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UNIFORM DOMAINS OF HIGHER ORDER III

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Abstract. We continue the study of (p,c)-uniform domains. Special emphasis is on (p,c)-NUD sets. The ambient space is assumed to be a Hilbert space.

1. Introduction

This paper is a continuation to [Al] and [AV], and we assume that the reader is familiar with these papers. They will be cited as I and II. For example, I.6.11 means the result 6.11 of [Al]. For basic notation and terminology, see I, p. 6, and II.1.4. However, we assume throughout this paper that the space E is *a Hilbert space* instead of a general normed space. This simplifies several proofs and also gives better estimates for various constants. The finite-dimensional case $E = R^n$ is probably the most interesting.

The main emphasis will be on the null-sets (NUD) for homotopically (htop) and homologically (hlog) (p,c)-uniform domains in E. In Section 2 we give elementary estimates for the uniformity constants of certain standard domains. Section 3 deals with cartesian products of NUD sets. The case where the set is contained in a linear subspace of E is studied in Section 4. In Section 5 we consider compact sets in infinite-dimensional spaces and an application of our theory to bilipschitz spheres.

We let $x \cdot y$ denote the inner product of two vectors $x, y \in E$. For $A \subset E$ and $r > 0$ we write

$$\overline{B}(A,r) = \{x : d(x,A) \le r\}, \qquad S(A,r) = \{x : d(x,A) = r\}.$$

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2. Estimates for uniformity constants

2.1. *Summary of Section* 2. We give estimates for the uniformity constants of various standard domains. All proofs are elementary. Remember that E is a Hilbert space. If E is infinite-dimensional, Theorem 5.2 will in certain cases give constants better than those obtained in this section.

2.2. **Lemma.** *Each bounded set $A \subset E$ is contained in a ball $\overline{B}(a,r)$ with $r = d(A)/\sqrt{2}$.*

Proof. If $\dim E = n < \infty$, this follows from Jung's theorem [Fe, 2.10.41]. Indeed, one can choose $r = d(A)\sqrt{n/2(n+1)}$. The infinite-dimensional case was proved by J. Daneš [Da, Th. 2]. □

2.3. **Theorem.** *Each ball and half space in E is htop and hlog (p,c)-uniform for all $0 \le p < \dim E - 1$ with $c = \sqrt{3/2} = 1.22474\ldots$.*

Proof. Suppose first that G is a ball, which can be assumed to be the unit ball $B(1)$. We show that G is htop (p,c)-uniform. Let $f\colon S^p \to G$ be continuous with $d = d(|f|) > 0$. If $d \ge 1$, we let $g\colon \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = 0$. The uniformity conditions are clearly satisfied with the constant 1. Assume that $d < 1$. By 2.2, there is a ball $B = \overline{B}(a, d/\sqrt{2})$ containing $|f|$. We may choose a so that $|a| \le 1$.

Case 1. $B \subset G$. Let $g\colon \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = a$. Each pair of points in $|g|$ lies in a triangle with side lengths at most d, and hence $d(|g|) \le d(|f|)$. If $x \in |g|$, then $x \in [a,y]$ for some $y \in |f|$, and we get

$$d(x,|f|) \le |x-y| \le d(x,\partial B) \le d(x,\partial G).$$

Again, the uniformity conditions hold with the constant 1.

Case 2. $B \not\subset G$. Choose $e \in \partial G$ with $a \in [0,e]$, and set $b = (1-d)e$. Then $|b| = 1-d < |a|$ and $B \cap G \subset \overline{B}(b, cd)$ with $c = \sqrt{3/2}$. Let $g\colon \overline{B}^{p+1} \to E$ be the cone extension of f with $g(0) = b$. Each pair of points in $|g|$ lies in a triangle with side lengths at most cd, and hence $d(|g|) \le cd$.

To prove the lens condition, let $x \in |g|$. Then $x = (1-t)y + tb$ for some $y \in |f|$ and $0 \le t \le 1$. We have

$$|x| \le (1-t)|y| + t|b| \le 1 - t + t(1-d) = 1 - td,$$

and hence $d(x,\partial G) = 1 - |x| \ge td$. This implies that

$$d(x,|f|) \le |x-y| = t|y-b| \le tdc \le cd(x,\partial G),$$

which is the lens condition.

Since the proof made use of solely cone extensions, it is valid with obvious modifications also in the homological case. The case where G is a half space follows from the above, since each compact set in G is contained in a ball $B \subset G$. □

2.4. **Theorem.** *Let B be a closed ball in E. Then the domain $G = E \setminus B$ is htop and hlog $(p, 5)$-uniform for all $0 \le p < \dim E - 1$.*

Proof. We may assume that B is the unit ball $\overline{B}(1)$. We show that G is htop $(p, 5)$-uniform. Let $f: S^p \to G$ be continuous with $d = d(|f|) > 0$. We consider two cases.

Case 1. $d \ge 1$. Set $r = \max\{|fx| : x \in S^p\}$. If $d < r - 1$, then $|f|$ is contained in a ball in G, and the desired extension of f is given by 2.3. Assume that $d \ge r - 1$. Set $R = r + \frac{1}{2}d$, and let $P: G \to S(R)$ be the radial projection $Px = Rx/|x|$. Since $p < \dim E - 1$, the map $Pf: S^p \to S(R)$ is null-homotopic. This homotopy and the segmental homotopy from f to Pf give an extension $g: \overline{B}^{p+1} \to G$ of f such that $|g|$ is contained in the union of $S(R)$ and all line segments $[x, Px]$, $x \in |f|$. Since

$$d(|g|) \le 2R = 2r + d \le 2(d + 1) + d \le 5d,$$

the turning condition holds with the constant 5. To prove the lens condition let $y \in |g|$. If $y \in [x, Px]$ for some $x \in |f|$, then

$$d(y, |f|) \le |y - x| \le |y| - 1 = d(y, B).$$

If $y \in S(R)$, then

$$d(y, |f|) \le |y| + r = 2r + \tfrac{1}{2}d.$$

Since $d(y, B) = r + \frac{1}{2}d - 1$, $d \ge 1$ and $r \ge 1$, we get $d(y, |f|) \le 5d(y, B)$, which is the lens condition.

Case 2. $d \le 1$. Applying 2.2 we choose a ball $\overline{B}(a, d/\sqrt{2}\,)$ containing $|f|$. In view of 2.3, we may assume that this ball meets B. Let L be the ray $\{ta : t \ge 0\}$, and write

$$R = 1 + d\sqrt{2}, \qquad C = \{x : d(x, L) \le d|x|/\sqrt{2}\,\}, \qquad F = C \cap S(R).$$

Let $P: G \to S(R)$ be the radial projection. There is an extension $g: \overline{B}^{p+1} \to G$ of f such that $|g|$ is contained in the union of F and all line segments $[x, Px]$, $x \in |f|$. It is easy to see that

$$|g| \subset C \cap \big(\overline{B}(R) \setminus B\big) \subset \overline{B}(w, 2d),$$

where w is the center of the sphere $S(R) \cap \partial C$. Hence $d(|g|) \le 4d$, which is the turning condition with the constant 4. To prove the lens condition let $y \in |g|$. The case where $y \in [x, Px]$ is treated as in Case 1. If $y \in F$, then

$$d(y, |f|) \le d(|g|) \le 4d = 2\sqrt{2}\, d(y, B).$$

Hence the lens condition holds with the constant $2\sqrt{2} < 5$. We have proved that G is htop $(p,5)$-uniform.

The proof for the homological case is essentially the same. Let z be a p-cycle in G and assume, for example, that $d(|z|) = d \le 1$. Choose again a ball $\overline{B}(a, d/\sqrt{2}\,)$ containing $|z|$ and assume that this ball meets B. With the notation of Case 2 above, $|z|$ is contained in the union A of F and all line segments $[x, Px]$, $x \in |z|$. Since A is contractible, $z = \partial g$ for some chain g in A. As in Case 2 above we see that g satisfies the uniformity conditions with the constant 5. □

2.5. **Theorem.** *The complement of a point $x_0 \in E$ is htop and hlog $(p,2)$-uniform for all $0 \le p < \dim E - 1$.*

Proof. We may assume that $x_0 = 0$. We prove only the homotopical case. Let $f\colon S^p \to G = E \setminus \{0\}$ be continuous with $d = d(|f|) > 0$, and set $r = \max\{|fx| : x \in S^p\}$. We may assume that $d \ge r$, since otherwise $|f|$ is contained in a ball in G and we can apply 2.3. Let $P\colon G \to S(r)$ be the radial projection. Then f has an extension $g\colon \overline{B}^{p+1} \to G$ such that $|g|$ is contained in the union of $S(r)$ and all line segments $[x, Px]$, $x \in |f|$. We have $d(|g|) \le 2r \le 2d$, which is the turning condition. Let $y \in |g|$. If $y \in [x, Px]$ for some $x \in |f|$, then $d(y, |f|) \le |x - y| \le |y| = d(y, \partial G)$. If $|y| = r$, then $d(y, |f|) \le 2r = 2d(y, \partial G)$. □

2.6. *Weak uniformity.* We recall that an open set $U \subset E$ is weakly hlog (p,c)-uniform if the uniformity conditions hold for every null-homologous p-cycle in U. Similarly, U is *weakly htop (p,c)-uniform* if each null-homotopic $f\colon S^p \to U$ has an extension $g\colon \overline{B}^{p+1} \to U$ satisfying the uniformity conditions.

2.7. **Theorem.** *Let T be a closed proper linear subspace of E. Then $U = E \setminus T$ is weakly htop and hlog $(p, \sqrt{5}\,)$-uniform for all $0 \le p < \dim E - 1$.*

Proof. We prove the homotopical case. Let $P\colon E \to T$ and $N\colon E \to T^{\perp}$ denote the orthogonal projections. Suppose that $f\colon S^p \to U$ is null-homotopic. We may assume that $0 \in P|f|$. We may also assume that $|Nfx| \le d$ for all $x \in S^p$, since otherwise $|f|$ is contained in a ball in U, and the desired extension is given by 2.3. Let $S(d)$ be the sphere $|x| = d$ in $T^{\perp}$, and let $Q\colon T^{\perp} \setminus \{0\} \to S(d)$ be the radial projection $Qx = dx/|x|$. For $x \in U$ consider the line segments $I_x = [x, Px + QNx]$ and $J_x = [Px + QNx, QNx]$. Then U deformation retracts along these segments onto $S(d)$. Thus $QNf\colon S^p \to S(d)$ is null-homotopic, and we obtain an extension $g\colon \overline{B}^{p+1} \to U$ of f such that $|g|$ is contained in the union of $S(d)$ and all I_x and J_x, $x \in |f|$.

Suppose that $y, y' \in |g|$. Then $Py \in [Px, 0]$ and $Py' \in [Px', 0]$ for some $x, x' \in |f|$. Since $0 \in P|f|$, we get $|Py - Py'| \le d(P|f|) \le d$. Since $|Ny - Ny'| \le 2d$, this implies the turning condition $d(|g|) \le d\sqrt{5}$.

To prove the lens condition let $y \in |g|$. If $y \in I_x$ for some $x \in |f|$, the lens condition holds with the constant 1. If $y \in J_x$, then

$$d(y, |f|) \le |x - y| \le d\sqrt{2} = \sqrt{2}\, d(y, T).$$

If $y \in S(d)$ and $x \in |f|$, then

$$|x - y|^2 = |Px - Py|^2 + |Nx - Ny|^2 \le d^2 + 4d^2 = 5d(y, T)^2.$$

Hence the lens condition holds with the constant $\sqrt{5}$. □

2.8. **Theorem.** *Let* T *be a closed linear subspace of* E *with* $\dim T \ge 1$, *and let* $S = S(1) \cap T$ *be the unit sphere of* T. *Then* $G = E \setminus S$ *is weakly htop and hlog* $(p, 5)$*-uniform for all* $0 \le p < \dim E - 1$.

Proof. We remark that the result follows with a larger constant from 2.7 and from the Möbius invariance of uniformity.

Let $P\colon E \to T$ and $N\colon E \to T^\perp$ be the orthogonal projections. For $x \in E \setminus T^\perp$ we have $x = Px + Nx = re + Nx$ for some $r > 0$ and $e \in S$, which are uniquely determined by x. We consider $T^\perp \times \mathbf{R}$ as a Hilbert space in the natural way, and let H denote the half space $T^\perp \times [0, \infty)$. Then $u = (Nx, r) \in H$. The pair (u, e) gives the *polar coordinates* of x with respect to T. If $x \in T^\perp$, then $u \in \partial H$ but e is undefined. We write $u_0 = (0, 1) \in H$. In polar coordinates we have $S = \{(u, e) : u = u_0\}$.

We prove that G is htop $(p, 5)$-uniform. Let $f\colon S^p \to G$ be null-homotopic with $d = d(|f|) > 0$. We may assume that $|f| \subset \overline{B}(S, d) = \{x : d(x, S) \le d\}$ and that $|f|$ meets $\overline{B}(1)$, since otherwise we can apply either 2.3 or 2.4. We consider three cases.

Case 1. $d \ge 1$. Now $|f| \subset \overline{B}(1 + d) \subset \overline{B}(2d)$. For $x \in B(S, 1) \setminus S$ let Qx be the point in $S(S, 1)$ closest to x. Explicitly, we can write in polar coordinates

$$Q(u, e) = \Big(u_0 + \frac{u - u_0}{|u - u_0|}, e\Big).$$

We extend Q to $E \setminus S$ by setting $Qx = x$ for $x \in A = \overline{B}(2d) \setminus B(S, 1)$ and $Qx = 2dx/|x|$ for $|x| \ge 2d$. Then $E \setminus S$ deformation retracts onto A along the line segments $[x, Qx]$. Hence Qf is null-homotopic in A, and we obtain an extension $g\colon \overline{B}^{p+1} \to G$ of f such that $|g|$ is contained in the union of A and all segments $[x, Qx]$, $x \in |f|$. Then $d(|g|) \le d\big(\overline{B}(2d)\big) = 4d$, which gives the turning condition.

To prove the lens condition let $y \in |g|$. The case $y \in [x, Qx]$ is again clear. Suppose that $y \in A$. Since $|f|$ meets $\overline{B}(1)$, we have $d(y, |f|) \le |y| + 1$. If $|y| \le 2$, this implies that $d(y, |f|) \le 3 \le 3d(y, S)$. If $|y| \ge 2$, we also get $d(y, |f|) \le |y| + 1 \le 3(|y| - 1) \le 3d(y, S)$.

Case 2. $1/\sqrt{2} \le d \le 1$. We use the same idea as in Case 1, but we let now $A = \overline{B}(1+d) \setminus B(S,d)$. We get an extension g with $d(|g|) \le 2+2d \le (2\sqrt{2}+2)d < 5d$. If $y \in A$, then

$$d(y,|f|) \le |y| + 1 \le 2 + d \le (2\sqrt{2}+1)d < 4d \le 4d(y,S).$$

Case 3. $d \le 1/\sqrt{2}$. Applying 2.2 we choose a ball $\overline{B}(a, d/\sqrt{2}\,)$ containing $|f|$. We may assume that there is a point $b \in S \cap \overline{B}(a, d/\sqrt{2}\,)$, since otherwise $|f|$ is contained in a ball in G. Then $|f| \subset \overline{B}(b, d\sqrt{2}\,)$. Set $\alpha = \arcsin d\sqrt{2}$ and

$$Z = \{(u,e) : |u - u_0| \le d,\ \operatorname{ang}(e,b) \le \alpha\}$$

in polar coordinates; here $\operatorname{ang}(e,b)$ is the angle between the unit vectors e and b. In the case $E = R^3$, $T = R^2$, Z is obtained from the solid torus $\overline{B}(S,d)$ by cutting it by two half planes whose boundary is the x_3-axis and which touch the ball $\overline{B}(b, d\sqrt{2}\,)$.

We have $|f| \subset \big(\overline{B}(S,d) \cap \overline{B}(b, d\sqrt{2}\,)\big) \setminus S \subset Z \setminus S$. Since $\dot{E} \setminus S$ deformation retracts onto $Z \setminus S$, f is null-homotopic in $Z \setminus S$. Let Q: $\overline{B}(S,d) \setminus S \to S(S,d)$ be as in the previous cases. Then $Z \setminus S$ deformation retracts onto $W = S(S,d) \cap Z$ along the segments $[x, Qx]$, and Qf is null-homotopic in W. Hence we obtain an extension g: $\overline{B}^{p+1} \to G$ of f such that $|g|$ is contained in the union of W and all line segments $[x, Qx]$, $x \in |f|$. We show that g satisfies the uniformity conditions.

The set $S \cap Z$ is a cap of S with diameter $2 \sin \alpha = 2d\sqrt{2}$. Since $Z \subset \overline{B}(S \cap Z, d)$, this gives the turning condition $d(|g|) \le d(Z) \le 2d\sqrt{2} + 2d < 5d$.

To prove the lens condition, it suffices to consider the case $y \in W$, and then $d(y,|f|) \le d(Z) < 5d = 5d(y,S)$. □

2.9. **Theorem.** *Let Z be the infinite cylinder $B(1) \times \mathbf{R} \subset E \times \mathbf{R}$. Then Z is htop and hlog $(p, \sqrt{2}\,)$-uniform for all $1 \le p < \dim E - 1$.*

Proof. Of course, we use the natural inner product in $E \times \mathbf{R}$; see 3.5. Let f: $S^p \to Z$ be continuous with $d = d(|f|) > 0$. If $d \le 1$, we choose a ball $B = B(a, d/\sqrt{2}\,)$ in $E \times \mathbf{R}$ containing $|f|$. We may assume that B does not lie in Z and that $a \in E \times \{0\}$ with $|a| \le 1$. The desired extension of f is obtained by coning with vertex at $(1-d)a$. If $d \ge 1$, let Q: $E \times \mathbf{R} \to \{0\} \times \mathbf{R}$ be the projection. The extension is now obtained by using a segmental homotopy from f to Qf and a homotopy of Qf to a point in the segment $Q|f|$. We omit the estimates for the uniformity constants, because they are rather similar to those in the previous theorems. In the homological case we can apply similar constructions to each component of the carrier of a given p-cycle. □

2.10. *Remark.* Observe that Z is not (0)-uniform.

3. Product sets

3.1. *Summary of Section* 3. Recall that an open set U is p-acyclic if it has trivial (reduced) homology groups $H_k(U)$ for $0 \le k \le p$. Suppose that $A \subset R^m$ and $B \subset R^n$ are closed sets without interior points. Assume that $R^m \setminus A$ is p-acyclic and that $R^n \setminus B$ is q-acyclic for some $p \le m-2$ and $q \le n-2$. It follows from the Alexander duality and from the Künneth formula for cohomology that $R^{m+n} \setminus (A \times B)$ is $(p+q+2)$-acyclic. In fact, one can show that it is also simply connected and hence, by the Hurewicz theorem, it is even $(p+q+2)$-connected. In this section we shall consider quantitative versions of this and related results.

3.2. *The homotopical case.* Recall that a closed set $A \subset E$ is htop (p,c)-porous if for all $r > 0$ and for all maps $f\colon \Delta^p \to E$ there is a map $g\colon \Delta^p \to E$ such that $\|g - f\| \le r$ and $d(|g|, A) \ge r/c$. This implies that A is htop (q,c)-porous for all $0 \le q \le p$; see I.3.3.3.

We shall prove in 3.5 that the product of a htop (p)-porous set and a htop (q)-porous set is htop $(p+q+1)$-porous. We need the following result, which shows that one can replace Δ^p in the definition of porosity by any p-dimensional polyhedron if the constant c is allowed to change.

3.3. **Lemma.** *Suppose that $A \subset E$ is htop (p,c)-porous and that P is a p-dimensional polyhedron. Suppose also that $f\colon P \to E$ is continuous and that $r > 0$. Then there is a map $g\colon P \to E$ such that*
(1) $\|g - f\| \le r$,
(2) $d(|g|, A) \ge r/c_1$,
where $c_1 = c_1(c,p)$.

Proof. Choose a triangulation K of P such that $d(f\Delta) \le r' = r/6^p$ for every $\Delta \in K$. Let K^k be the k-skeleton of K and set $f_k = f \mid |K^k|$. Since A is htop $(0,c)$-porous, we can define $g_0\colon |K^0| \to E$ such that $|g_0 v - f v| \le r'$ and $d(g_0 v, A) \ge r'/c$ for each vertex v of K. Proceeding inductively, assume that $0 \le k < p$ and that we have defined a map $g_k\colon |K^k| \to E$ satisfying the following two conditions:

$$(\alpha_k) \qquad \|g_k - f_k\| \le 6^k r',$$

$$(\beta_k) \qquad d(|g_k|, A) \ge r'/c(c+2)^k.$$

Let $\Delta \in K$ be a $(k+1)$-simplex. Using the cone construction we extend $g_k \mid \partial\Delta$ to a map $u_\Delta\colon \Delta \to E$ such that $d(|u_\Delta|) = d(g_k \partial\Delta)$. Applying I.3.4 and (β_k) we find an extension $g_\Delta\colon \Delta \to E$ of $g_k \mid \partial\Delta$ such that

$$\|g_\Delta - u_\Delta\| \le r'/(c+2)^{k+1}, \qquad d(|g_\Delta|, A) \ge r'/c(c+2)^{k+1}.$$

Setting $g_{k+1} \mid \Delta = g_\Delta$ we obtain a map $g_{k+1}\colon |K^{k+1}| \to E$ satisfying (β_{k+1}). To prove (α_{k+1}), observe that

$$\begin{aligned} d(|g_\Delta|) &\le d(|u_\Delta|) + 2\|g_\Delta - u_\Delta\| \le d(g_k\partial\Delta) + 2r'/(c+2)^{k+1} \\ &\le d(f\partial\Delta) + 2\|f_k - g_k\| + r' \le r' + 2\cdot 6^k r' + r' \le 4\cdot 6^k r'. \end{aligned}$$

Let $x \in \Delta$ and choose a vertex y of Δ. Then

$$\begin{aligned} |g_{k+1}x - fx| &\le |g_{k+1}x - g_{k+1}y| + |g_{k+1}y - fy| + |fy - fx| \\ &\le d(|g_\Delta|) + \|g_0 - f_0\| + d(f\Delta) \le 4\cdot 6^k r' + r' + r' \le 6^{k+1} r'. \end{aligned}$$

Hence (α_{k+1}) is true.

The last step gives a map $g = g_p\colon P \to E$ satisfying (1) and (2) with $c_1 = 6^p c(c+2)^p$. □

3.4. *Remark.* We consider the special case of 3.3 where $\dim E \ge p+1$ and A consists of a single point, say $A = \{0\}$. Then g can be constructed directly as follows: Let $0 < \varepsilon < r$. Choose a piecewise linear map $f_1\colon P \to E$ with $\|f_1 - f\| < \varepsilon$. By general position, we may assume that $0 \notin |f_1|$. Let P be the radial retraction of $E \setminus \{0\}$ onto $E \setminus B(r-\varepsilon)$. Then $g = Pf_1$ satisfies (1) and (2) of 3.3 with a constant c_1 arbitrarily close to one. In particular, c_1 does not depend on p.

3.5. *Preparations.* In the product $E_1 \times E_2$ of two inner products spaces we use the inner product

$$(x,y)\cdot(x',y') = x\cdot x' + y\cdot y'.$$

We recall the concept of a *dual skeleton.* Let K be a finite simplicial complex of dimension n. For $0 \le p \le n$, the dual skeleton $D(K^p)$ of the p-skeleton K^p of K is the subcomplex of the barycentric subdivision $\operatorname{Sd} K$ of K consisting of all simplexes that do not meet $|K^p|$. The dimension of $D(K^p)$ is $n-p-1$. Thus $D(K^n) = \emptyset$, and $D(K^{n-1})$ is the finite set of the barycenters $b(\Delta)$ of all n-simplexes $\Delta \in K$. If Δ is an n-simplex of K, the $(n-p-1)$-simplexes of $D(K^p)$ contained in Δ are of the form $[b(\sigma_{p+1}), \dots, b(\sigma_n)]$, where $\sigma_n = \Delta$ and each σ_i is an i-face of σ_{i+1}. The most important property of the dual skeleton is that given $0 \le p \le n$, each n-simplex $\Delta' \in \operatorname{Sd} K$ can be uniquely written as $\Delta' = \sigma * \tau$, where $\sigma \in \operatorname{Sd} K^p$ and $\tau \in D(K^p)$. Here $\sigma * \tau$ denotes the join of two joinable simplexes σ and τ; see [Ru, p. 10].

3.6. **Theorem.** *Let* $A_1 \subset E_1$ *be htop* (p,c)*-porous and let* $A_2 \subset E_2$ *be htop* (q,c)*-porous. Then* $A_1 \times A_2$ *is htop* $(p+q+1, c_1)$*-porous in* $E = E_1 \times E_2$ *with* $c_1 = c_1(c,p,q)$.

Proof. Let $r > 0$ and let $f\colon \Delta^{p+q+1} \to E$ be a map. Choose a triangulation K of Δ^{p+q+1} such that $d(f\Delta) \le \frac{1}{4}r$ for all $\Delta \in K$. Write $f = (f_1, f_2)$ with $f_i\colon \Delta^{p+q+1} \to E_i$, $P_1 = |K^p|$ and $P_2 = |D(K^p)|$. By 3.3 we find maps $h_i\colon P_i \to E_i$ such that

$$\|h_i - f_i|P_i\| \le r/4, \qquad d(|h_i|, A_i) \ge r/4c_2$$

for $i = 1, 2$, where $c_2 = c_2(c,p,q)$.

Let Δ be a $(p+q+1)$-simplex of $\operatorname{Sd} K$. Then Δ can be uniquely expressed as $\sigma * \tau$, where $\sigma \in \operatorname{Sd} K^p$ and $\tau \in D(K^p)$. Each point $z \in \Delta$ can be written as $z = (1-t)x + ty$ with $x \in \sigma$, $y \in \tau$, $0 \le t \le 1$. The representation is unique except for $z \in \sigma \cup \tau$, which occurs only if $t = 0$ or $t = 1$. We define a map $g_\Delta\colon \Delta \to E$ as follows: For $x \in \sigma$, $y \in \tau$ we set

$$g_\Delta x = (h_1 x, f_2 x), \qquad g_\Delta y = (f_1 y, h_2 y), \qquad g_\Delta\big(\tfrac{1}{2}(x+y)\big) = (h_1 x, h_2 y).$$

On the line segments $[x, \frac{1}{2}(x+y)]$ and $[\frac{1}{2}(x+y), y]$, g_Δ is defined to be affine. Explicitly, for $z = (1-t)x + ty$ we set

$$g_\Delta z = \begin{cases} \big(h_1 x, (1-2t) f_2 x + 2t h_2 y\big) & \text{if } 0 \le t \le \frac{1}{2}, \\ \big((2-2t) h_1 x + (2t-1) f_1 y, h_2 y\big) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The maps g_Δ clearly define a map $g\colon \Delta^{p+q+1} \to E$. We show that this is the desired map.

Let $z = (1-t)x + ty$ be as above. If $0 \le t \le \frac{1}{2}$, then

$$\begin{aligned} |h_1 x - f_1 z| &\le |h_1 x - f_1 x| + |f_1 x - f_1 z| \le \tfrac{1}{4}r + d(f\Delta) \le \tfrac{1}{2}r, \\ |(1-2t) f_2 x + 2t h_2 y - f_2 z| &\le (1-2t)|f_2 x - f_2 z| + 2t|h_2 y - f_2 z| \\ &\le (1-2t) d(f\Delta) + 2t|h_2 y - f_2 y| + 2t|f_2 y - f_2 z| \\ &\le (1-2t) d(f\Delta) + \tfrac{1}{4}r + 2t d(f\Delta) \le \tfrac{1}{2}r. \end{aligned}$$

These inequalities imply that $|gz - fz| \le r/\sqrt{2} < r$. Similar arguments show that this is true also if $\frac{1}{2} \le t \le 1$. Thus $\|g - f\| \le r$.

Consider again the point $z = (1-t)x + ty \in \Delta = \sigma * \tau$. If $0 \le t \le \frac{1}{2}$, we have

$$d(gz, A_1 \times A_2) \ge d(h_1 x, A_1) \ge r/4c_2.$$

If $\frac{1}{2} \le t \le 1$, then

$$d(gz, A_1 \times A_2) \ge d(h_2 y, A_2) \ge r/4c_2.$$

Hence $d(|g|, A_1 \times A_2) \ge r/4c_2$, and we have proved the theorem with $c_1 = 4c_2 = 4 \cdot 6^s c(c+2)^s$, $s = p \vee q$. □

Recall from II.1.4 that a set $A \subset E$ has a property $Q(p)$ involving an integer $p \geq 0$ *completely* if A has property $Q(k)$ for all $0 \leq k \leq p$. For example, A is completely htop (p,c)-NUD if A is htop (k,c)-NUD for all $0 \leq k \leq p$. By I.4.9, this is quantitatively equivalent to htop $(p+1,c)$-porosity. Hence Theorem 3.6 has the following corollary:

3.7. **Theorem.** *Let $A_1 \subset E_1$ be completely htop (p,c)-NUD and let $A_2 \subset E_2$ be completely htop (q,c)-NUD. Then $A_1 \times A_2$ is completely htop $(p+q+2, c_1)$-NUD in $E_1 \times E_2$ with $c_1 = c_1(c,p,q)$.* □

3.8. *Remark.* We consider the special case of 3.6 where $\dim E_2 \geq q+1$ and A_2 consists of a single point, say $A_2 = \{0\}$. Let $c_0 > 1$. By 3.4 we can choose the map h_2 in the proof of 3.6 so that $d(|h_2|, 0) \geq r/c_0$. It follows that $A_1 \times \{0\}$ is htop $(p+q+1, c_1)$-porous in E with $c_1 = c_1(c,p) = 4 \cdot 6^p c(c+2)^p$.

3.9. *The homological case.* It is natural to conjecture that 3.6 and 3.7 are true if htop is replaced by hlog. In fact, we believe that they are true in the stronger form where the hlog properties of A_1 and A_2 imply the corresponding htop property of $A_1 \times A_2$. We prove this for finite-dimensional spaces:

3.10. **Theorem.** *Let $A_1 \subset R^m$ be completely hlog (p,c)-NUD and let $A_2 \subset R^n$ be completely hlog (q,c)-NUD. Then $A_1 \times A_2$ is completely htop $(p+q+2, c_1)$-NUD in R^{m+n} with $c_1 = c_1(c,m,n)$.*

Proof. The proof is based on compact families of sets, and it does not give an explicit estimate for the constant c_1. Since the proof is rather long, we give only a sketch. The topological fact behind the proof is the inequality $\dim(A_1 \times A_2) \leq \dim A_1 + \dim A_2$ for the topological dimension.

Part 1. The compactness theory of [Vä$_2$] cannot be directly applied, because we cannot allow rotations in $R^m \times R^n$. We must therefore rewrite part of this theory replacing the group sim of all similarities of R^n by the subgroup sim^* consisting of all maps $f\colon R^n \to R^n$ of the form $fx = \lambda x + a$, $\lambda > 0$, $a \in R^n$.

Fix an integer $n \geq 1$. As before, we let K^n denote the family of all nonempty compact subsets of $\dot{R}^n$, and we write $K^n_\infty = \{A \in K^n : \infty \in A\}$. If $L \subset K^n$, we set $L^* = \{A \in L : 0 \in \partial A \text{ and } S^{n-1} \cap \partial A \neq \emptyset\}$. We say that L is $*$-stable if $\mathrm{sim}^* L = L$ and if L^* is compact. For $H \subset K^n$, we let $\sigma^*(H)$ denote the union of all $*$-stable subfamilies of H. Compared with the theory of [Vä$_2$], a stable family is always $*$-stable, and we have $\sigma(H) \subset \sigma^*(H)$. A $*$-filtration of $\sigma^*(H)$ is a function $c \mapsto L_c$, defined for $c \geq 1$, such that

(1) $c < d$ implies $L_c \subset L_d$,
(2) each L_c is contained in a $*$-stable subfamily of H,
(3) each $*$-stable subfamily of H is contained in some L_c.

In this case, $\sigma^*(H) = \cup\{L_c : c \geq 1\}$.

We next give the following modification of I.6.4: Let $n \geq 2$, $0 \leq p \leq n-1$, and let $M \subset K^n_\infty$ be a $*$-stable family such that the reduced Čech cohomology groups $H^{n-p-1}(A)$ and $H^{n-1}(A)$ are trivial for all $A \in M$. Then for each $c \geq 1$ there is $c_1 = c_1(c, M) \geq 1$ such that $R^n \setminus A$ satisfies the condition $\mathrm{HT}^*(p, c, c_1)$ of II.3.4 for all $A \in M$. The proof of this is obtained by modifying the proof of I.6.4.

Using this result and II.3.10, we can show as in I.6.6 that, with the notation of II.3.16, we have $\sigma^*(D_p) = \mathrm{HU}(p)$ and that $c \mapsto \mathrm{HU}(p, c)$ is a $*$-filtration of $\sigma^*(D_p)$. Furthermore, with the notation of II.3.16 we have $\sigma^*(L_{n-p-2}) = \mathrm{HN}(p)$, and $c \mapsto \mathrm{HN}(p)$ gives a $*$-filtration of $\sigma^*(L_{n-p-2})$. Indeed, I.6.10 implies that $\mathrm{HN}(p, c) \subset \sigma^*(L_{n-p-2})$, and the opposite inclusion follows rather easily from [HW, VIII 4F]. With the convention of II.3.10, the statement is also valid for $p = -1$; then one may have $n = 1$.

Part 2. We turn to the proof of 3.10. Since htop and hlog (0)-uniformity coincide, the case $p \leq 0$, $q \leq 0$ is a special case of 3.7. We may thus assume that $p \geq 1$. From 3.7 it follows that $A_1 \times A_2$ is htop $(1, c_1)$-NUD. By the quantitative Hurewicz theorem II.4.5, it suffices to prove that $A_1 \times A_2$ is completely hlog $(p + q + 2, c_2)$-NUD with c_2 depending only on (c, m, n).

Let M be the family of all sets $A_1 \times A_2$ where A_1 and A_2 satisfy the hypotheses of 3.10. It is rather easy to see that $\overline{M}$ is $*$-stable. We next show that $\dim F \leq m + n - p - q - 4$ for all $F \in \overline{M}$. Indeed, F is of the form $B_1 \times B_2$, where B_1 and B_2 are limits of sets satisfying the hypotheses of 3.10. From I.6.3 and from I.6.10 it follows that $\dim B_1 \leq m - p - 2$ and $\dim B_2 \leq n - q - 2$. By [HW, p. 33] this implies that $\dim F \leq \dim B_1 + \dim B_2 \leq m + n - p - q - 4$.

We have proved that $\overline{M}$ is a $*$-stable family in $L_{m+n-p-q-4}$. By Part 1, M is contained in some $\mathrm{HN}(p + q + 2, c_2)$, and the theorem is proved. □

Theorem 3.10 can also be expressed in terms of hlog porosity, defined in II.6.2. By II.6.15, the complete hlog (c, p)-NUD property is quantitatively equivalent to complete hlog $(c, p+1)$-porosity. In view of I.4.9, Theorem 3.10 has the following corollary:

3.11. **Theorem.** *Let $A_1 \subset R^n$ be completely hlog (p, c)-porous and let $A_2 \subset R^n$ be completely hlog (q, c)-porous. Then $A_1 \times A_2$ is htop $(p + q + 1, c_1)$-porous in R^{m+n} with $c_1 = c_1(c, m, n)$.* □

4. Subsets and changing dimension

4.1. *Summary of Section* 4. We first show that the htop and hlog porosity and complete uniformity properties of a set $A \subset E$ are inherited by closed subsets. Next we consider the case where A is contained in a linear subspace E_1 of E, and we compare the properties of A in E_1 and in E. As a corollary we obtain results on planar sections of A.

4.2. **Theorem.** *Suppose that A is htop or hlog (p,c)-porous in E. Then each closed subset of A has the same property.*

Proof. This follows at once from the definition of porosity. □

4.3. **Theorem.** *Suppose that A is completely htop or hlog (p,c)*-NUD *in E. Then each closed $B \subset A$ is completely htop or hlog (p,c')*-NUD *for all $c' > c$, respectively.*

Proof. We remark that in view of I.4.9 and II.6.15, we obtain from 4.2 the weaker result where c' depends on p and c.

We first consider the homotopical case. If suffices to show that B is htop (p,c')-NUD. Let $f\colon S^p \to E \setminus B$ be a map and let $\varepsilon > 0$. Since A is completely htop (p,c)-NUD, A is LC^p rel E; see II.5.2 and II.5.3.1. By II.5.4 there is a map $f_1\colon S^p \to E \setminus A$ with $\|f_1 - f\| < \varepsilon$. Since A is htop (p,c)-NUD, there is an extension $g_1\colon \overline{B}^{p+1} \to E \setminus A$ of f_1 satisfying the uniformity conditions in $E \setminus A$. Let $e \in S^p$ and $0 \le t \le 1$. Setting

$$g(te) = \begin{cases} (2t-1)f(e) + (2-2t)f_1(e) & \text{for } \frac{1}{2} \le t \le 1, \\ g_1(2te) & \text{for } 0 \le t \le \frac{1}{2}, \end{cases}$$

we obtain an extension $g\colon \overline{B}^{p+1} \to E \setminus B$ of f. It is easy to see that g satisfies the uniformity conditions in $E \setminus B$ with a constant $c' = c'(c,\varepsilon)$ such that $c' \to c$ as $\varepsilon \to 0$. This proves the homotopical case of the theorem.

The homological case is proved similarly, using II.5.10 instead of II.5.4. Indeed, let z be a p-cycle in E and let $\varepsilon > 0$. Then II.5.10 gives $z_1 \in Z_p\bigl(\overline{B}(|z|,\varepsilon) \setminus A\bigr)$ and $g_1 \in S_{p+1}\bigl(\overline{B}(|z|,\varepsilon)\bigr)$ such that $\partial g_1 = z - z_1$. Since A is hlog (p,c)-NUD, $z_1 = \partial g_2$ for some g_2 satisfying the uniformity conditions in $E \setminus A$. Then $z = \partial(g_1 + g_2)$, and $g_1 + g_2$ satisfies the uniformity conditions in $E \setminus B$ with a constant c' arbitrarily close to c. □

4.4. *Raising dimension.* Suppose that E_1 is a closed linear subspace of E and that A is a closed subset of E_1. We want to compare the properties of A in E_1 and in E. It turns out that, roughly speaking, raising the dimension by one improves the order of porosity and NUD by one. An early example of this phenomenon was given in 1982 by S. Granlund, P. Lindqvist and O. Martio [GLM, 4.18], who observed that a closed set $A \subset S^n$ is (0)-porous if and only if it is (0)-NUD in R^{n+1}.

The converse problem is discussed in 4.6.

4.5. **Theorem.** *Suppose that E is a Hilbert space and that E_1 is a closed linear subspace of E of codimension at least $k \ge 1$. Let A be closed in E_1.*

(1) *If A is htop (p,c)-porous in E_1, then A is htop $(p+k,c_1)$-porous in E with $c_1 = c_1(c,p)$.*

(2) *If A is completely htop (p,c)-NUD in E_1, then A is completely htop $(p+k,c_1)$-NUD in E with $c_1=c_1(c,p,k)$.*

Suppose also that $\dim E = n < \infty$.

(3) *If A is completely hlog (p,c)-porous in E_1, then A is htop $(p+k,c_1)$-porous in E with $c_1=c_1(c,n)$.*

(4) *If A is completely hlog (p,c)-NUD in E_1, then A is completely htop $(p+k,c_1)$-NUD in E with $c_1=c_1(c,n)$.*

Proof. Writing $E_2 = E_1^{\perp}$ we can identify E with $E_1\times E_2$ and A with $A\times\{0\}$. Now (1) follows directly from Remark 3.8, and (2) follows from (1) and from I.4.9. The statements (3) and (4) follow from 3.10 and 3.11, respectively, since $\{0\}$ is completely hlog $(k-1)$-porous in R^k by 2.5 and II.6.15. □

4.6. *Lowering dimension.* Let us consider the situation $A\subset E_1\subset E$ as in 4.4 and in 4.5. We ask whether the properties of A in E imply lower-order properties of A in E_1. It turns out that in the homotopical case the answer is negative. This will be shown by counterexamples 4.12 and 4.13, but a partially positive answer will be given in 4.14. The next result gives a positive answer in the homological case. In view of an application in 5.5, we give the result also in terms of weak uniformity; see 2.6.

By a *hyperplane* in a Hilbert space E we mean the orthogonal complement $b^{\perp}=\{x : x\cdot b = 0\}$ of any nonzero vector $b\in E$.

4.7. **Theorem.** *Suppose that T is a hyperplane in E and that $A\subsetneq T$ is [weakly] hlog (p,c)-NUD in E with $p\ge 0$. If $p\ge 1$, then $T\setminus A$ is [weakly] hlog $(p-1,c_1)$-uniform with any $c_1>c\sqrt{2}$; if $p=0$, then A is $2c^2$-porous in T.*

Proof. Write $T=e^{\perp}$ with $|e|=1$ and $H_1=\{x\in E : x\cdot e>0\}$, $H_2=\{x\in E : x\cdot e<0\}$. As in [Do$_2$, Prop. 1] we can make use of a Mayer–Vietoris sequence to obtain an isomorphism $H_{p-1}(T\setminus A)\to H_p(E\setminus A)$. Hence it suffices to prove the weak version.

Suppose first that $p=0$, and let $x\in T$, $r>0$. We may assume that $x=0$. Writing $t=\sqrt{4c^2-1}$, $a=re/t$, $b=-re/t$, we can find an arc $\gamma\subset E\setminus A$ joining a to b and satisfying the c-uniformity conditions in $E\setminus A$. Choose a point $y\in\gamma\cap T$. Since $|y|^2+(r/t)^2\le(2rc/t)^2$ by the turning condition, we have $|y|\le r$. On the other hand, the cigar condition implies that $d(y,A)\ge|y-a|/c\ge r/ct>r/2c^2$. Thus A is $2c^2$-porous in T.

Suppose then that $p\ge 1$. Assume that $z\in Z_{p-1}(T\setminus A)$ with $d(|z|)=d>0$ and that $z\sim 0$ in $T\setminus A$. By 2.2, $|z|$ is contained in a ball $\overline{B}(a,s)$ with $a\in T$ and $s=d/\sqrt{2}$. Setting $b_1=se$ and $b_2=-se$ we consider the cones $z_1=b_1\cdot z$ and $z_2=b_2\cdot z$; see [Do$_1$, III.(4.7)]. Then $\partial z_1=\partial z_2=z$ by [Do$_1$, III.(4.8)]. Since $z\sim 0$ in $T\setminus A$, we have $z=\partial h$ for some $h\in S_p(T\setminus A)$. Setting $z'=z_1-z_2$ and $h'=b_2\cdot h-b_1\cdot h$ we get $\partial h'=h-b_2\cdot z-h+b_1\cdot z=z'$, and thus $z'\sim 0$ in $E\setminus A$.

Since A is weakly hlog (p,c)-NUD, we have $z'=\partial g'$ for some $g'\in S_{p+1}(E\setminus A)$ satisfying the uniformity conditions

$$\begin{aligned} d(|g'|) &\le cd(|z'|)\le 2cs = cd\sqrt{2},\\ d(x,|z'|) &\le cd(x,A) \end{aligned} \tag{4.8}$$

for all $x\in|g'|$.

Let $\varepsilon>0$ and set $B=\overline{B}(|g|\cap T,\varepsilon)\cap T$. We first show that z bounds in B. Choose a positive integer j such that $d(|\sigma|)<\varepsilon$ for all $\sigma<\mathrm{Sd}^j g'$; for notation, see II.1.4. Next choose $0<\delta<\varepsilon$ such that $d(|\sigma|,T)>\delta$ for all $\sigma<\mathrm{Sd}^j g'$ with $|\sigma|\cap T=\emptyset$. Finally, choose an integer $k\ge j$ such that $d(|\tau|)\le\delta$ for all $\tau<\mathrm{Sd}^k g'$.

Define a map $\varphi\colon \mathbf{R}\to\mathbf{R}$ by $\varphi(t)=0$ for $|t|\le\delta$ and by $\varphi(t)=t-\delta\,\mathrm{sgn}\,t$ for $|t|\ge\delta$. For $y\in T$ and $t\in\mathbf{R}$ we write $r(y+te)=y+\varphi(t)e$. We obtain a map $r\colon E\to E$, which retracts the layer $T+\overline{B}(\delta)$ onto T.

Let $\mathrm{Sd}^k g'=\sum_{\sigma\in J} n_\sigma\sigma$ be the normal representation. Write

$$g_1=\sum\{n_\sigma\sigma:\sigma\in J,\ |\sigma|\cap\overline{H}_1\ne\emptyset\},\quad g_2=\mathrm{Sd}^k g'-g_1,\quad g=r_\#(\mathrm{Sd}^k z_1-\partial g_1).$$

Then $|g|\subset\overline{H}_1$. Since

$$\partial g_1+\partial g_2=\partial\mathrm{Sd}^k g'=\mathrm{Sd}^k z_1-\mathrm{Sd}^k z_2,$$

we have $\mathrm{Sd}^k z_1-\partial g_1=\partial g_2-\mathrm{Sd}^k z_2$, and hence $g=r_\#(\partial g_2-\mathrm{Sd}^k z_2)$, which implies that $|g|\subset\overline{H}_2$. Thus $|g|\subset\overline{H}_1\cap\overline{H}_2=T$.

Since $r\mid T=\mathrm{id}$, we have

$$\partial g=r_\#\mathrm{Sd}^k\partial z_1=r_\#\mathrm{Sd}^k z=\mathrm{Sd}^k z.$$

Hence z bounds in $|g|\cup|z|$; cf. I.1.3.1. Since $|z|=|z'|\cap T\subset|g'|\cap T\subset B$, it suffices to show that $|g|\subset B$. Let $x\in|g|$. Then $x=ry$ for some $y=x+te\in|\partial g_1|\cup|\mathrm{Sd}^k z_1|$ with $|t|\le\delta$. Since $|\partial g_1|\cup|\mathrm{Sd}^k z_1|\subset|\mathrm{Sd}^k g'|\subset|\mathrm{Sd}^j g'|$, there is $\sigma<\mathrm{Sd}^j g'$ with $y\in|\sigma|$. Then $|\sigma|$ meets T by the choice of δ. Choosing a point $x_1\in|\sigma|\cap T$ we have $x_1\in|g'|\cap T$. Since $|x-x_1|\le|y-x_1|\le d(|\sigma|)\le\varepsilon$, we have $x\in B$. We have proved that z bounds in B.

Let $c_1>c\sqrt{2}$. It suffices to show that for sufficiently small ε, B satisfies the uniformity conditions

$$\begin{aligned} d(B) &\le c_1 d,\\ d(x,|z|) &\le c_1 d(x,A) \end{aligned}$$

for all $x\in B$. Since $d(B)\le d(|g'|)+2\varepsilon\le cd\sqrt{2}+2\varepsilon$ by (4.8), the first condition is clear. To prove the second one, it suffices to show that $d(x,|z|)\le cd(x,A)\sqrt{2}$ for all $x\in|g'|\cap T$. This follows from (4.8) and from the elementary lemma (4.9) below. □

4.9. **Lemma.** *Let $T = b^\perp$ be a hyperplane in E, $b \neq 0$, and let $Y \subset \overline{B}(|b|) \cap T$. Then $d(x, Y) \le d(x, b * Y)\sqrt{2}$ for all $x \in T$. Here $b * Y$ is the cone of Y with vertex b.*

Proof. Let $x \in T$, let $y \in Y$, and let L be the line through y and b. It suffices to show that $|x - y| \le d(x, L)\sqrt{2}$. Write $x = y' + v$ with $y' \in \operatorname{span}(y)$ and $v \perp y$. Let P be the orthogonal projection of E onto L. Set $z = Px = Py'$. From similar triangles we obtain

$$\frac{|z-y|}{|y'-z|} = \frac{|y|}{|b|} \le 1.$$

Hence

$$|x-y|^2 = |v|^2 + |y'-z|^2 + |z-y|^2 \le 2|v|^2 + 2|y'-z|^2 = 2|x-z|^2 = 2d(x,L)^2. \ \square$$

4.10. **Theorem.** *Suppose that $E_1 \subset E$ is a closed linear subspace of finite codimension k. Let $A \subsetneq E_1$ be [weakly] hlog (p, c)-NUD in E with $p \ge k - 1$. If $p \ge k$, then $E \setminus A$ is [weakly] hlog $(p - k, c_1)$-uniform in E_1 for all $c_1 > 2^{k/2}c$, and hence with $c_1 = (\frac{3}{2})^k c$. If $p = k - 1$, then A is $2^k c^2$-porous in E_1.*

Proof. This follows from 4.7 by induction. To obtain the required constant in the case $p = k-1$, notice that the proof of 4.7 actually gives the porosity constant $ct = c\sqrt{4c^2 - 1}$ in the case $p = 0$. $\square$

4.11. *Remark.* There are several obvious corollaries of Theorem 4.10. For example, suppose that A is completely hlog (p, c)-NUD in E with $p \ge k$. If $A \subset E_1$, then A is completely hlog $(p - k, c_1)$-NUD in E_1. If $A \not\subset E_1$, then still $A \cap E_1$ is completely hlog $(p - k, c_1)$-NUD in E_1 by the monotonicity result 4.3.

4.12. *Example.* We show that 4.7 is not true if hlog is replaced by htop. Indeed, we construct a set $A \subset R^3$ such that A is completely htop (2)-NUD in R^4 but not htop (1)-NUD in R^3. This set is the famous Antoine's necklace constructed in a self-similar manner. Related considerations appear in the recent work of S. Semmes [Se].

Let S be a circle of radius r in R^3. For $0 < t < r$, the set $T = \overline{B}(S, t)$ is a *solid* 3*-torus* with core S. We say that a line $L \subset R^3$ is a *piercing line* of T if L meets S in two diametrically opposite points.

The building block of the Antoine set A is a solid 3-torus T together with a collection of solid 3-tori $T_1, \dots, T_m$, contained in T and linked with each other in the well-known manner. For details, see [Mo, Section 18] and [Se, Section 3]. We assume that the sets $T_1, \dots, T_m$ are mutually congruent and similar to T.

We show that A is completely htop (2)-NUD in R^4. By I.2.2 it suffices to find a quasiconformal map $f\colon R^4 \to R^4$ carrying A to a completely htop (2)-NUD set. We show that fA can be chosen to be a porous subset of a line. This will prove the assertion by I.4.15 or by 4.5(2).

Fix a piercing line L of T. We fatten the solid 3-tori T and T_j to solid 4-tori C and C_j in the natural way. Since the sets C_j are no longer linked in R^4, there is a bilipschitz homeomorphism $f_0\colon R^4 \to R^4$ such that

(1) $f_0 = \mathrm{id}$ outside C,
(2) $f_0 \mid C_j$ is a similarity for $1 \le j \le m$,
(3) L is a piercing line of $f_0 C_j$ for $1 \le j \le m$.

The map f will agree with f_0 outside the sets C_j. In the sets C_j, we iterate the construction in the obvious manner. A limiting process gives a quasiconformal homeomorphism f, which maps the Antoine set A onto a Cantor type subset of L. It is easy to see that fA is porous in L. We have thus proved that A is completely htop (2)-NUD in R^4. However, it is not htop (1)-NUD in R^3, since $R^3 \setminus A$ is not simply connected. Observe that A is hlog (1)-NUD in R^3 by 4.7.

We can modify this example to get a set $A' \subset R^3$ such that (1) A' is completely htop (2)-NUD in R^4, (2) A' is not htop (1)-NUD in R^3, (3) $R^3 \setminus A'$ is simply connected. This set A' is a subset of A, obtained by removing a thin "slice of sausage" from each of the infinite number of solid 3-tori in the construction of A. Then A' is no longer topologically wild. But if the slices become relatively thinner and thinner along the construction, A' cannot be htop (1)-NUD in R^3. On the other hand, A' is completely htop (2)-NUD in R^4 by 4.3.

4.13. *Example.* Assume that A is completely hlog (p)-NUD in R^n. Then A is completely htop $(p+1)$-NUD in R^{n+1} by 4.5(4), but A need not be htop (p)-NUD in R^n. This is the case in Example 4.12 with $n = 3$, $p = 1$. We give another example with $n = 4$, $p = 1$.

Let $A \subset R^4$ be a BT arc such that $R^4 \setminus A$ is not simply connected. Such an arc exists by [Vä$_1$, 6.3] and [Bl, Th. 3E]. Then A is not htop (1)-NUD in R^4. However, it is completely hlog (1)-NUD in R^4 by I.5.6 and [MV, 3.9]. Thus A is completely htop (2)-NUD in R^5.

However, the following homotopical version of 4.7 was pointed to us by the referee.

4.14. **Theorem.** *Suppose that T is a hyperplane in R^n and that $A \subsetneq T$ is completely htop (p,c)-NUD in R^n and htop $(1,c)$-NUD in T with $p \ge 1$. Then A is completely htop $(p-1,c_1)$-NUD in T with $c_1 = c_1(c,n)$.*

Proof. By II.4.2, the set A is completely hlog (p,c_2)-NUD in R^n with $c_2 = c_2(c,p)$. From 4.7 it follows that A is completely hlog $(p-1,2c_2)$-NUD in T. Since A is also htop $(1,c)$-NUD in T, the theorem follows from II.4.5. □

5. Miscellaneous results

5.1. *Summary of Section* 5. This section consists of two parts. In the first part we consider the case where $\dim E = \infty$ and $A \subset E$ is compact or, more generally, *boundedly compact*, which means that A meets every ball $\overline{B}(r)$ in a compact set. It turns out that A has all NUD and porosity properties with universal constants. We prove in detail the htop NUD case in 5.2 and consider the other cases in 5.3. We thank E. Saksman for useful discussions concerning this part.

In the second part we consider L-bilipschitz maps $f\colon S^k \to R^n$ and give a new proof for the fact that fS^k is completely weakly hlog $(n-2,c)$-NUD with $c = c(L,n)$.

5.2. **Theorem.** *Let* $\dim E = \infty$ *and let* $A \subset E$ *be boundedly compact. Then* A *is htop* (p,c)-NUD *for each* $p \ge 0$ *and for each* $c > \sqrt{3/2} = 1.22474\ldots$.

Proof. Let I^{p+1} be the cube $[-1,1]^{p+1}$, and let $f : \partial I^{p+1} \to E \setminus A$ be continuous. We may assume that f is piecewise linear and that $d(|f|) = d > 0$. By Lemma 2.2 there is $a \in E$ with $|f| \subset \overline{B}(a, d/\sqrt{2}\,)$. We may assume that $a = 0$. Let $\varepsilon > 0$. Since the set $A' = A \cap \overline{B}(2d)$ is compact, there is a finite-dimensional linear subspace E_1 of E such that $|f| \subset E_1$ and $A' \subset \overline{B}(E_1, \varepsilon)$. Let b be a vector perpendicular to E_1 with $|b| = d$. Let $g\colon I^{p+1} \to E$ be the cone of f with $g(0) = b$. We show that g satisfies the uniformity conditions in $E \setminus A$ if ε is small enough.

Let $x_1, x_2 \in |g|$. Then $x_j \in [b, y_j]$ for some $y_j \in |f|$. Since $|b - y_j|^2 = d^2 + |y_j|^2 \le 3d^2/2$ and $|y_1 - y_2| \le d$, we obtain $|x_1 - x_2| \le d\sqrt{3/2}$, and hence $d(|g|) \le d\sqrt{3/2}$, which is the turning condition.

To prove the lens condition let $c > \sqrt{3/2}$ and let $x \in |g|$, $z \in A$. Then $x \in [b, y]$ for some $y \in |f|$, and hence $d(x, |f|) \le |x - y| \le d\sqrt{3/2}$. If $z \notin A'$, we have $|x - z| \ge |z| - |x| \ge 2d - d = d$, and hence $d(x, |f|) \le |x - z|\sqrt{3/2}$. Assume that $z \in A'$, set $r = d(|f|, A)$, and write $x = (1-t)y + tb$ with $0 \le t \le 1$. Then $|x - y| = t|y - b| \le td\sqrt{3/2}$ and $d(x, E_1) = td(b, E_1) = td$. If $td\sqrt{3/2} \le \frac{1}{2}r$, then

$$2|x - y| \le r \le |y - z| \le |y - x| + |x - z|,$$

and hence $|x - y| \le |x - z|$. If $td\sqrt{3/2} \ge \frac{1}{2}r$, then for small ε we get

$$\frac{|x - y|}{|x - z|} \le \frac{td\sqrt{3/2}}{d(x, E_1) - \varepsilon} = \frac{\sqrt{3/2}}{1 - \varepsilon/td} \le \frac{\sqrt{3/2}}{1 - \varepsilon\sqrt{6}/r} < c.$$

These estimates yield $d(x, |f|) \le cd(x, A)$. □

5.3. *Variations.* Theorem 5.2 remains true if htop is replaced by hlog. The proof is almost the same. Furthermore, the set A is htop and hlog (p,c)-porous for all $p \geq 0$ and $c > 1$. The proof is easier than that of 5.2, since we can make use of a translation in a direction perpendicular to a suitable finite-dimensional subspace.

5.4. *Bilipschitz spheres.* Suppose that $f\colon S^{n-1} \to R^n$ is θ-quasimöbius. It is well known but not easy to prove that the components of $R^n \setminus fS^{n-1}$ are c-uniform domains with $c = c(\theta, n)$. A proof based on compactness was announced in [Vä$_2$, 5.10], and a direct proof with c independent of n was recently given by P. MacManus [Ma$_2$]. He gives $c = 322\theta(8)$, but his definition for the uniformity constant is different from ours. A third proof is given in [Vä$_3$, 5.25]. It gives an explicit estimate depending on n. On the other hand, it shows that these domains are completely hlog $(n-2,c)$-uniform.

We shall give a fourth proof, which is valid only in the case where f is bilipschitz. On the other hand, it is valid for maps $f\colon S^k \to R^n$, $0 \leq k \leq n-1$, and gives weak hlog (p)-uniformity for all relevant p. It was inspired by the proof of A. Dold [Do$_2$] for the Jordan–Brouwer separation theorem. The basic idea is to extend f to a bilipschitz map $R^{n+k} \to R^{n+k}$. This is possible by the following Lipschitz version of the well-known Klee trick; see [Ru, p. 74] and [Ma$_1$, 1.3].

5.5. **Theorem.** *Let $A \subset R^k$ and let $f\colon A \to R^n$ be M-bilipschitz. Set $A_1 = A \times \{0\} \subset R^k \times R^n = R^{k+n}$, and define $f_1\colon A_1 \to R^{k+n}$ by $f_1(x,0) = (0, fx)$. Then f_1 extends to an M'-bilipschitz map $F\colon R^{k+n} \to R^{k+n}$ with $M' = M^2\sqrt{7}$.*

Proof. By the Kirszbraun theorem [Fe, 2.10.43], there are M-Lipschitz extensions $g\colon R^k \to R^n$ and $h\colon R^n \to R^k$ of f and f^{-1}, respectively. Define $G, H\colon R^{k+n} \to R^{k+n}$ by $G(x,y) = (x, y + gx)$ and $H(x,y) = (x - hy, y)$. The map $F = HG$ is an extension of f_1. We show that it is M'-bilipschitz. Let $z = (x,y)$, $z' = (x',y') \in R^{k+n}$. Since

$$F(x,y) = \bigl(x - h(y+gx), y + gx\bigr),$$

we obtain by the Schwarz inequality

$$\begin{aligned} |Fz - Fz'|^2 &\leq \bigl(|x-x'| + |h(y+gx) - h(y'+gx')|\bigr)^2 + (|y-y'| + |gx - gx'|)^2 \\ &\leq \bigl(|x-x'| + M(|y-y'| + M|x-x'|)\bigr)^2 + (|y-y'| + M|x-x'|)^2 \\ &\leq \bigl((1+M^2)^2 + M^2\bigr)|z-z'|^2 + (1+M^2)|z-z'|^2 \\ &= (2 + 4M^2 + M^4)|z-z'|^2 \leq 7M^4|z-z'|^2. \end{aligned}$$

Hence F is M'-bilipschitz. Since $F^{-1}(x,y) = \bigl(x + hy, y - g(x+hy)\bigr)$, similar estimates show that F^{-1} is M'-Lipschitz.

5.6. *Question.* Is there a quasisymmetric version of 5.5? A direct analogue is false, since a quasisymmetric map $f: A \to R^n$, $A \subset R^k$, need not be Hölder continuous. However, this cannot happen if, for example, A is connected.

5.7. **Theorem.** *Let $0 \le k \le n-1$ and let $f: S^k \to R^n$ be M-bilipschitz. Then fS^k is completely weakly hlog $(n-2,c)$-NUD with $c = 35 \cdot (\frac{3}{2})^{k+1} M^4$.*

Proof. By 2.8, the set S^k is completely weakly hlog $(k+n-1,5)$-NUD in R^{k+1+n}. Since $S^k \subset R^{k+1}$, 5.5 gives an $M^2\sqrt{7}$-bilipschitz extension $F: R^{k+1+n} \to R^{k+1+n}$ of f_1. Since an L-bilipschitz map increases uniformity constants at most by a factor L^2, fS^k is completely weakly hlog $(k+n-1, 35M^4)$-NUD in R^{k+1+n}. Applying Theorem 4.10, with $E = R^{k+1+n}$, $E_1 = \{0\} \times R^n$, $p = k+n-1$, gives the theorem. □

5.8. *Remarks.* 1. Theorem 5.7 remains true if S^k is replaced by R^k and c by $16 \cdot (\frac{3}{2})^k M^4$. In the proof we replace 2.8 by 2.7.

2. In these results, the set fS^k or fR^k is hlog (p,c)-NUD for $p \ne n-k-1$, $0 \le p \le n-2$.

5.9. **Corollary.** *If $f: S^{n-1} \to R^n$ is M-bilipschitz, the components of $R^n \setminus fS^{n-1}$ are c-uniform domains with $c = 35 \cdot (\frac{3}{2})^n M^4$.*

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