CARLESON MEASURE, ATOMIC DECOMPOSITION
AND FREE INTERPOLATION FROM BLOCH SPACE

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Abstract. Several characterizations, Carleson measures and atomic decomposition for the
Bloch space \( B \) are given. For their applications, free interpolations from \( B \) are also discussed.

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the unit disk in the finite complex plane \( \mathbb{C} \) and
\[
dm_\alpha(z) = (1 - |z|^2)^\alpha \, dm(z)
\]
the two-dimensional Lebesgue measure with weight \((1 - |z|^2)^\alpha\), \( \alpha > -1 \). Denote by \( A \) and \( H^\infty \) the sets of functions analytic and
boundedly analytic on \( D \), respectively. For \( f \in A \) we say \( f \in B \) if
\[
\|f\|_B = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty;
\]
also \( f \in A^1_\alpha \) if
\[
\|f\|_{1,\alpha} = \int_D |f(z)| \, dm_\alpha(z) < \infty.
\]

\( B \) and \( A^1_0 \) are the so-called Bloch space and the Bergman space weighted by
\((1 - |z|^2)^\alpha\), \([6],[15]\).

It is well known that the dual space of \( A^1_0 \) is identified with \( B \) under the following inner product:
\[
\langle f, g \rangle = \frac{1}{\pi} \lim_{t \to 1} \int_{tD} f(z)\overline{g(z)} \, dm(z)
\]
\[
= \frac{1}{\pi} \int_D (\vee f)(z)(1 - |z|^2)\overline{g'(z)} \, dm(z) + f(0)\overline{g(0)}
\]
for \( f \in A^1_0 \) and \( g \in B \), where \( t \in (0,1) \), \( tD = \{ z : |z| < t \} \) and \( (\vee f)(z) = [f(z) - f(0)]/z \); see \([3]\).
In [11] we discussed the atomic decomposition and the free interpolation on the Bergman space \( A^1_0 \). Since \((A^1_0)^* = B\), it is very natural to consider similar problems on the Bloch space. As far as we know, these questions have not been thoroughly dealt with yet ([8], [13]), which is what we try to do in this paper. First, in Section 2, we give several characterizations of \( B \) as well as relations between \( B \) and Carleson measure. Next, in Section 3, we obtain an atomic decomposition of \( B \) by means of the pseudohyperbolic metric. Finally, in Section 4, we study the free interpolations by functions from \( B \) by means of the direct construction and the operator theory.

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2. Bloch space and Carleson measure

There are many works on the Bloch space, [1], e.g. [2], [7], [10]. Here we will give several interesting characterizations, some of which are new.

For \( z \) and \( w \) in \( D \), let \( \varphi_w(z) = (w - z)/(1 - \overline{w}z) \), \( \varrho(w, z) = |\varphi_w(z)| \) and \( d(w, z) = \frac{1}{2} \log \{1 + \varrho(w, z) /[1 - \varrho(w, z)]\} \). Here \( \varrho(\cdot, \cdot) \) and \( d(\cdot, \cdot) \) are called the pseudohyperbolic and hyperbolic distances, respectively. Also, denote the measure of set \( E \subset D \), relative to \( dm_\alpha(z) \), by \( m_\alpha(E) = \int_{E} dm_\alpha(z) = \int_{E} (1 - |z|^2)^\alpha dm(z) \).

Then we have the following result.

**Theorem 2.1.** Let \( f \in A \). Then the following statements are equivalent:

(i) \( f \in B \);

(ii) \( \sup_{w, z \in D} |f(w) - f(z)|/d(w, z) < \infty \);

(iii) there is a constant \( C > 0 \) such that

\[
\sup_{w \in D} \int_D \exp\left[C \left( |f \circ \varphi_w(z) - f(w)| \right) \right] dm_\alpha(z) < \infty.
\]

**Proof.** We will show this fact according to (i) \( \implies \) (ii) \( \implies \) (iii) \( \implies \) (i).

Firstly, (i) \( \implies \) (ii). Let \( f \in B \), and \( g_w(\lambda) = (f \circ \varphi_w)(\lambda) - f(w), \lambda, w \in D \). Then \( g_w(0) = 0 \) and \( \|g_w\|_B = \sup_{z \in D} (1 - |z|^2) |f'(z)| \leq \|f\|_B < \infty \). Further,

\[
|g_w(\lambda)| = \left| \int_0^\lambda g'_w(\zeta) d\zeta \right| \leq \frac{1}{2} \|f\|_B \log \frac{1 + |\lambda|}{1 - |\lambda|}.
\]

Setting \( z = \varphi_w(\lambda) \) we obtain

\[
|f(z) - f(w)| \leq \frac{1}{2} \|f\|_B \log \frac{1 + \varrho(z, w)}{1 - \varrho(z, w)} = \|f\|_B d(z, w),
\]

i.e., (ii) holds.
Secondly, \((\text{ii}) \implies (\text{iii})\). Suppose that
\[
0 < \|f\|'_B = \sup_{w, z \in D} \left| f(w) - f(z) \right| / d(w, z) < \infty;
\]
then for \(t \geq 0\),
\[
\left\{ z : z \in D, \ |g_w(z)| > t \right\} \subset \left\{ z : z \in D, \ |z| > \left[ \exp\left( \frac{2t}{\|f\|'_B} \right) - 1 \right] / \left[ \exp\left( \frac{2t}{\|f\|'_B} \right) + 1 \right] \right\}.
\]
Moreover, when \(0 < C < \left[ 2(\alpha + 1) / \|f\|''_B \right] \),
\[
\int_D \exp[C|g_w(z)|] \, dm_\alpha(z) = C \int_0^\infty (\exp Ct) \cdot m_\alpha(\left\{ z : z \in D, \ |g_w(z)| > t \right\}) \, dt
\leq C \int_0^\infty (\exp Ct) \frac{4\pi}{\alpha + 1} \exp\left( -\frac{2(\alpha + 1)t}{\|f\|'_B} \right) \, dt
= \frac{4\pi C}{(\alpha + 1)[2(\alpha + 1) - C\|f\|''_B]}.
\]
Thirdly, \((\text{iii}) \implies (\text{i})\). Let
\[
\|f\|''_B = \sup_{w \in D} \int_D \exp[C|(f \circ \varphi_w)(z) - f(w)|] \, dm_\alpha(z) < \infty
\]
for some constant \(C > 0\). Then
\[
\|(f \circ \varphi_w)(z) - f(w)\|_{1,\alpha} \leq \frac{\|f\|''_B}{C} < \infty.
\]
Since \(g_w\) has a Taylor series \(\sum_n a_n z^n\) which converges uniformly on \(tD \ (0 < t < 1)\), a simple calculation gives
\[
a_1 = g_w'(0) = \frac{(\alpha + 1)(\alpha + 2)}{1 - [1 + (\alpha + 1)t^2](1 - t^2)^{\alpha + 1}} \int_{tD} g_w(z) z \, dm_\alpha(z).
\]
By letting \(t \to 1\) we get
\[
|g_w'(0)| \leq (\alpha + 1)(\alpha + 2) \int_D |g_w(z)| \, dm_\alpha(z),
\]
i.e.,
\[
(1 - |w|^2)|f'(w)| \leq \left[ \frac{(\alpha + 1)(\alpha + 2)}{C} \right] \cdot \|f\|''_B.
\]
So, \(f \in B\).
Remark. This theorem tells us that $B$ is Lipschitz’s class, relative to the hyperbolic metric $d(\cdot, \cdot)$. However, we know that $B$ can be identified with the Zygmund class (see [1], [5]). Hence our result is much clearer than the one in [1].

In what follows we characterize connection between the Bloch space and Carleson measure.

For $w \in D$ let $D(w, r) = \{ z : z \in D, g(w, z) < r \}$, $r \in (0, 1)$. $D(w, r)$ is called the pseudohyperbolic disk. It is more convenient to use $D(w, r)$ (not Carleson square) for discussing Borel measure on the Bergman space $A_1^1$; see [6], [12]. Similarly, we have the following theorem.

Theorem 2.2. Let $p \in (0, \infty)$ and $r \in (0, 1)$, and let $\mu$ be a nonnegative Borel measure on $D$. Then the following statements are equivalent:

(i) $\sup_{w \in D \atop 0 \neq f \in B} \left[ \frac{1}{\|f\|_B^p} \int_D \left| f(z) - f(w) \right|^p \frac{(1 - |w|^2)^{2+\alpha}}{1 - \overline{w}z|^{4+2\alpha}} \, d\mu(z) \right]^{1/p} < \infty$;

(ii) $\sup_{w \in D \atop \mu(D(w, r))} \left[ \frac{\mu(D(w, r))}{m_\alpha(D(w, r))} \right] < \infty$;

(iii) $\sup_{w \in D \atop \int_D \frac{(1 - |w|^2)^{2+\alpha}}{1 - \overline{w}z|^{4+2\alpha}} \, d\mu(z)} < \infty$.

Proof. (ii) $\Leftrightarrow$ (iii) has been derived in [13], so we only need to claim (i) $\Leftrightarrow$ (ii).

On the one hand, if (ii) is true, it follows by Theorem 2.1 that

$$\left[ \int_D \left| (f \circ \varphi_w)(z) - f(w) \right|^p \, dm_\alpha(z) \right]^{1/p} \leq C\|f\|_B$$

for $f \in B$, where $C > 0$ is a constant independent of $f$. Further, by [15], [12] and [6] it yields another constant $C_0$ depending on the condition (ii) such that

$$\left[ \int_D \left| f(z) - f(w) \right|^p \frac{(1 - |w|^2)^{2+\alpha}}{1 - \overline{w}z|^{4+2\alpha}} \, d\mu(z) \right]^{1/p} \leq C_0 \left[ \int_D \left| f(z) - f(w) \right|^p \frac{(1 - |w|^2)^{2+\alpha}}{1 - \overline{w}z|^{4+2\alpha}} \, dm_\alpha(z) \right]^{1/p}$$

$$= C_0 \left[ \int_D \left| (f \circ \varphi_w)(z) - f(w) \right|^p \, dm_\alpha(z) \right]^{1/p} \leq C_0 C\|f\|_B.$$ 

On the other hand, let (i) hold. Taking $f_0(z) = [1/(1 - \overline{w_0}z)] - 1$ for $w_0 = \left(-\frac{1}{2}(r + 1) + w\right)/(1 - \frac{1}{2}(r + 1) \cdot \overline{w})$, $\frac{1}{2}(r + 1) \neq w \in D$, $r \in (0, 1)$, we get
\[ \|f_0\|_B = |w_0|/(1 - |w_0|^2), \quad |f_0(z) - f_0(w_0)| = |w_0||z - w_0|/|1 - \overline{w_0}z|(1 - |w_0|^2) \text{ and} \]

\[
\sup_{z \in D(w, r)} |1 - \overline{w_0}z| = \sup_{\lambda \in rD} \left| 1 - \frac{w - \frac{1}{2}(r + 1)}{1 - \overline{\lambda}(r + 1)} \cdot \frac{w - \lambda}{1 - \overline{w}\lambda} \right| \leq \frac{(2 + r^2 + r)(1 - |w|^2)}{(1 - r)^2}.
\]

Also, there are two constants \( C_1 > 0 \) and \( C_2 > 0 \) depending only on \( \alpha \) and \( r \) such that (see [14])

\[ C_1 \cdot (1 - |w|^2)^{2+\alpha} \leq m_\alpha(D(w, r)) \leq C_2 \cdot (1 - |w|^2)^{2+\alpha}. \]

We also have

\[
\infty > \sup_{\lambda \in D, 0 \neq f \in B} \left[ \frac{1}{\|f\|_B^p} \cdot \int_D |f(z) - f(\lambda)|^p \cdot \frac{(1 - |\lambda|^2)^{2+\alpha}}{|1 - \lambda z|^{4+2\alpha}} d\mu(z) \right]^{1/p} \\
\geq \left[ \frac{1}{\|f_0\|_B^p} \cdot \int_D |f_0(z) - f_0(w_0)|^p \cdot \frac{(1 - |w_0|^2)^{2+\alpha}}{|1 - \overline{w_0}z|^{4+2\alpha}} d\mu(z) \right]^{1/p} \\
\geq \left[ \frac{1}{|w_0|^2} \cdot \int_{D(w, r)} \left( \frac{|w_0|}{1 - |w_0|^2} \right)^p \left[ \varrho(z, w_0) \right]^p \cdot \frac{(1 - r)^{4(2+\alpha)}}{(2 + r + r^2)^{4+2\alpha}} \left[ \frac{1}{(1 - |w|^2)^{2+\alpha}} d\mu(z) \right]^{1/p} \\
\geq \frac{(1 - r)^{4(2+\alpha)/p}}{2^{(4\alpha+8+p)/p}} C_1^{1/p} \cdot \left[ \frac{\mu(D(w, r))}{m_\alpha(D(w, r))} \right]^{1/p}
\]

Therefore

\[
\sup_{w \in D} \left[ \frac{\mu(D(w, r))}{m_\alpha(D(w, r))} \right] < \infty. \quad \Box
\]

The measure \( \mu \) satisfying one of the three statements in Theorem 2.2 is said to be \( \alpha \)-Carleson measure. The following fact is interesting.

**Theorem 2.3.** Let \( f \in A \). Then the following statements are equivalent:

(i) \( f \in B \);
(ii) \( |f'(z)|^2 (\log 1/|z|)^2 \, dm(z) \) is 0-Carleson measure;
(iii) \( |f'(z)|^2 (1 - |z|^2)^2 \, dm(z) \) is 0-Carleson measure.
Proof. We will give the whole claim in accordance with the order (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

First of all, (i) $\implies$ (ii). Under $f \in B$, we consider the integral below:

$$I_1 = \int_D \frac{(1 - |w|^2)^2}{|1 - w^2|^4} |f'(z)|^2 \left( \log \frac{1}{|z|} \right)^2 \, dm(z) = \left( \int \{ |z| > \frac{1}{4} \} + \int \{ |z| \leq \frac{1}{4} \} \right) \{ \cdots \} \, dm(z).$$

Since $\log(1/|z|) \leq C_1 (1 - |z|^2)$ when $|z| > \frac{1}{4}$,

$$\int \{ |z| > \frac{1}{4} \} \{ \cdots \} \, dm(z) \leq C_1^2 \int \frac{|f'(z)|^2 (1 - |w|^2)^2 (1 - |z|^2)^2}{|1 - w^2|^4} \, dm(z) \leq C_1^2 \cdot \| f \|_B^2 \int_D \frac{(1 - |w|^2)^2}{|1 - w^2|^4} \, dm(z) \leq \pi C_1^2 \| f \|_B^2,$$

where $C_1 > 0$ is an absolute constant. At the same time

$$\int \{ |z| \leq \frac{1}{4} \} \{ \cdots \} \, dm(z) \leq \left( \frac{16}{15} \right)^2 \| f \|_B^2 \int \frac{(1 - |w|^2)^2}{|1 - w^2|^4} \left( \log \frac{1}{|z|} \right)^2 \, dm(z) \leq \left( \frac{16}{15} \right)^2 \cdot \frac{4^4}{3^4} \| f \|_B^2 \int \{ |z| \leq \frac{1}{4} \} \left( \log \frac{1}{|z|} \right)^2 \, dm(z) = C_2 \| f \|_B^2,$$

where $C_2 > 0$ is an absolute constant. Consequently

$$I_1 \leq (\pi C_1^2 + C_2) \| f \|_B^2.$$

So, from Theorem 2.2 (iii) we see that $|f'(z)|^2 \left( \log(1/|z|) \right)^2 \, dm(z)$ is 0-Carleson measure.

Next (ii) $\implies$ (iii). This is obvious, since $(1 - |z|^2)^2 \leq 4(\log(1/|z|))^2$ for all $z \in D$.

Finally (iii) $\implies$ (i). Assuming that $|f'(z)|^2 (1 - |z|^2)^2 \, dm(z)$ is 0-Carleson measure, we have

$$I_2 = \sup_{w \in D} \int_D |f'(z)|^2 \cdot \frac{(1 - |z|^2)^2 (1 - |w|^2)^2}{|1 - w^2|^4} \, dm(z) < \infty,$$

and obviously $\infty > I_2 \geq \int_D |f'(z)|^2 (1 - |z|^2)^2 \, dm(z)$. Moreover,

$$(f'(w))^2 = \frac{3}{\pi} \int_D (f'(\lambda))^2 \cdot \frac{(1 - |\lambda|^2)^2}{(1 - w\lambda)^4} \, dm(\lambda).$$

Hence

$$(1 - |w|^2)^2 |f'(w)|^2 \leq \frac{3}{\pi} \int_D |f'(\lambda)|^2 \cdot \frac{(1 - |\lambda|^2)^2 (1 - |w|^2)^2}{|1 - w\lambda|^4} \, dm(\lambda) \leq I_2 < \infty,$$

i.e., $f \in B$.

Supposing $g_D(z, w) = \log \left| (1 - w^2)/(w - z) \right|$ (the Green’s function on $D$), we just have
Corollary 2.4. Let \( f \in A \). Then \( f \in B \) if and only if

\[
(2.1) \quad \sup_{w \in D} \int_D \left| f'(z) \right|^2 g_D^2(z, w) \, dm(z) < \infty.
\]

Proof. This fact is readily derived from the equivalence between \( \| f \circ \varphi_w \|_B \) and \( \| f \|_B \), and Theorem 2.3 (ii). Nevertheless, the result can also be shown by Theorem 2.1 and 2.3. \( \square \)

3. Atomic decomposition

To begin with, we let \( 1^1 \) and \( 1^\infty \) stand for the usual sequence spaces as follows:

\[
(3.1) \quad 1^1 = \left\{ \{ c_n \} : \{ c_n \} \subset \mathbb{C}, \| \{ c_n \} \|_1 = \sum_n |c_n| < \infty \right\},
\]

\[
(3.2) \quad 1^\infty = \left\{ \{ c_n \} : \{ c_n \} \subset \mathbb{C}, \| \{ c_n \} \|_\infty = \sup_n |c_n| < \infty \right\}.
\]

Both are Banach spaces. Also, suppose that \( \{ z_n \} \) is a sequence of points on \( D \). A sequence of points \( \{ z_n \} \) is called \( \delta \)-weakly separated if \( \delta = \inf_{m \neq n} \varrho(z_m, z_n) > 0 \) and \( \eta \)-uniformly separated if \( \eta = \inf_n \prod_{m \neq n} \varrho(z_m, z_n) > 0 \). Clearly an \( \eta \)-uniformly separated sequence must be \( \delta \)-weakly separated. A sequence of points \( \{ z_n \} \) is said to be \( \varepsilon \)-dense if \( D = \bigcup_n D(z_n, \varepsilon) \), where \( D(z_n, \varepsilon) = \{ z : z \in D, \varrho(z_n, z) < \varepsilon \} \) and \( \varepsilon \in (0, 1) \).

Luecking [6] and Xiao [12] proved the quasi-atomic decomposition theorem of \( A_{\alpha}^1 \) as follows.

Lemma 3.1. Let \( \{ z_n \} \) be a sequence of points on \( D \), \( \alpha > -1 \) and \( f \in A_{\alpha}^1 \). If \( \{ z_n \} \) is \( \delta \)-weakly separated, there is a constant \( C_1 > 0 \) depending only on \( \delta \) and \( \alpha \) so that

\[
(3.3) \quad \| f \|_{1, \alpha} \geq C_1 \cdot \sum_n (1 - |z_n|^2)^{2+\alpha} |f(z_n)|.
\]

Furthermore, there are an \( \varepsilon_0 > 0 \) and a constant \( C_2 > 0 \) depending only on \( \delta \) and \( \alpha \) so that

\[
(3.4) \quad \| f \|_{1, \alpha} \leq C_2 \sum_n (1 - |z_n|^2)^{2+\alpha} |f(z_n)|
\]

if \( \{ z_n \} \) is also \( \varepsilon \)-dense with \( 0 < \varepsilon \leq \varepsilon_0 \).
After the above lemma, we can state an atomic decomposition theorem on the Bloch space.

**Theorem 3.2.** Let \( \{z_n\} \) be a sequence of points on \( D \). If \( \{z_n\} \) is \( \delta \)-weakly separated, the function of the form (3.5) is in \( B \) for any \( \{c_n\} \in 1^\infty \)

\[
(3.5) \quad f(z) = \sum_n c_n \cdot \left( \frac{1 - |z_n|^2}{1 - \overline{z_n}z} \right)^2.
\]

Moreover, there is an \( \varepsilon_0 > 0 \) such that every \( f \in B \) has the form (3.5) for some \( \{c_n\} \in 1^\infty \) if \( \{z_n\} \) is also \( \varepsilon \)-dense with \( 0 < \varepsilon \leq \varepsilon_0 \).

*Proof.* Let \( \{z_n\} \) be \( \delta \)-weakly separated. Then \( T \), defined as follows, is a bounded linear operator from \( A_1^0 \) to \( 1^1 \),

\[
(3.6) \quad Tf = \{(Tf)_n\} = \{ (1 - |z_n|^2)^2 f(z_n) \},
\]
in that (3.3) holds under \( \{z_n\} \) being \( \delta \)-weakly separated. Thus \( T^* \), the adjoint operator of \( T \) given by (3.7), is a bounded linear operator from \( 1^\infty \) (= \( (1^1)^* \)) to \( B \) (= \( A_1^0)^* \)),

\[
(3.7) \quad \langle Tf, y \rangle = \langle f, T^* y \rangle, \quad f \in A_1^0, \quad y \in 1^\infty,
\]
where the left \( \langle \cdot, \cdot \rangle \) is just the usual inner product between \( 1^1 \) and \( 1^\infty \).

To compute \( T^* \), we take

\[
y = e_n, \quad (e_n)_m = \begin{cases} 1, & m = n \\ 0, & m \neq n, \end{cases}
\]

so

\[
\langle Tf, e_n \rangle = (Tf)_n = (1 - |z_n|^2)^2 f(z_n) = (1 - |z_n|^2)^2 \langle f, K_{z_n} \rangle,
\]

where \( K_{z_n}(z) = 1/(1 - \overline{z_n}z)^2 \) is the reproducing kernel for \( A_1^0 \). Hence

\[
T^* e_n = (1 - |z_n|^2)^2 K_{z_n}(z)
\]

and

\[
T^* y = \sum_n c_n \cdot \frac{(1 - |z_n|^2)^2}{(1 - \overline{z_n}z)^2} \quad \text{for } y = \{c_n\} \in 1^\infty,
\]
i.e., the function in the form (3.5) is in \( B \). Indeed, it is easy to derive \( T^* y \in B \) by means of the direct computation.

Now we turn to showing the second part of Theorem 3.2. In fact, it is only necessary to claim \( T^* \) to be surjective. However, \( T^* \) is onto if and only if \( T \) is bounded below. By Lemma 3.1, there exists an \( \varepsilon_0 > 0 \) such that \( T \) is bounded below if \( \{z_n\} \) is \( \varepsilon \)-dense with \( 0 < \varepsilon \leq \varepsilon_0 \). That is to say, there is an \( \varepsilon_0 > 0 \) such that every \( f \in B \) has the form (3.5) for some \( \{c_n\} \in 1^\infty \) as \( \{z_n\} \) is \( \varepsilon \)-dense with \( 0 < \varepsilon \leq \varepsilon_0 \). Therefore the proof is completed. \( \Box \)
4. Free interpolation

As is well-known, a given sequence of points \( \{z_n\} \) on \( D \) is called an \( H^\infty \)-interpolating sequence if for any \( \{c_n\} \in 1^\infty \) there exists \( f \in H^\infty \) satisfying \( f(z_n) = c_n \) for all \( n \). Carleson stated in [4] that \( \{z_n\} \) is an \( H^\infty \)-interpolating sequence if and only if \( \{z_n\} \) is \( \eta \)-uniformly separated. Here we want to extend this fact to the Bloch space. Yet, it is unfortunate that the \( \eta \)-uniformly separated property is only a sufficient condition for \( B \). A sequence of points \( \{z_n\} \) is said to be a \( B \)-interpolating sequence if there is \( f \in B \) such that \( f(z_n) = c_n \) for all \( n \) and any \( \{c_n\} \in 1^\infty \).

**Theorem 4.1.** Let \( \{z_n\} \) be a sequence of points on \( D \). If \( \{z_n\} \) is a \( B \)-interpolating sequence, \( \{z_n\} \) is \( \delta \)-weakly separated. Conversely, if \( \{z_n\} \) is \( \delta \)-weakly separated and (4.1) or (4.2) is true, then \( \{z_n\} \) is a \( B \)-interpolating sequence where

\[
\begin{align*}
\text{(4.1)} & \quad \sup_n \sum_{m \neq n} \frac{(1 - |z_m|^2)(1 - |z_n|^2)}{|1 - \overline{z_n}z_m|^2} < \infty, \\
\text{(4.2)} & \quad \sup_n \sum_{m \neq n} \frac{(1 - |\overline{z}_m|^2)^2}{|1 - \overline{z}_nz_m|^2} < 1.
\end{align*}
\]

**Proof.** Firstly, if \( \{z_n\} \) is a \( B \)-interpolating sequence, then \( 1^\infty \subset T_\infty B \), where \( T_\infty f = \{f(z_n)\} \). Since \( B \) is a Banach space, relative to \( \|\cdot\|_B \), it follows from the open mapping theorem that there is a uniform constant \( C_1 > 0 \) and \( f \in B \) so that \( \|f\|_B \leq C_1 \) with \( f(z_n) = w_n \) for all \( n \) and \( \|\{w_n\}\|_\infty \leq 1 \). Picking \( w_m = 0 \), \( m \neq n \); \( w_m = 1 \), \( m = n \), there exist \( f_n \in B \), \( \|f_n\|_B \leq C_1 \) satisfying \( f_n(z_n) = 1 \); \( f_n(z_m) = 0 \), \( m \neq n \). Theorem 2.1 yields

\[
\frac{|f_n(z_n) - f_n(z_m)|}{d(z_n, z_m)} \leq C_1, \quad m \neq n,
\]

and so

\[
\inf_{m \neq n} d(z_n, z_m) \geq 1/C_1 > 0, \quad \text{i.e.,}
\]

\[
\delta = \inf_{m \neq n} g(z_m, z_n) \geq (e^{2/C_1} - 1)/(e^{2/C_1} + 1) > 0.
\]

Conversely, let \( \{z_n\} \) be \( \delta \)-weakly separated. If (4.1) is true, \( \{z_n\} \) is \( \eta \)-uniformly separated and hence \( 1^\infty = T_\infty H^\infty \subset T_\infty B \) since \( H^\infty \) is a proper subspace of \( B \). Furthermore, if (4.2) holds, we consider the linear operator \( T^* \), given by \( T^* \{c_n\} = \sum_n c_n \cdot \left( (1 - |z_n|^2)/(1 - \overline{z}_nz) \right)^2 \), \( \{c_n\} \in 1^\infty \). Clearly, \( T^* \) is bounded from \( 1^\infty \) to \( B \) (by Theorem 3.2), while

\[
\| (T_\infty T^* - I) \{c_n\} \|_\infty = \sup_n \left| \sum_{m \neq n} c_m \cdot \frac{(1 - |z_m|^2)^2}{1 - \overline{z}_m z_n} \right| \leq \| \{c_n\} \|_\infty \cdot \sup_n \sum_{m \neq n} \frac{(1 - |z_m|^2)^2}{|1 - \overline{z}_n z_m|^2}.
\]
So, \( \|(T_\infty T^* - I)\| < 1 \) where \( I \) is the identify operator, i.e., \( T_\infty T^* \) has an inverse, denoted by \( (T_\infty T^*)^{-1} \). Further, \( T_\infty \) has a right inverse \( T^*(T_\infty T^*)^{-1} \), that is to say, \( T_\infty (T^*(T_\infty T^*)^{-1}) = I \), and thus \( 1^\infty \subset T_\infty B \). So, \( \{z_n\} \) is a \( B \)-interpolating sequence. \( \Box \)

Note that \( T_\infty H^\infty \subset T_\infty B \). In general, it is necessary to take into consideration the generic free interpolation problem from \( B \). That is, for which \( \{w_n\} \subset C \) there is \( f \in B \) satisfying \( \{f(z_n)\} = T_\infty f = \{w_n\} \). For this we obtain the following fact.

**Theorem 4.2.** Let \( \{z_n\} \) be a \( \delta \)-weakly separated sequence of points on \( D \). If \( \{f(z_n)\} = T_\infty f = \{w_n\} \) is solvable in \( B \) for \( \{w_n\} \subset C \), the following assertions (i) and (ii) hold:

(i) there are a constant \( C_1 > 0 \) and a function \( \beta(z) \) such that

\[
\sup_{z \in D} \sum_n [1 - \varrho^2(z, z_n)]^\gamma \exp[C_1 |w_n - \beta(z)|] < \infty
\]

for \( \gamma > 1 \);

(ii) there is a constant \( C_2 > 0 \) such that

\[
\sup_{z \in D} \sum_n [1 - \varrho^2(z, z_n)]^\gamma \exp[C_1 |w_n - h(z)|] < \infty
\]

for \( \gamma > 1 \), where \( h(z) = \left\{ \sum_n w_n [1 - \varrho^2(z, z_n)]^\gamma \right\} / \sum_n [1 - \varrho^2(z, z_n)]^\gamma \).

Conversely, if (i) or (ii) holds for \( \gamma = 1 \), then \( \{f(z_n)\} = T_\infty f = \{w_n\} \) is solvable in \( B \).

**Proof.** First we consider the case (i). If \( \{f(z_n)\} = T_\infty f = \{w_n\} \) is solvable in \( B \), then Theorem 2.1 yields

\[
\sup_{z \in D} \int_D \exp[C_1 |g_z(w)|] \, dm_\alpha(w) < \infty
\]

for \( C_1 < 2(\alpha + 1) \|f\|_B \) (\( \|f\|_B > 0 \) is naturally assumed), where \( g_z(w) = (f \circ \varphi_z)(w) - f(z) \). The above statement means that \( \exp(C_1 g_z) \) is in \( A^1_\alpha \). Consequently, by Lemma 3.1,

\[
\sup_{z \in D} \sum_n \exp[C_1 |g_z(\tilde{z}_n)|] \left(1 - |\tilde{z}_n|^2\right)^{\alpha+2} \leq C \sup_{z \in D} \|\exp C_1 g_z\|_{1,\alpha} < \infty,
\]

where \( \{\tilde{z}_n\} = \{\varphi_z(z_n)\} \), \( C > 0 \) is a constant independent of \( g_z \), and \( \{\tilde{z}_n\} \) is also a \( \delta \)-weakly separated sequence of points on \( D \) since \( \{z_n\} \) is such a sequence. Thus (4.5) means that (i) holds for \( \gamma = \alpha + 2 > 1 \) and \( \beta(z) = f(z) \).

Now let us consider (ii).
Because \( \{z_n\} \) is \( \delta \)-weakly separated, we get \( \sum_n [1 - \theta^2(z, z_n)]^{\alpha + 2} < \infty \) by Lemma 3.1. By (4.5) we further have

\[
\sum_{\{n:|w_n - f(z)| > t\}} [1 - \theta^2(z, z_n)]^{\alpha + 2} \leq C_2 \exp(-C_1 t)
\]

for \( t \geq 0 \), where \( C_1 \) and \( C_2 \) are constants with \( C_1 < 2(\alpha + 1)/\|f\|_B \), \( \|f\|_B > 0 \), and \( f \) is the interpolating function for \( \mathbf{T}_\infty f = \{w_n\} \) in \( B \).

Thus, for \( \gamma = \alpha + 2 > 1 \)

\[
|h(z) - f(z)| \leq \frac{1}{\sum_n [1 - \theta^2(z, z_n)]^{\gamma}} \sum_n |w_n - f(z)| [1 - \theta^2(z, z_n)]^{\gamma}
\]

\[
= \frac{1}{\sum_n [1 - \theta^2(z, z_n)]^{\gamma}} \int_0^\infty \left\{ \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \right\} dt
\]

\[
\leq \frac{1}{\sum_n [1 - \theta^2(z, z_n)]^{\gamma}} \int_0^\infty \min \left\{ \sum_n [1 - \theta^2(z, z_n)]^{\gamma}, C_2 \exp(-C_1 t) \right\} dt
\]

\[
\leq \frac{1}{C_1} \left\{ 1 + \log \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \right\}
\]

and, consequently,

\[
\sup_{z \in D} \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \exp \left[ \frac{C_1}{2} |w_n - h(z)| \right]
\]

\[
\leq \sup_{z \in D} \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \exp \left[ \frac{C_1}{2} |w_n - f(z)| \exp \left[ \frac{C_1}{2} |f(z) - h(z)| \right] \right]
\]

\[
\leq \sup_{z \in D} \left\{ \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \exp \left[ C_1 |w_n - f(z)| \right] \right\}^{1/2} \cdot \left\{ \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \right\}^{1/2}
\]

\[
= \sqrt{C_2} \cdot \left\{ \sup_{z \in D} \sum_n [1 - \theta^2(z, z_n)]^{\gamma} \exp \left[ C_1 |w_n - f(z)| \right] \right\}^{1/2} < \infty.
\]

That is to say, (ii) is true for \( \gamma = \alpha + 2 > 1 \).

Next we show the contrary assertion. If (i) or (ii) holds for \( \gamma = 1 \), then it follows from (4.4) that

\[
\sup_{z \in D} \sum_n [1 - \theta^2(z, z_n)] < \infty.
\]
This, together with \( \{z_n\} \) being \( \delta \)-weakly separated, shows that \( \{z_n\} \) is \( \eta \)-uniformly separated. Also, it follows by \cite{9} that there is \( f \in \text{BMOA}(\partial D) \) to make \( T_\infty f = \{f(z_n)\} = \{w_n\} \) for \( \{w_n\} \subset \mathbb{C} \) which is satisfied with (4.3) or (4.4) for \( \gamma = 1 \). Since \( \text{BMOA}(\partial D) \subsetneq B \), there exists \( f \in B \) such that \( T_\infty f = \{f(z_n)\} = \{w_n\} \) under the previous assumption. Thus the theorem is proved. \( \square \)

References


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