary: Algebraic limits of Kleinian groups which rearrange the pages of a book. - Invent. Math. 126, 1996, 205–214.

[2] Anderson, J. W., R. D. Canary, and D. McCullough: The topology of deformation spaces of Kleinian groups. - Ann. of Math. 152, 2000, 693–741.

[3] Bers, L.: On boundaries of Teichmüller spaces and on Kleinian groups: I. - Ann. of Math. 91, 1970, 570–600.

[4] Bromberg, K. and J. Holt: Self-bumping of deformation spaces of hyperbolic 3-manifolds. - J. Differential Geom. 57, 2001, 47–65.

[5] Buser, B.: Geometry and spectra of compact Riemann surfaces. - Progr. Math. 106, Birkhäuser, 1992.

[6] Epstein, D. B. A., A. Marden, and V. Markovic: Quasiconformal homeomorphisms and the convex hull boundary. - Ann. of Math. 159, 2004, 305–336.

[7] Keen, L., and C. Series: Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori. - Topology 32, 1993, 719–749.

[8] Keen, L., and C. Series: How to bend pairs of punctured tori. - Lipa's legacy, Contemp. Math. 211, 359–387, Amer. Math. Soc., 1997.

[9] Keen, L., and C. Series: Pleating invariants for punctured torus groups. - Topology 43:2, 2004, 447–491.

[10] Komori, Y., and T. Sugawa: Bers embedding of the Teichmüller space of a once-punctured torus. - Conform. Geom. Dyn. 8, 2004, 115–142.

[11] Komori, Y., and Y. Yamashita: Linear slices of the quasifuchsian space of punctured tori. - Preprint, 2000.

[12] Kourouniotis, C.: Complex length coordinates for quasi-Fuchsian groups. - Mathematika 41, 1994, 173–188.

[13] Maskit, B.: Kleinian groups. - Springer-Verlag, 1988.

[14] Matsuzaki, K., and M. Taniguchi: Hyperbolic manifolds and Kleinian groups. - Oxford University Press, 1998.

[15] McMullen, C.: Complex earthquakes and Teichmüller theory. - J. Amer. Math. Soc. 11, 1998, 283–320.

[16] McShane, G., J. R. Parker, and I. Redfern: Drawing limit sets of Kleinian groups using finite state automata. - Experiment. Math. 3, 1994, 153–170.

[17] Minsky, Y.: The classification of punctured torus groups. - Ann. of Math. 149, 1999, 559–626.

[18] Miyachi, H.: On the horocyclic coordinate for the Teichmüller space of once-punctured tori. - Proc. Amer. Math. Soc. 130, 2002, 1019–1029.

[19] Miyachi, H.: Cusps in complex boundaries of one-dimensional Teichmüller space. - Conform. Geom. Dyn. 7, 2003, 103–151.

[20] Parker, J. R., and J. Parkkonen: Coordinates for quasi-Fuchsian punctured torus space. - Geom. Topol. Monogr. 1, 1998, The Epstein Birthday Schrift, 451–478.

[21] Parker, J. R., and C. Series: Bending formulae for convex hull boundaries. - J. Anal. Math. 67, 1995, 165–198.

[22] Parkkonen, J.: Geometric complex analytic coordinates for deformation spaces of Koebe groups. - Ann. Acad. Sci. Fenn. Ser. I A Math. Dissertationes 102, 1995.

[23] Parkkonen, J.: The outside of the Teichmüller space of punctured tori in Maskit's embedding. - Ann. Acad. Sci. Fenn. Math. 24, 1999, 305–342.

[24] Tan, S. P.: Complex Fenchel–Nielsen coordinates for quasi-Fuchsian structures. - Internat. J. Math. 5, 1994, 239–251.

[25] Wright, D. J.: The shape of the boundary of Maskit's embedding of the Teichmüller space of punctured tori. - Preprint, 1990.

Received 7 December 2005