COORDINATES FOR TEICHMÜLLER SPACES OF $b$-GROUPS WITH TORSION

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Abstract. In this paper we will use $b$-groups to construct coordinates for the Teichmüller spaces of 2-orbifolds. The main technical tool is the parametrization of triangle groups, which allows us to compute explicitly formulae for generators of $b$-groups uniformizing orbifolds. In this way, we obtain a technique to pass from the abstract objects of deformation spaces to concrete calculations with Möbius transformations. We explore this computational character of our work by finding the expressions of certain classical isomorphisms between Teichmüller spaces.

0. Introduction

One of the most active lines of research in geometric function theory nowadays deals with the problem of finding parametrizations of Teichmüller spaces that are useful for computations ([4], [7], [9], [15], [16], [17], [20]). This paper makes a two fold contribution to this aspect of complex analysis: we will give coordinates for the Teichmüller spaces of $b$-groups with torsion (or equivalently, for the Teichmüller spaces of 2-orbifolds) and we will use our coordinates to compute explicit formulae of well known isomorphisms between deformation spaces. Our coordinates are good from a computational point of view because, given a point $\alpha$ in the Teichmüller space of a $b$-group, $T(p, n; \nu_1, \ldots, \nu_n)$, we have a technique to compute a set of Möbius transformations that generate a Kleinian group $\Gamma$, whose coordinates in the space $T(p, n; \nu_1, \ldots, \nu_n)$ are $\alpha$.

Given an orbifold $S$, defined over a surface of genus $p$ with $n$ special points, and a maximal partition $\mathcal{C}$ on $S$ (that is, a way of decomposing $S$ into ‘pairs of pants’), we can find a Kleinian group $\Gamma$, acting discontinuously on a simple connected open set $\Delta$ of $\hat{\mathcal{C}}$, such that $\Delta/\Gamma \cong S$. Besides $S$, the group $\Gamma$ uniformizes a finite number of rigid orbifolds. Therefore, the Teichmüller space of $\Gamma \ T(\Gamma)$ is a model for $T(S)$, the Teichmüller space of the orbifold $S$. This last set is important in the study of Riemann surfaces because it is the universal covering space of the Riemann space $R(S)$, which parametrizes the biholomorphic classes of complex structures on $S$. The advantage of using the deformation space of the group over the deformation space of the orbifold lies in the fact that one

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can do explicit computations with Möbius transformations, obtaining in this way some properties of $T(S)$.

Using the partition $\mathcal{C}$, we can decompose the group $\Gamma$ into a set of subgroups, $\Gamma_1, \ldots, \Gamma_{3p-3+n}$, with the property that $T(\Gamma_j)$ has dimension 1. By a theorem of B. Maskit, the restriction mapping $T(\Gamma) \to \prod_{j=1}^{3p-3+n} T(\Gamma_j)$ is one-to-one and holomorphic. Therefore, to give coordinates on $T(\Gamma)$ it suffices to study the one-dimensional cases. This is done in detail in Section 3 of this paper. Putting together our computations with the above embedding, we get the following general result, which generalizes the torsion-free case studied by I. Kra.

**Theorem 10** ([7] and Section 3.7). Let $S$ be an orbifold with hyperbolic signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$, and let $\mathcal{C}$ be a maximal partition on $S$. Then there exists a set of global coordinates for the deformation space $T(S) = T(p, n; \nu_1, \ldots, \nu_n)$, say $(\alpha_1, \ldots, \alpha_d)$, where $d = 3p-3+n$, and a set of non-negative numbers, $y_1^1, \ldots, y_1^d, y_2^1, \ldots, y_2^d$, which depends only on the signature $\sigma$ and the partition $\mathcal{C}$, such that

$$\{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d; \ \text{Im}(\alpha_i) > y_1^i, \ \forall 1 \leq i \leq d\} \subset T(p, n; \nu_1, \ldots, \nu_n)$$

and

$$T(p, n; \nu_1, \ldots, \nu_n) \subset \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d; \ \text{Im}(\alpha_i) > y_2^i, \ \forall 1 \leq i \leq d\}.$$  

Given a point $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $T(p, n; \nu_1, \ldots, \nu_n)$, it is possible to find explicitly a set of $2p+n$ Möbius transformations that generate a group $\Gamma$, whose coordinates in that Teichmüller space are $\alpha$.

The Kleinian groups that we will use are known as terminal regular $b$-groups (see Section 1 for the definition). They are built from triangle groups by a finite number of applications of the Klein–Maskit combination theorems. Therefore, to prove the above theorem, we first need a way of computing generators for triangle groups. This is the content of the following result, which is the main technical tool of our work.

**Theorem** (Sections 2.2 and 2.6). Given three distinct points $(a, b, c)$ in the Riemann sphere, and a signature $\sigma = (0, 3; \nu_1, \nu_2, \nu_3)$, which is either hyperbolic or $(0, 3; \infty, 2, 2)$, there exists a pair of Möbius transformations $A$ and $B$, uniquely determined by the parameters $(a, b, c)$, such that the group generated by them is a triangle group of signature $\sigma$. The transformations $A$ and $B$ are explicitly computable from $(a, b, c)$ and $\sigma$.

The explicit character of our coordinates allows us to compute some classical isomorphisms between different Teichmüller spaces as follows. Let $S$ be a surface of genus 2, uniformized by a $b$-group $\Gamma$. Since all surfaces of genus 2 are
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hyperelliptic, we have a conformal involution \( j \) on \( S \) with 6 fixed points. The quotient orbifold \( S' = S/\langle j \rangle \) has signature \((0, 6; 2, \ldots, 2)\). It is a classical result ([19]) that the spaces \( T(S) \) and \( T(S') \) are biholomorphically equivalent. To find a mapping between them, we first find the lifting of \( j \), say \( A_2^{1/2} \), to the covering determined by \( \Gamma \). We have that the group generated by \( \Gamma \) and \( A_2^{1/2} \) is a \( b \)-group uniformizing \( S' \). Using explicit calculations on \( T(\Gamma) \) and \( T(\Gamma') \), we get the following result.

Theorem 11 (Section 4.1). The mapping

\[
(\tau_1, \tau_2, \tau_3) \mapsto (\frac{1}{2} \tau_1, 1 + \tau_2, 1 + \frac{1}{2} \tau_3)
\]

gives an isomorphism between \( T(2,0) \) and \( T(0,6; 2,2,2,2,2) \).

This paper is organized as follows. Section 1 contains non-standard background on Kleinian groups and Teichmüller spaces. In Section 2 we compute generators for triangle groups. These computations are used in Section 3 to develop coordinates for Teichmüller spaces of terminal regular \( b \)-groups; in particular we prove Theorem 10. In Section 4 we prove Theorem 11, and indicate how our methods can be used to prove similar results.

The content of this paper is part of the author’s Ph.D. thesis. I would like to thank my advisor, Irwin Kra, for all his help during my years as a graduate student; and thank Chaohui Zhang and Suresh Govindarajan for many useful conversations and comments on my work. I also want to thank the referee for making many useful comments which have improved a first, and very hard to understand, version of this paper.

1. Background

1.1. Throughout this paper, we will identify the group of Möbius transformations with the projective special linear group, \( \text{PSL}(2, \mathbb{C}) \). The mapping \( z \mapsto (az + b)/(cz + d) \) will be identified with \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). The square brackets denote an element of \( \text{PSL}(2, \mathbb{C}) \); that is, a class of matrices of \( \text{SL}(2, \mathbb{C}) \). If we take a particular lifting of an element of \( \text{PSL}(2, \mathbb{C}) \) to \( \text{SL}(2, \mathbb{C}) \), we will use parenthesis.

An elliptic transformation of finite order \( q \) is called geometric if it can be conjugated in \( \text{PSL}(2, \mathbb{C}) \) to \( z \mapsto \exp(\pm 2\pi i/q)z \). Observe that only ‘minimal rotations’ are geometric. For example, every element of order 5 is conjugate to a transformation of the form \( z \mapsto (\exp(2k\pi i/q))z \), for \( k = 1, \ldots, 4 \), but only those with \( k = 1, 4 \) are geometric.

An element \( A \) of a group \( G \) of Möbius transformations is called primitive if the only solutions of the equation \( B^n = A \), with \( B \in G \) and \( n \in \mathbb{Z} \), are given by \( B = A^{\pm 1} \).
1.2. Let $G$ be a non-elementary finitely generated Kleinian group. An isomorphism $\theta: G \to \theta(G) \subset \text{PSL}(2, \mathbb{C})$ is called geometric if there exists a quasiconformal homeomorphism of the Riemann sphere $w$, such that $\theta(g) = w \circ g \circ w^{-1}$ for all $g \in G$. Two geometric isomorphisms $\theta_1, \theta_2$, are equivalent if there exists a Möbius transformation $A$, such that $\theta_1(g) = A \circ \theta_2(g) \circ A^{-1}$, for all $g \in G$. The set of equivalence classes of geometric isomorphisms of $G$ is the Teichmüller (or deformation) space of $G$, $T(G)$. It is a well known fact that $T(G)$ is a complex manifold ([2], [5], [11]).

1.3. A signature is a set of numbers $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ such that $p, n \in \mathbb{Z}^+$, $\nu_j \in \mathbb{Z}^+ \cup \{\infty\}$, $\nu_j \geq 2$. We will say the signature is hyperbolic, parabolic or elliptic if the quantity $2p - 2 + n - \sum_1^n (1/\nu_j)$ is positive, zero or negative, respectively. The pair $(p, n)$ is called the type of the signature.

A (2-)orbifold $\mathcal{S}$ of signature $\sigma$, is a compact surface of genus $p$ from which finitely many points have been removed (as many as $\infty$’s are in $\sigma$) and such that $\mathcal{S}$ has a covering which is locally $\nu_j$-to-1 over certain points, where the $\nu_j$’s are the finite values appearing in $\sigma$. The $\nu_j$’s are called ramification values of the signature or of the orbifold. A maximal partition $\mathcal{C}$ on an orbifold $\mathcal{S}$ with hyperbolic signature $\sigma$ is a set of $3p - 3 + n$ simple unoriented disjoint curves on $\mathcal{S}' = \mathcal{S} - \{x_j; \nu_j < \infty\}$, such that no two curves of $\mathcal{C}$ are freely homotopically equivalent on $\mathcal{S}'$, and no curve of $\mathcal{C}$ is freely homotopically equivalent to a loop around a point or a puncture of $\mathcal{S}'$. To avoid trivial cases, when we talk about maximal partitions we will assume that the signature of the orbifold satisfies $3p - 3 + n > 0$.

1.4. The following result of B. Maskit provides us with a uniformization of orbifolds by Kleinian groups that are better for computational purposes than Fuchsian groups.

**Theorem 1** (Maskit [10], [13]). Given an orbifold $\mathcal{S}$ with hyperbolic signature $\sigma$ and maximal partition $\mathcal{C}$, there exists a (unique up to conjugation in $\text{PSL}(2, \mathbb{C})$) geometrically finite Kleinian group $\Gamma$, called a terminal regular $b$-group, such that:

1. $\Delta/\Gamma$ is conformally equivalent to $\mathcal{S}$;
2. for each element $a_j$ of the partition $\mathcal{C}$, there is a curve $\tilde{a}_j$ in $\Delta$, precisely invariant under a cyclic subgroup $\langle A_j \rangle$ of $G$, generated by an accidental parabolic transformation $A_j$, and such that $\Delta \supset \tilde{a}_j \overset{\pi}{\rightarrow} \pi(\tilde{a}_j) = a_j \subseteq \mathcal{S}$, where $\pi: \Delta \to \mathcal{S}$ is the natural projection.
3. $(\Omega(\Gamma) - \Delta)/\Gamma$ is the union of the orbifolds of type $(0, 3)$ obtained by squeezing each curve of $\mathcal{C}$ to a puncture and discarding all orbifolds of signature $(0, 3; 2, 2, \infty)$ that appear.

From a Teichmüller theory point of view, the only interesting surface uniformized by a terminal regular $b$-group $\Gamma$ is the one corresponding to the invariant component, since the deformation spaces of orbifolds of type $(0, 3)$ are points. The
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space \( T(\Gamma) \) is then equivalent to the deformation space of the orbifold \( T(S) \). When convenient, we will denote \( T(\Gamma) \) by \( T(p, n; \nu_1, \ldots, \nu_n) \). Its complex dimension is \( 3p - 3 + n \).

1.5. The uniformization theorem of Maskit allows us to embed \( T(\Gamma) \) into a product of one dimensional Teichmüller spaces as follows. Let \( T_j \) be the connected component of \( S - \{a_k; a_k \in \mathcal{C}, k \neq j\} \) containing the curve \( a_j \). Let \( D_j \) be a connected component of \( \pi^{-1}(T_j) \), and let \( \Gamma_j \) be its stabilizer in \( \Gamma \); that is, \( \Gamma_j = \{\gamma \in \Gamma; \gamma(D_j) = D_j\} \). The \( \Gamma_j \)'s are terminal regular \( b \)-groups of type \((0, 4)\) or \((1, 1)\), and therefore \( T(\Gamma_j) \) is a one-dimensional manifold ([6]). It is clear that any geometric isomorphism of \( \Gamma \) induces an geometric isomorphism of \( \Gamma_j \) by restriction.

**Theorem 2** (Maskit [12], Kra [6]). The mapping defined by restriction, \( T(\Gamma) \hookrightarrow \prod_{j=1}^{3p-3+n} T(\Gamma_j) \), is holomorphic and injective with open image.

We will call this mapping the **Maskit embedding** of the group \( \Gamma \).

1.6. Throughout this paper, for a signature \((p, n; \nu_1, \ldots, \nu_n)\), \( q_j \) and \( p_j \) will denote \( \cos(\pi/\nu_j) \) and \( \sin(\pi/\nu_j) \), respectively, \( j = 1, \ldots, n \).

2. Triangle groups

2.1. A triangle group is a Kleinian group \( \Gamma \) generated by two Möbius transformations, \( A \) and \( B \), such that \( |A| = \nu_1 \), \( |B| = \nu_2 \) and \( |AB| = \nu_3 \). Here \( |T| \) denotes the order of the transformation \( T \), with parabolic elements considered as elements of order equal to \( \infty \). A triangle group \( \Gamma \) with hyperbolic signature can be constructed, for example, by taking a triangle on \( \mathbb{H} \) with angles \( \pi/\nu_j \), \( j = 1, 2, 3 \), and considering the group of transformations \( \Gamma^* \) generated by reflections on the sides of the triangle. Then \( \Gamma \) is the index 2 subgroup consisting of the orientation preserving transformations. A similar construction can be carried out for the case of parabolic groups. Our goal is to correctly choose the position of the vertices of such triangles.

**Hyperbolic groups.** 2.2. Let \((a, b, c)\) be three distinct points on \( \hat{\mathbb{C}} \). Let \( \Lambda \) be the circle determined by these points. Let \( \Delta = \{z \in \mathbb{C}; \text{Im}(cr(a, b, c, z)) > 0\} \), where \( cr \) denotes the cross ratio of four different points of the Riemann sphere, chosen such that \( cr(\infty, 0, 1, z) = z \) (remember that there are six possible definitions of cross ratio). Let \( L \) and \( L' \) be the circles orthogonal to \( \Lambda \) and passing through \{a, b\} and \{a, c\}, respectively.

**Definition 1.** Let \( z_1 \) and \( z_2 \) be two distinct points in \( L \cap \overline{\Delta} \). We will say that they are **well ordered**, with respect to \((a, b, c)\), if one of the following set of conditions is satisfied (they are not mutually exclusive):
1. \( z_1 = a \);
2. \( z_2 = b \);
3. \( z_1 \neq a, z_2 \neq b \) and \( cr(a, z_1, z_2, b) \) is real and strictly bigger than 1.
For example, if \( a = \infty \), \( b = 0 \) and \( c = 1 \), we have that \( L \) is the imaginary axis, and \( \Delta \) is the upper half plane. If \( z_1 = \lambda i \) and \( z_2 = \nu i \), then these two points are well ordered with respect to \((\infty, 0, 1)\) if and only if \( \lambda > \nu > 0 \).

Given this definition, we can state the concept of ‘good’ generators.

**Definition 2.** Let \((a, b, c)\) be three distinct points of \( \hat{\mathbb{C}} \), and let \( \Lambda, \Delta, L \) and \( L' \) be as previously defined. Suppose that \( \Gamma \) is a triangle group with hyperbolic signature \((0; 3; \nu_1, \nu_2, \nu_3)\) and whose limit set is \( \Lambda \). Let \( A \) and \( B \) be elements of \( \Gamma \). We will say that \((A, B)\) are canonical generators of \( \Gamma \) for the parameters \((a, b, c)\) if they generate the group \( \Gamma \) and the following conditions are satisfied:

1. \( |A| = \nu_1, |B| = \nu_2, |AB| = \nu_3 \),
2. \( A \) and \( B \) have their fixed points on \( L \), and \( AB \) on \( L' \),
3. if \( z_1 \) and \( z_2 \) are the fixed points of \( A \) and \( B \) on \( L \cap \Delta \), then they are well ordered with respect to \((a, b, c)\),
4. \( A \) and \( B \) are primitive elements, and geometric whenever elliptic.

Our main result about existence and uniqueness of canonical generators for hyperbolic triangle groups is the following:

**Theorem 3.** Given three different points \((a, b, c)\) in the Riemann sphere, and a hyperbolic signature \( \sigma = (0; 3; \nu_1, \nu_2, \nu_3) \), there exists a unique pair of Möbius transformations \((A, B)\), such that they are canonical generators of a triangle group with signature \( \sigma \) and for the given parameters.

In the case \((a = \infty, b = 0, c = 1)\), these generators are given by:

1. Signature \((0; 3; \infty, \nu_1, \nu_2)\), \( \nu_i \leq \infty \):

   \[
   A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_1 & b \\ q_1 + q_2 & -q_1 \end{bmatrix}, \quad b = \frac{q_1^2 - 1}{q_1 + q_2}.
   \]

2. Signature \((0; 3; \nu_1, \nu_2, \nu_3)\), where all the \( \nu_i \) are finite:

   \[
   A = \begin{bmatrix} -q_1 & -kp_1 \\ k^{-1}p_1 & -q_1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_2 & -hp_2 \\ h^{-1}p_2 & -q_2 \end{bmatrix}.
   \]

Here \( k = (q_2 + q_1q_3 + q_1l)/(p_1l) \), \( h = kp_1p_2/(q_1q_2 + q_3 + l) \), and \( l = \sqrt{q_1^2 + q_2^2 + q_3^2 + 2q_1q_2q_3 - 1} \) with the square root chosen to be positive.

For any other set of parameters, \((a, b, c)\), the generators are given by conjugating the above transformations by the unique Möbius transformation \( T \) that maps \( a, b, c \) to \( \infty, 0, 1 \) respectively.

Observe that the transformations in the second case of the above theorem converge to those of the first case when \( \nu_1 \) goes to \( \infty \).

**2.3.** The following two technical results are needed to prove Theorem 3.
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Proposition 1. Let $\Gamma$ be a triangle group with signature $(0, 3; \mu_1, \mu_2, \mu_3)$ and let $l^2 = q_1^2 + q_2^2 + q_3^2 + 2q_1q_2q_3 - 1$. Then $l^2$ is positive, zero or negative if and only if the signature is hyperbolic, parabolic or elliptic, respectively.

Proof. The elliptic and parabolic cases can be checked by computing the values of $l^2$. In the hyperbolic case, first observe that the expression of $l^2$ is symmetric on $\nu_j$'s, so we can assume, without loss of generality, that $\nu_1 \leq \nu_2 \leq \nu_3$. It is also clear that $l^2$ is increasing with $\nu_j$, so we need to compute its values only for the cases of small signatures. More precisely, it suffices to consider the cases $(0, 3; 2, 3, 7)$ and $(0, 3; 3, 3, 4)$. For the first of these signatures we have $l^2 = \cos^2(\pi/7) - \frac{3}{4} \cos^2(\pi/6) - \frac{4}{9} = 0$, while for the second signature we get $l^2 = 1 + \frac{3}{2}\sqrt{2} > 0$. □

Proposition 2. Let $A$ and $B$ be canonical generators for the group $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$. Assume that $\tilde{A}$ and $\tilde{B}$ are liftings of $A$ and $B$, respectively, to $\text{SL}(2, \mathbb{C})$, both having negative trace. Then the product $\tilde{A}\tilde{B}$ also has negative trace.

Proof. We start with the observation that all the ramification values should be bigger than 2, since involutions have matrix representatives with zero trace, and then the proposition would not make sense. Since the trace of a matrix is invariant under conjugation, we are free to choose $(\infty, 0, 1)$ as parameters for the group; this will simplify our computations.

Let us first look at the case of $\Gamma(\infty, \nu_2, \nu_3; \infty, 0, 1)$. Assume that $\tilde{A}$ and $\tilde{B}$ have negative trace but $\tilde{A}\tilde{B}$ has positive trace. We have the following expressions for the liftings of $A$ and $B$:

$$\tilde{A} = \begin{pmatrix} -1 & -\alpha \\ 0 & -1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} a & b \\ c & -a - 2q_2 \end{pmatrix}.$$ 

From the definition of canonical generators for the above parameters we get the following conditions:

$$\begin{cases} \text{trace } (\tilde{A}\tilde{B}) = -2q_1 \leq 0 & \Leftrightarrow 2q_2 - \alpha c = 2q_3 \\ \text{Re}(\text{fixed points of } B) = 0 & \Leftrightarrow 2a + 2q_2 = 0 \\ \text{Re}(\text{fixed points of } AB) = 1 & \Leftrightarrow -\alpha c - 2q_2 - 2a = -2c. \end{cases}$$

Solving these equations we get

$$\tilde{A} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} -q_1 \\ q_1 - q_2 \end{pmatrix}, \quad b = \frac{q_1^2 - 1}{q_1 - q_2}.$$ 

The Schimizu–Leutbecher lemma [14, p. 18] implies that $|q_2 - q_3| \geq \frac{1}{2}$ or $q_2 = q_3$. The second situation cannot happen as it would imply that the group is
elementary. In the first case, the only possible solutions are $q_2 = 1, q_3 = \frac{1}{2}$ or vice versa. This implies that the possible signatures of the group are $(0, 3; \infty, \infty, 3)$ or $(0, 3; \infty, 3, \infty)$. But in both cases we obtain an element of order 2, namely $ABA$ and $AB^{-1}$ respectively, which is not possible because of the signatures.

The case of the group $\Gamma(\nu_1, \infty, \nu_3; \infty, 0, 1)$, with $\nu_1 < \infty$, can be reduced to the previous situation as follows. Let $x \in \mathbb{R} - \{0\}$ be the end point of the geodesic in $H$ joining 0 and the fixed point of $AB$. Then $B$ and $(AB)^{-1}$ are canonical generators for $\Gamma(\infty, \nu_3, \nu_1; 0, x, \infty)$. Let $T$ be the Möbius transformation that takes $0, x, \infty$ to $\infty, 0, 1$ respectively. Then $TT(\infty, \nu_3, \nu_1; 0, x, \infty) T^{-1} = \Gamma(\infty, \nu_3, \nu_1; \infty, 0, 1)$, and we are in the situation already discussed.

Consider now the case of $\Gamma(\nu_1, \nu_2, \nu_3; \infty, 0, 1)$, where all the ramification values are finite. Assume again that $AB$ has negative trace. Then an easy computation shows that the matrix representatives of $A$ and $B$ are given by the following expressions:

$$
\tilde{A} = \begin{pmatrix} -q_1 & -mp_1 \\ m^{-p_1} & -q_1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} -q_2 & -np_2 \\ n^{-p_2} & -q_2 \end{pmatrix},
$$

where $m = (q_2 - q_1 q_3 + q_1 r)/p_1 r$, $n = (mp_1 p_2)/(q_1 q_2 - q_3 + r)$, and $r$ is the positive square root of $q_1^2 + q_2^2 + q_3^2 - 2q_2 q_3 - 1$.

If $m > 0$, then consider the triangle of Figure 1, where the vertices $v_1$, $v_2$ and $v_3$ are fixed by $A$, $B$ and $AB$, respectively, and the angle at $v_j$ is $\pi/\nu_j$. It is not hard to see that $A(\infty) < 0$ and $AB(\infty) < 1$, so the action of the transformations $A$ and $AB$ are as indicated in the figure. Reflect the triangle on the geodesic joining $v_1$ and $v_3$ to get a similar triangle with vertices $v_1$, $v_3$ and $v_4$. Since $A$ and $B$ are isometries in $H$, a look at the figure shows that $ABA$ fixes $v_2$. This implies that $ABA = B^n$ for some integer $n$. If $n = 0$, we have that the group is elementary, which is not possible. If $n \neq 0$, we use the fact that $AB = (B^{-1}A^{-1})^{\nu_3 - 1}$, to get $B^n = ABA = (B^{-1}A^{-1})^{\nu_3 - 1} A = (B^{-1}A^{-1})^{\nu_3 - 2} B^{-1}$. Therefore $B^{n+1} = (AB)^{\nu_3 - 2}$. But this equality can be satisfied only if $\nu_3 = 2$, contrary to our assumptions.

The case of $m < 0$ is solved as above by considering the element $A^{-1}B^{-1}A^{-1}$ instead of $ABA$. □
We are now in a position to prove Theorem 3.

2.4. Proof of Theorem 3. First case: the signature is \((0, 3; \infty, \nu_1, \nu_2)\). By the same trick used in the proof of Proposition 2, we can reduce all the signatures with punctures to this case. The element \(A\) is a parabolic transformation fixing \(\infty\), so it is of the form \(A(z) = z + \alpha\), with \(\alpha \in \mathbb{C} - \{0\}\). Consider now the element \(B(z) = (az + b)/(cz + d)\). Choose liftings of \(A\) and \(B\) to \(\text{SL}(2, \mathbb{C})\) with negative traces. We then have the following equations:

\[
\begin{align*}
\text{trace}(\tilde{B}) = -2q_2 &\leq 0 & \iff a + d = -2q_2, \\
\text{Re}(\text{fixed points of } B) = 0 & \iff a - d = 0, \\
\text{trace}(\tilde{AB}) = -2q_3 &\leq 0 & \iff -a - \alpha c - d = 2q_3, \\
\text{Re}(\text{fixed points of } AB) = 1 & \iff a + \alpha c - d = 2c, \\
\text{det} \tilde{B} = 1 & \iff ad - bc = 1.
\end{align*}
\]

Solving these equations we get the matrices of Theorem 1.

Second case: the signature is \((0, 3; \nu_1, \nu_2, \nu_3)\), where all the ramification values are finite. The canonical generators will be of the form \(A(z) = (az + \beta)/(\gamma z + \delta)\) and \(B(z) = (az + b)/(cz + d)\). Choosing matrix representatives of these transformations with negative traces, we get the equations:

\[
\begin{align*}
\text{trace}(\tilde{A}) = -2q_1 &\leq 0 & \iff \alpha + \delta = -2q_1, \\
\text{Re}(\text{fixed points of } A) = 0 & \iff \alpha - \delta = 0, \\
\text{trace}(\tilde{B}) = -2q_2 &\leq 0 & \iff a + d = -2q_2, \\
\text{Re}(\text{fixed points of } B) = 0 & \iff a - d = 0, \\
\text{trace}(\tilde{AB}) = -2q_3 &\leq 0 & \iff \alpha a + \beta c + \gamma b + \delta d = -2q_3, \\
\text{Re}(\text{fixed points of } AB) = 1 & \iff \alpha a + \beta c - \gamma b - \delta d = 2(\gamma a + \delta c).
\end{align*}
\]

Using the fact that the matrices involved in these equations have determinant equal to 1, it is not hard to see that the only solution is the one given in Theorem 3. \(\blacksquare\)

2.5. The following technical result will be needed in Section 2.7.

Proposition 3. If \((A, B)\) and \((A, D)\) are two pairs of canonical generators for a hyperbolic triangle group with signature \((0, 3; \infty, \nu_1, \nu_2)\) then there exists an integer number, \(n\), such that \(D = A^{n/2}BA^{-n/2}\).

In general a Möbius transformation has several square roots, but in the case of parabolic elements such a situation does not happen. So we have that the transformation \(A^{n/2}\) is well defined for any integer \(n\).

Proof. By conjugation we may assume that \(A\) and \(B\) are canonical generators for the parameters \((\infty, 0, 1)\), whose expressions are given in Theorem 1. Let \(\tilde{A}\) and \(\tilde{D}\) be liftings of \(A\) and \(D\) to \(\text{PSL}(2, \mathbb{C})\) respectively, with negative trace. By Proposition 2 we have that \(\tilde{A}\tilde{D}\) has also negative trace. Computations show that, under these conditions, \(A(z) = z + 2\) and \(D(z) = (az + \beta)/(q_1 + q_2 z - 2q_1 - \alpha)\),
Let $T(z) = z + h$, where $h = (\alpha + q_1)/(q_1 + q_2)$ (this is the real part of the fixed points of $AD$). Then $TBT^{-1} = D$. So $T$ belongs to the normalizer of $\Gamma$ in $\text{PSL}(2, \mathbb{C})$, and induces an automorphism of the quotient surface that fixes one puncture (since $TAT^{-1} = A$). This means that either $T \in \Gamma$ or $T^2 \in \Gamma$, giving us $T = A^n$ or $T = A^{n/2}$ as desired.

**Parabolic groups. 2.6.** The only case of parabolic triangle groups that we need in this paper is the one of groups with signature $(0, 3; \infty, 2, 2)$. For a treatment of the general case, as well as the elliptic signatures, see [1].

**Definition 3.** Let $\Gamma$ be a triangle group with signature $(0, 3; \infty, 2, 2)$. Let $A$ and $B$ be two generators of the group. We will say that they are canonical for the parameters $(a, b, c)$ if the following conditions are satisfied:

1. $|A| = \infty$, $|B| = 2$ and $|AB| = 2$,
2. $A(a) = a$, $B(b) = b$ and $AB(c) = c$,
3. $A$ and $B$ are primitive.

**Theorem 4.** Given three different points $(a, b, c)$ in the Riemann sphere, there exists a unique pair of Möbius transformations $(A, B)$, such that they are the canonical generators of a triangle group with signature $(0, 3; \infty, 2, 2)$ and for the given parameters. We will denote the triangle group with the pair of canonical generators by $\Gamma(\infty, 2, 2; a, b, c)$.

**Proof.** Taking parameters $(\infty, 0, 1)$, the proof is reduced to a simple computation.

Proposition 4 also holds for this type of groups. Since the proof is the same, we will not reproduce it again.

**The geometry of the quotient orbifolds. 2.7.** Our next goal is to produce coordinates on the orbifolds uniformized by the groups studied above. We will use these coordinates to explore the relation between parameters on Teichmüller spaces and the construction of Riemann surfaces (see Theorems 6 and 9).

Let $\Gamma = \Gamma(\infty, \nu_2, \nu_3; a, b, c)$ be a group with hyperbolic signature. Let $\Delta$ be defined as in Section 2.1. Put on $\Delta$ the metric of constant curvature $-1$. Since $\Gamma$ acts on $\Delta$ by isometries, we have a metric on $S \cong \Delta/\Gamma$ obtained by projection. A geodesic on $S$ is just a curve that lifts to a geodesic on $\Delta$.

**Proposition 4.** Let $\mathcal{I}$ be a hyperbolic orbifold with signature $(0, 3; \infty, \nu_1, \nu_2)$. Let $P \notin \mathcal{I}$ be the puncture corresponding to the first $\infty$ in the signature, and $Q$ the puncture or branched point corresponding to $\nu_1$. Then there exists a unique simple geodesic $c: I \rightarrow \mathcal{I}$, joining $P$ and $Q$ such that, if $c$ is parametrized by arc length $s$, then:

1. if $\nu_1 = \infty$, $I = \mathbb{R}$, $\lim_{s \to +\infty} c(s) = P$, and $\lim_{s \to -\infty} c(s) = Q$.
2. if $\nu_1 < \infty$, $I = [0, +\infty)$, $\lim_{s \to +\infty} c(s) = P$, and $c(0) = Q$. 


Proof. The existence part is easy. These orbifolds have no moduli, so we can assume that the covering space is $H$, and the covering group $\Gamma$ has parameters $(\infty, 0, 1)$. As a fundamental domain for our group we can choose (depending on the signature of the group) one of those in Figure 2 above. The projection of the part of the imaginary axis that lies in the boundary of that fundamental domain gives a geodesic on $\mathcal{S}$ that satisfies the conditions of the statement of the proposition.

For the uniqueness part, let us assume that there is another geodesic, satisfying the properties stated in the proposition. We lift it to $H$ and we can assume that the lifting is a half-vertical line. We want to prove that this second vertical line is simply a translate of the imaginary axis under a power of $A$, and therefore the projection of the two lines will be the same geodesic in the orbifold.

The end point of our line, say $x_0$, which corresponds to $Q$, has to be fixed by a transformation $B_1$. Now, if we remove on the orbifold the point corresponding to $\nu_2$ (if $\nu_2 = \infty$, then we do not have to remove anything, since punctures are not in the orbifold), we are in a situation like the torsion free case, and we get that $A$ and $B_1$ generate the group $\Gamma$ (see [7]). Therefore $A$ and $B_1$ will be canonical generators for some parameters. By Proposition 4, we have that there is an integer $n \in \mathbb{Z}$, such that $A^{n/2}BA^{-n/2} = B_1$. Our proof will be complete if we show that $n$ is even.

If $n$ is odd, then $B_1 = A^{1/2}BA^{-1/2}$ is conjugate to $B$ in the group $\Gamma$, since both transformations, $B$ and $B_1$, correspond to the same point $Q$ in $S$. This implies that the element $A^{1/2}$ belongs to the normalizer of $\Gamma$ in PSL$(2, \mathbb{R})$ and therefore it induces an automorphism of the quotient orbifold that fixes at least one puncture (the one represented by $A$). Since $A^{1/2}$ does not belong to $\Gamma$, the induced automorphism is not the identity, and so it has to interchange the other two ramification points. This implies that $B$ and $AB$ are transformations of the same order. It is easy to see that, in such case, $A^{1/2}BA^{-1/2} = (AB)^{-1}$. Then we would have that $B$ and $(AB)^{-1}$ are conjugate in the group $\Gamma$, which is not true since they correspond to different branch points. Therefore $x_0$ is an even integer and the geodesics are the same. \[\square\]

2.8. We can use the geodesic of the above proposition to construct coordinates
on the orbifold $\mathcal{S}$. In this paper we will use only coordinates around punctures, and the main result in that line is in [7]. His proof can be applied to our situation since the argument is local. We copy the result here for the convenience of the reader. For a more general situation see [1].

**Proposition 5.** Let $\Gamma(\infty, \nu_1, \nu_2; a, b, c)$ be a hyperbolic triangle group. Let $\mathcal{S}$ be an orbifold uniformized by this group, and suppose that $\mathcal{S}$ has the metric of constant curvature $-1$ that comes from its universal (branched) covering space. Let $P_1 \notin \mathcal{S}$ the puncture corresponding to the $\infty$ in the signature, and let $P_2$ be the ramification point corresponding to $\nu_1$. Let $c$ be the geodesic on $\mathcal{S}$ given in Proposition 4. Then there exists a biholomorphism $z$, defined in a neighborhood $N$ of $P_1$, such that $z(P_1) = 0$ and $z$ maps isometrically the portion of $c$ inside $N$ into the positive real axis, with the metric of the punctured disc. These properties define the germ of $z$ uniquely.

**2.9.** We have similar results for the parabolic case.

**Proposition 6.** Let $\mathcal{S}$ be an orbifold of signature $(0, 3; \infty, 2, 2)$. Let $P \notin \mathcal{S}$ be the puncture, and let $Q$ be one of the branched points. Then there exists a unique geodesic $c: [0, \infty) \rightarrow \mathcal{S}$, such that $c(0) = Q$, $\lim_{s \rightarrow \infty} c(s) = P$, for $s$ the arc length parametrization, and $c$ realizes the distance between any two points in it.

**Proof.** We first note that there is no loss of generality in taking the group with parameters $(\infty, 0, 1)$, and in assuming that $Q$ lifts to $0$. By our definition of geodesics, any straight line joining zero and infinity will project onto a geodesic of the orbifold. It is easy to see that the imaginary axis $i\mathbb{R}$ satisfies all the required properties.

To prove uniqueness, suppose first that the geodesic on $S$ lifts to another vertical line, say $L$, whose point of intersection with the real axis is $x_0$. Then, since $x_0$ and 0 project to the same point on $S$, we must have that $x_0 = 2n$, for some integer $n$. This implies that $L$ is the image of the imaginary axis under the mapping $A^n$, and therefore $i\mathbb{R}$ and $L$ give the same geodesic on $S$.

To study the cases of non-vertical lines, identify the complex plane with $\mathbb{R}^2$, with coordinates $(x, y)$. Then we can write the lifting of our geodesic as $L' = \{y = mx\}$, with $m$ a real number. The slope $m$ cannot be zero because the real axis projects onto a line the joins that two branch points, but it stays away from the puncture. So we have that $m \neq 0$. Consider the points $2i$ on the imaginary axis and $2 + 2im$ on $L'$. Both points project onto the same point in the orbifold, since they are equivalent under the transformation $A$. The distance from 0 to $2i$ along $i\mathbb{R}$ is 2, while the distance from 0 to $2 + 2im$ along $L'$ is $2\sqrt{1 + m^2}$. Therefore $L'$ does not satisfy the properties of the proposition. □

**2.10.** We can construct local coordinates on the quotient orbifold as in the hyperbolic case, but we do not have a defining property as the one in Proposition 8,
due to the curvature constraint. The expression \( f_{12}(z) = e^{\pi i \varphi^{-1}(z)} \) gives a local coordinate on \( \mathcal{S} = \mathbb{C}/\Gamma(\infty, 2, 2; \infty, 0, 1) \) around the puncture. Here \( \varphi: \mathbb{C} \rightarrow \mathcal{S} \) is the natural projection mapping. We will say that the germ of holomorphic functions defined by \( f_{12} \) is a horocyclic coordinate.

3. Coordinates for the Teichmüller spaces of \( b \)-groups with torsion

3.1. This section is devoted to computing coordinates for the Teichmüller spaces of orbifolds with hyperbolic signature. As we said in the introduction, to parametrize the Teichmüller spaces of orbifolds, it suffices to consider in detail the one-dimensional cases. These correspond to the orbifolds of type \((0, 4)\) and \((1, 1)\). Then, the Maskit embedding theorem gives coordinates in the general deformation space.

Before starting our computations we need to have a convention about orientation of curves on orbifolds. Assume that \( a \) is a simple loop contractible to a puncture on an orbifold \( S \). Then we orient \( a \) by requiring that the puncture lies to the left of the curve.

Similarly, if \( A \) is a parabolic transformation fixing \( z_0 \in \hat{\mathbb{C}} \), and \( L \) is a horocircle through \( z_0 \), we orient \( L \) by choosing a point \( z \neq z_0 \) in \( L \), and requiring that \( z, A(z) \) and \( A^2(z) \) follow each other in the positive orientation. Observe that a horocircle passing through \( \infty \) can be understood as a circle on the Riemann sphere, and therefore it makes sense to talk about its positive orientation.

3.2. Let us start with the case of an orbifold \( S \) of signature \((0, 4; \nu_1, \ldots, \nu_4)\).

A maximal partition on \( S \) consists of a curve \( a \) that divides \( S \) into two subsets \( S_1 \) and \( S_2 \), each of them with two ramification points. Without loss of generality, we can assume that the points with ramification values \( \nu_1 \) and \( \nu_2 \) are in \( S_1 \), and this set lies to the right of \( a \). Cut \( S \) along \( a \) and glue to each resulting boundary curve a punctured disc. In this way we complete \( S_j \) to an orbifold of type \((0, 3)\). Denote these new orbifolds by the same letter, and assume for the moment that both of them, \( S_1 \) and \( S_2 \), are of hyperbolic type. We have that \( S_1 \) is uniformized by a triangle group that can be taken to be \( \Gamma_1 = \Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1) \). Since we are interested on parameters for Teichmüller spaces, we are free to conjugate by a Möbius transformation, which explains our choice of \( \Gamma_1 \). Its canonical generators are given in Theorem 3. Since \( S_1 \) lies to the right of \( a \), our orientation requirements imply that we are considering the action of \( \Gamma_1 \) on the upper half plane.

The orbifold \( S_2 \) is uniformized by a triangle group \( \Gamma_2 = \Gamma(\infty, \nu_3, \nu_4; d, e, f) \). We have that the transformation that corresponds to the curve \( a \) is the canonical generator \( A(z) = z + 2 \). Since \( S_1 \) and \( S_2 \) come both from the same orbifold, and we want to glue them to obtain \( S \), we must have that the element that corresponds to \( a \) in \( \Gamma_2 \) must be either \( A \) or \( A^{-1} \). But \( S_2 \) lies to the left of the partition curve, so we get that one of the canonical generators of \( \Gamma_2 \) is \( A^{-1} \). This implies that \( \Gamma_2 \) is conjugate to \( \Gamma(\infty, \nu_3, \nu_4; \infty, 0, -1) \) by a transformation \( T \), such
that $TA^{-1}T^{-1} = A^{-1}$. Therefore $T(z) = z + \alpha$, $\alpha \in \mathbb{C} - \{0\}$. The generators of $\Gamma_2$ are then given by $A^{-1}$ and

$$B_\alpha^{-1} = \begin{bmatrix} -q_3 - \alpha(q_3 + q_4) & -b^* - \alpha^2(q_3 + q_4) \\ -(q_3 + q_4) & -q_3 + \alpha(q_3 + q_4) \end{bmatrix}, \quad b^* = \frac{1 - q_3^2}{q_3 + q_4}.$$ 

We have $\text{Im}(\alpha) > 0$, since $S_1$ is uniformized by $\Gamma_1$ in the upper half plane. Suppose now that this imaginary part is big enough, say bigger than 2. Then the curve $\{\text{Im}(z) = \text{Im}(\alpha)/2\}$ is invariant under $A$ and lies in the intersection of the discontinuity sets of $\Gamma_1$ and $\Gamma_2$. So we can use this curve to apply the first Klein–Maskit combination theorem [14, p. 149]. In this way we obtain that the group $\Gamma_\alpha = \Gamma_1 *_{\langle A \rangle} \Gamma_2$ is a terminal regular $b$-group uniformizing an orbifold with the above signature. By the classical theory of quasiconformal mappings, we have that any orbifold of the above signature is uniformized by one such group $\Gamma_\alpha$.

The parameter $\alpha$ is then a global coordinate for the space $T(0, 4; \nu_1, \ldots, \nu_4)$. It can be expressed in an invariant way as follows. Suppose

$$\Gamma = \Gamma(\infty, \nu_1, \nu_2; a, b, c) *_{\langle C \rangle} \Gamma(\infty, \nu_3, \nu_4; d, e, f)$$

is a terminal regular $b$-group uniformizing an orbifold with the above signature. Then the point corresponding to $\Gamma$ in the deformation space is $\alpha = cr(a, b, c, f)$.

**Theorem 5.** $\alpha$ is a global coordinate, called a horocyclic coordinate, for $T(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$. The following inclusions are satisfied:

$$\{\alpha; \text{Im}(\alpha) > y_1\} \subset T(0, 4; \nu_1, \nu_2, \nu_3, \nu_4) \subset \{\alpha; \text{Im}(\alpha) > y_2\},$$

where

$$y_1 = \frac{1}{q_1 + q_2} + \frac{1}{q_3 + q_4}, \quad y_2 = \max \left(\frac{1}{q_1 + q_2}, \frac{1}{q_3 + q_4}\right).$$

**Proof.** We must show only the inclusions. The first one follows from the above reasoning about how to apply the first combination theorem.

For the second inclusion, we must use the fact that the lower half plane $X$ is precisely invariant under $\Gamma_1$ in $\Gamma_\alpha$; that is, if $\gamma \in \Gamma_\alpha$ and $\gamma(X) \cap X \neq \emptyset$, then $\gamma \in \Gamma_1$. The point $z = (-q_3 - \alpha(q_3 + q_4) - i)/(q_3 + q_4)$, has negative imaginary part; and the imaginary part of $B_\alpha(z)$ is equal to $\text{Im}(\alpha) - 1/(q_3 + q_4)$. This last number should be positive, giving one condition on the imaginary part of $\alpha$. Similarly, we have that the set $Y = \{z; \text{Im}(z) > \text{Im}(\alpha)\}$ is precisely invariant under $\Gamma_2$ in $\Gamma_\alpha$. Consider the point $w = (q_1 + i)/(q_1 + q_2)$. Its image under $B$ has imaginary part equal to $1/(q_1 + q_2)$. So, if $\text{Im}(\alpha) < 1/(q_1 + q_2)$, we then get that both, $w$ and $B(w)$, belong to $Y$. □
3.3. We can relate the above group theoretical computations to a more geometric construction by means of the plumbing constructions. This is a well known technique, and we will not explain it here in detail. See [7] for a careful treatment of it. We will only say that, given two orbifolds $S_1$ and $S_2$, with punctures $P_1$ and $P_2$ respectively, one can construct a new orbifold as follows. Remove neighborhoods $V_j$ of $P_j$, $j = 1, 2$, and identify the boundaries of the resulting orbifolds $S_j - V_j$. If such identification is given by an expression of the form $z_1 z_2 = t$, where $z_j$ is a horocyclic coordinate around $P_j$, then we say that the resulting orbifold has been constructed by plumbing techniques with parameter $t$. One can as well do plumbing constructions in one single orbifold.

To compute the plumbing parameter in the above construction of groups of type $(0, 4)$, for the group $\Gamma_1$ we take the coordinate $z(\xi) = e^{\pi i \xi}$, with $\xi$ in the upper half plane. Similarly, for the orbifold $S_2$ we take $w(\xi) = e^{\pi i (\alpha - \xi)}$, with $\text{Im}(\xi) < \text{Im}(\alpha)$. Then we get $zw = t = e^{\pi i \alpha}$. We have proven the following results, up to the (easy) computation on the bounds for plumbing parameters.

**Theorem 6.** The orbifold corresponding to the point $\alpha$ in $T(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$ is conformally equivalent to the orbifold constructed by plumbing with parameter $t = e^{\pi i \alpha}$. We have that $0 < |t| < e^{\pi y_2}$, with $y_2$ as in Theorem 5.

3.4. The case of one of the orbifolds, say $S_2$, having signature $(0, 3; \infty, 2, 2)$ is treated in a similar way. We will leave the computations to the reader, writing only the final results. The group $\Gamma_1$ is given by $\Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$, while $\Gamma_2 = \Gamma(\infty, 2, 2; \infty, \alpha, \alpha - 1)$. The generators for $\Gamma_1$ are given in Theorem 3; those of $\Gamma_2$ are

$A^{-1} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B_\alpha = B_\alpha^{-1} = \begin{bmatrix} -i & 2i \alpha \\ 0 & i \end{bmatrix}$.

**Theorem 7.** $\alpha$ is a global coordinate for $T(0, 4; \nu_1, \nu_2, 2, 2)$, and the following inclusions are satisfied:

$\{\alpha; \text{Im}(\alpha) > 0\} \subset T(\Gamma(0, 4; \nu_1 \nu_2, 2, 2)) \subset \{\alpha; \text{Im}(\alpha) > y_1\}$,

where $y_1 = 1/(q_1 + q_2)$.

We also have a result about plumbing parameters similar to the one in Theorem 6.

3.5. The other deformation spaces of dimension one correspond to orbifolds of signature $(1, 1; \nu)$. Let $S$ be an orbifold with this signature, and let $a$ be a maximal partition on $S$. If we cut $S$ along $a$ and glue punctured discs to the boundary curves, what we get is a single orbifold $S_1$ with signature $(0, 3; \infty, \infty, \nu)$. Let $\Gamma_1 = \Gamma(\infty, \infty, \nu; \infty, 0, 2)$ be a triangle group uniformizing $S_1$. To obtain $S$, what we have to do is to identify the elements $A$ and $B$ corresponding to the two punctures on $S_1$. Due to the orientation of the curves around punctures, the correct
identification is carried by a Möbius transformation $C$ such that $CB^{-1}C^{-1} = A$. The expressions of $A$ and $B$ are again given in Theorem 1. An easy computation shows that $C$ has the form

$$C = \begin{bmatrix} i\tau & i\sqrt{2/(1+q)} \\ i\sqrt{(1+q)/2} & 0 \end{bmatrix}.$$ 

Applying the second Klein–Maskit combination theorem we get that the group $\Gamma_\tau = \Gamma_1*C := \langle \Gamma_1, C \rangle$ is a terminal regular $b$-group of the desired signature.

**Theorem 8.** $\tau$ is a global coordinate for $T(1, 1; \nu)$, and we have the following inclusions:

$$\{\tau; \Im(\tau) > 2\} \subset T(\Gamma(1, 1; \nu)) \subset \{\tau; \Im(\tau) > 0\}.$$ 

**Proof.** Observe that $C$ maps horocircles at $0$ (that is, circles passing through zero) to straight lines (horocircles at $\infty$). In particular we have

$$C(\{z; |z-ri| = r\}) = \{z; \Im(z) = \sqrt{\frac{2}{1+q}} \Im(\tau) - \frac{1}{r(1+q)} \}. $$

If these two circles are disjoint, the second combination theorem can be applied. Therefore we want

$$\sqrt{\frac{2}{1+q}} \Im(\tau) - \frac{1}{r(1+q)} > 2r, \quad \text{or} \quad \Im(\tau) > \frac{1+q}{2}\left(\frac{1}{r(1+q)} + 2r\right).$$

The minimum value of the last expression is 2. This gives the first inclusion of the theorem. For the other inclusion we just need to use the fact that the lower half plane is precisely invariant under $\Gamma_1$ in $\Gamma_\tau$. So for any point $z$ with negative imaginary part we should have

$$\Im(C(z)) = \sqrt{\frac{2}{1+q}} \Im(\tau) - \frac{2}{1+q} \frac{\Im(z)}{|z|} > 0,$$

which gives the desired result.

**3.6.** As in the $(0, 4)$ case, we have a plumbing construction for these orbifolds. Take the coordinates on $S_1$ given by $z = e^{\pi i \zeta}$ and $w = -\exp((-2\pi i)/(1+q)\zeta)$, near the punctures determined by $A$ and $B$, respectively. The identification we have to make in this case is given by

$$z(C(\zeta))w(\zeta) = z\left(\sqrt{\frac{2}{1+q}} + \frac{2}{1+q} \frac{1}{\zeta}\right)w(\zeta) = \exp\left(\frac{2\pi i}{1+q}\right)\exp\left(\sqrt{\frac{2}{1+q}} \pi i \tau\right) = t.$$

This proves the following result:
The orbifold corresponding to the point $\tau$ in $T(1, 1; \nu)$ is conformally equivalent to the orbifold constructed by plumbing with parameter $t$, with $0 < |t| < 1$.

3.7. Let $S$ now be a hyperbolic orbifold with a maximal partition $S$, uniformized by the terminal regular $b$-group $\Gamma$. As we noted in Section 3.1, we can decompose the group into subgroups with one dimensional deformation spaces. We have explained in detail how to parametrize these simpler Teichmüller spaces. Together with the Maskit embedding, we obtain the following results for the general situation.

**Theorem 10.** Let $S$ be an orbifold of hyperbolic type with signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ and let $\mathcal{C}$ be a maximal partition on $S$, uniformized by the terminal regular $b$-group $\Gamma$. Then there exists a set of (global) coordinates, $(\alpha_1, \ldots, \alpha_d)$, called horocyclic coordinates, for the Teichmüller space $T(\Gamma) \cong T(p, n; \nu_1, \ldots, \nu_n)$, where $d = 3p - 3 + n$, and a set of complex numbers, $(y_1^1, y_1^2, \ldots, y_2^d)$, which depends on the signature $\sigma$ and the partition $\mathcal{C}$, such that

$$\{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d; \text{Im}(\alpha_i) > y_i^1, \forall 1 \leq i \leq d\} \subset T(p, n; \nu_1, \ldots, \nu_n)$$

and

$$T(p, n; \nu_1, \ldots, \nu_n) \subset \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d; \text{Im}(\alpha_i) > y_2^i, \forall 1 \leq i \leq d\}.$$  

Moreover, the surface corresponding to the point $(\alpha_1, \ldots, \alpha_d)$ is conformally equivalent to a surface constructed by plumbing techniques with parameters $(t_1, \ldots, t_d)$, obtained as in Theorems 5 and 8.

It is clear that given a point $\alpha$ in $T(p, n; \nu_1, \ldots, \nu_n)$, one can construct a set of Möbius transformations that generate a group $\Gamma$, which corresponds to $\alpha$. The one dimensional case has been done explicitly. In the general case, one has simply to iterate the constructions explained above. For a more detailed description of this process, see [7, § 7.5], with the necessary modifications to include finite order transformations. The above techniques can be generalized to constructions of Kleinian groups without parabolic elements as well; see [1].

4. The Patterson isomorphisms in the horocyclic coordinates

4.1. One of the most natural questions one may ask about Teichmüller spaces is that under what circumstances two such spaces are biholomorphic. A result of Patterson ([19], [3]) states that all possible isomorphisms between Teichmüller space of hyperbolic orbifolds of different type (with $2p - 2 + n > 0$) are $T(2, 0) \cong T(0, 6; 2, 2, 2, 2, 2), \ T(1, 2; \infty, \infty) \cong T(0, 5; \infty, 2, 2, 2, 2)$ and $T(1, 1; \infty) \cong T(0, 4; \infty, 2, 2, 2)$. The existence of these isomorphisms is based on the fact that
all surfaces of genus 2, or of genus 1 with either two or one punctures, have a conformal involution (hyperelliptic involution); the quotient of the surface by that involution is a sphere with six, five or four points, with ramification values as above. Our main result is as follows.

**Theorem 11.** The mapping

\[
(\tau_1, \tau_2, \tau_3) \mapsto (\frac{1}{2} \tau_1, 1 + \tau_2, 1 + \frac{1}{2} \tau_3)
\]

gives an isomorphism between \(T(0,2)\) and \(T(0,6;2,\ldots,2)\) in the horocyclic coordinates \(\tau_j, j = 1,2,3\), corresponding to the partition given in Figure 3.

4.2. We start with a surface of genus 2 with a maximal partition as shown in Figure 3.

Let \(\Gamma\) denote a terminal regular \(b\)-group uniformizing the surface and the partition in the simply connected invariant component \(\Delta\). A presentation for \(\Gamma\) can be found in [7]; we copy it here for the convenience of the reader. \(\Gamma = \{ A_1, C_1, A_3, C_3; A_1, A_2 = [C_1^{-1}, A_1], A_3 \text{ are accidental parabolic, } [A_1, C_1^{-1}] \circ [A_3^{-1}, C_3^{-1}] = I \}\), where \([A,B] = ABA^{-1}B^{-1}\). The elements \(A_i\) correspond to the curves \(a_i\), while \(C_i\) correspond to \(c_i\). These transformations have the following expressions:

\[
A_1 = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix},
\]
\[
A_3 = \begin{bmatrix} -1 - 2\tau_2(1 - \tau_2) & -2(1 - \tau_2)^2 \\ 2\tau_2^2 & -1 + 2\tau_2(1 - \tau_2) \end{bmatrix},
\]
\[
C_1 = i \begin{bmatrix} \tau_1 & 1 \\ 1 & 0 \end{bmatrix},
\]
\[
C_3 = i \begin{bmatrix} \tau_3 \tau_2^2 + 2(1 - \tau_3)\tau_2 + \tau_3 - 2 & -\tau_3 \tau_2^2 + (3\tau_3 - 2)\tau_2 - 2\tau_3 + 3 \\ \tau_3 \tau_2^2 + (2 - \tau_3)\tau_2 - 1 & -\tau_3 \tau_2^2 - 2(1 - \tau_3)\tau_2 + 2 \end{bmatrix}.
\]

\(\tau_1, \tau_2, \text{ and } \tau_3\) are complex numbers.

The key ingredients in the proof of our theorem is the fact that the hyperelliptic involution lifts to a Möbius transformation in the covering determined by \(\Gamma\) and \(\Delta\). More precisely, we have that such lifting is given by the transformation \(A_2^{1/2} = 1/(-z + 2)\) (see [7]).
**Proposition 7.** The subgroup $\tilde{\Gamma}$ of $\text{PSL}(2, \mathbb{C})$ generated by $\Gamma$ and $A_2^{1/2}$ is a terminal regular $b$-group of signature $(0, 6; 2, 2, 2, 2, 2, 2)$.

**Proof.** The facts that $\tilde{\Gamma}$ is Kleinian and geometrically finite follow from [14, V.E.10, p. 98, and V.I.E.6, p. 132, respectively]. We also get that $\Omega(\tilde{\Gamma}) = \Omega(\Gamma) = \Omega$. Recall that $\Delta$ is the invariant component of $\Gamma$. If $A_2^{1/2}(\Delta) = U$, where $U$ is some component of $\tilde{\Gamma}$, then for all $\gamma \in \Gamma$ we have $A_2^{1/2} \gamma A_2^{-1/2}(U) = U$. This implies that $A_2^{1/2}(\Delta) = \Delta$, since $A_2^{1/2} \gamma A_2^{-1/2} \in \Gamma$ and $\Delta$ is the unique invariant component of this group. The signature of $\Delta/\Gamma$ is a consequence of the fact that the hyperelliptic involution fixes six points on the surface $\Delta/\Gamma$. The statement about the accidental parabolic elements of $\tilde{\Gamma}$ is trivial.

To finish the proof we need to show that $(\Omega - \Delta)/\tilde{\Gamma}$ is a union of orbifolds of type $(0, 3)$ (of certain signatures). Let $\Omega_0$ be a component of $\Omega - \Delta$, and let $\Gamma_0 = \text{stab}(\Omega_0, \Gamma) = \{ \gamma \in \Gamma; \gamma(\Omega_0) = \Omega_0 \}$ be the stabilizer of $\Omega_0$ in $\Gamma$. Since $\Gamma$ is a terminal regular torsion $b$-group of signature $(2, 0)$, we have that $\Omega_0/\Gamma_0$ is an orbifold of signature $(0, 3; \infty, \infty, \infty)$. If $A_2^{1/2}(\Omega_0) \neq \Omega_0$, then $\Gamma_0 = \tilde{\Gamma}_0$, where $\tilde{\Gamma}_0 = \text{stab}(\Omega_0, \tilde{\Gamma})$, and therefore the quotient orbifolds $S_0 = \Omega_0/\Gamma_0$ and $\Omega_0/\tilde{\Gamma_0}$ are equal. If, to the contrary, $A_2^{1/2}$ fixes $\Omega_0$, then it will induce a biholomorphic mapping in $S_0$, say $f$. Since $A_2^{1/2} \notin \Gamma$ but $A_2 \in \Gamma$, we have that $f$ is not trivial and has order 2. We also have that $f$ fixes the puncture determined by $A_2$, since $A_2^{1/2}$ commutes with that element. Therefore, $f$ has to interchange the other two punctures and this implies that $\Omega_0/(\Gamma_0, A_2^{1/2}) = S_0/(f)$ has signature $(0, 3; \infty, \infty, 2)$. Observe that we have used the fact that $\tilde{\Gamma}_0$ is generated by $\Gamma_0$ and $A_2^{1/2}$. \qed

The group $\tilde{\Gamma}$ has the following presentation $\tilde{\Gamma} = \{ A_1, A_3, C_1, C_3, A_2^{1/2}; A_1, A_2^{1/2}, A_3, A_2^{-1}, A_3^{-1} \}$ are accidental parabolics, $A_2^{-1/2} C_1^{-1}, C_1 A_2^{1/2} A_1, A_1^{-1} A_2^{-1/2}, A_2^{1/2} A_3, C_3 A_2^{-1/2}, A_2^{-1} C_3^{-1} A_3^{-1}$ are elliptic elements of order 2}. We will write only the expressions of the generators of $\tilde{\Gamma}$ that we will use in this proof:

\[
(C_1 A_2^{1/2}) = i \begin{bmatrix} -1 & 2 + \tau_1 \\ 0 & 1 \end{bmatrix}, \quad A_1^{-1} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad A_2^{1/2} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},
\]
\[
A_3^{-1} = \begin{bmatrix} -1 + 2\tau_2(1 - \tau_2) & 2(1 - \tau_2)^2 \\ -2\tau_2^2 & -1 - 2\tau_2(1 - \tau_2) \end{bmatrix},
\]
\[
C_3 A_2^{-1/2} = i \begin{bmatrix} -1 + 2\tau_2 - \tau_2^2 \tau_3 + \tau_2^2 \tau_3 & 2 - \tau_3 - 2\tau_2 - \tau_2^2 \tau_3 + 2\tau_2 \tau_3 \\ 2\tau_2 + \tau_2^2 \tau_3 & 1 - 2\tau_2 + \tau_2 \tau_3 - \tau_2^2 \tau_3 \end{bmatrix}.
\]

**4.3.** Let $\mathcal{F}$ be a terminal regular $b$-group with signature $(0, 6; 2, 2, 2, 2, 2, 2)$ constructed by the techniques of Section 3, and corresponding to the orbifold shown in Figure 4.
Remark. It is easy to see that if we apply the hyperelliptic involution to the surface of Figure 3 we obtain a partition in the quotient surface as given in Figure 4. That is why we have chosen this partition among all the possible ones on an orbifold with signature \((0, 6; 2, 2, 2, 2, 2)\).

A set of generators for \(\mathcal{F}\) (equivalent to the generators of \(\tilde{\Gamma}\) computed above) consists of the following transformations:

\[
D_1 = i \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 - \alpha & \alpha^2 \\ -1 & -1 + \alpha \end{bmatrix},
\]

\[
B_3 = \begin{bmatrix} -1 + 2\beta + 2\alpha\beta^2 & -2(1 + \alpha\beta)^2 \\ 2\beta^2 & -1 - 2\beta - 2\alpha\beta^2 \end{bmatrix},
\]

\[
D_4 = i \begin{bmatrix} -1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} - 2(1 + \alpha\beta)(-\alpha\beta^2 + \gamma + \alpha\beta\gamma)\beta^{-2} \\ 2\gamma - 2\beta \end{bmatrix} \begin{bmatrix} -1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} - 2(1 + \alpha\beta)(-\alpha\beta^2 + \gamma + \alpha\beta\gamma)\beta^{-2} \\ 1 + 2\alpha\beta - 2\alpha\gamma - 2\gamma\beta^{-1} \end{bmatrix}.
\]

\(\alpha, \beta, \text{ and } \gamma\) are three complex numbers, chosen so that the above matrices have nice expressions.

To complete the proof of the theorem, we have to find a Möbius transformation \(E\) such that \(E\mathcal{F}E^{-1} = \tilde{\Gamma}\), and then we have to express the coordinates of \(\mathcal{F}\) in terms of the numbers \(\alpha, \beta\) and \(\gamma\). Topological considerations give that \(EB_1E^{-1} = A_1\) and \(EB_2E^{-1} = A_2^{-1/2}\). This implies that \(E(z) = -z + 1 + \alpha\).

We also have \(ED_1E^{-1} = A_2^{-1/2}C_1^{-1}\) which gives \(\alpha = \frac{1}{2}\tau_1\). It is a matter of computation to see that the conjugation \(EB_3E^{-1} = A_3^{-1}\) gives the following four equations, whose unique answer is \(\beta = \tau_2\):

\[
\begin{align*}
-1 + 2\beta + 2\alpha\beta^2 + 2\beta^2b & = 1 + 2\tau_2(1 - \tau_2), \\
-2(1 + \alpha\beta)^2 + b(-1 - 2\beta - 2\alpha\beta^2) & = -b(-1 - 2\tau_2 + 2\tau_2^2) - 2(1 - \tau_2)^2, \\
-2\beta^2 & = -2\tau_2^2, \\
1 + 2\beta + 2\alpha\beta^2 & = -2\tau_2^2b - 1 + 2\tau_2(1 - \tau_2).
\end{align*}
\]

The relation \(ED_4E^{-1} = C_3A_2^{-1/2}\) gives the set of equations

\[
\begin{align*}
-1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} + b(2\gamma - 2\beta) & = -1 + 2\tau_2 - \tau_2\tau_3 + \tau_2^2\tau_3, \\
2\beta - 2\gamma & = 2\tau_2 + \tau_2^2\tau_3,
\end{align*}
\]

Figure 4. A surface with signature \((0, 6; 2, 2, 2, 2, 2)\).
whose answer is $\gamma = -\frac{1}{2} \tau_2 \tau_3$.

In [7] is proven that $(\tau_1, \tau_2, \tau_3)$ is a set of coordinates for $T(2,0)$. The coordinates for $T(0,6; 2, 2, 2, 2, 2, 2)$ are given by $z_1 = \alpha$, $z_2 = 1 + \beta$ and $z_3 = 1 - (\gamma/\beta^2)$. Substituting the values of $\alpha$, $\beta$ and $\gamma$ obtained above we get that the expression of the isomorphism is the one given in the statement of Theorem 11.

4.4. As a corollary of the proof of Theorem 11 we obtain the other isomorphisms of Section 4.4.

**Corollary 1.** The mappings

$$
\tau_1 \mapsto \frac{1}{2} \tau_1 \quad \text{and} \quad (\tau_1, \tau_2) \mapsto (\frac{1}{2} \tau_1, 1 + \tau_2),
$$

give the isomorphisms $T(1, 1; \infty) \cong T(0, 4; \infty, 2, 2, 2)$ and $T(1, 2; \infty, \infty) \cong T(0, 5; \infty, 2, \ldots, 2)$, respectively, for some choice of horocyclic coordinates.

**Proof.** The argument goes as follows: to construct the surface of genus 2 with the partition given in the Figure 3, we start with a thrice punctured sphere, $S_1$; then we glue two of the punctures, obtaining a surface $S_2$ with signature $(1, 1; \infty)$. This construction uses only the coordinate $\tau_1$, and therefore it gives the first isomorphism of the theorem. The next step is to glue to the puncture of $S_2$ a three times punctured sphere to get a surface $S_3$, with signature $(1, 2; \infty, \infty)$. For this construction we need the coordinates $(\tau_1, \tau_2)$. Thus we obtain the second isomorphism, completing the proof of the theorem.

This technique can be used to compute different isomorphisms between Teichmüller spaces. For example, there is another partition of a surface of genus 2; the hyperelliptic involution can be found in [7]. With computations similar to the ones described in this section, one can find the Patterson isomorphisms in that case. By a result of Kravetz, the set of fixed points of a biholomorphic map of finite order on a Teichmüller space is isomorphic to another Teichmüller space of lower dimension (see [8] or [18, pp. 259–260]). Some of those isomorphisms can also be studied with our techniques.

**References**


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