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RECOGNIZING THE THREE BEST OF A SEQUENCE

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1. Introduction

In this work we consider a multi-choice variant of the classical secretary problem. The standard secretary problem can be described as follows. A known number, say N, of items is to be presented one by one in random order so that all $N!$ possible orders are equally probable. We assume that it is possible to rank all applicants from best to worst without restrictions. The decision to accept or reject an item is to be based only on the relative ranks of those applicants interviewed so far. If the observer has just rejected an applicant, he cannot recall her later. The purpose is to maximize the probability of choosing the best candidate of the N applicants. For an extensive review on the secretary problem cf. [2].

The solution of the standard problem is simple. The optimal strategy is to reject about $[N/e]$ candidates and then to choose the next candidate who is best in the relative ranking of the observed applicants. It turns out that the expected payoff by adopting this strategy tends to $1/e \approx 0.3679$ as $N \to \infty$.

The multi-choice variant we are going to investigate is as follows. The decision-maker is allowed to have just three choices with aim to choose exactly the three best candidates from a group of N applicants for a fixed $N \in \mathbf{N}$.

We want to find the optimal rule to select three candidates so that the probability that they are the three best ones among the N applicants is maximum. We shall use what we have called the 'nonfeasible domain' method (NFD) to find an expression of the probability of winning. The optimal stopping rule can be determined by maximizing the probability of winning.

2. The structure of the optimal stopping rule

We call an applicant a 'first candidate' if she is the best among applicants already appeared. Respectively, we give the name 'second candidate' to any applicant who is at least the second-best among applicants so far. In the same way, the name 'third candidate' is given to any applicant who turns out to be at least the third-best among applicants presented so far. Note that the set of the 1st candidates is contained in the set of the 2nd candidates which in turn is contained in the set of the 3rd candidates.

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First, the structure of the optimal stopping rule should be known in order to determine the respective probability (function) of winning. By using the same kind of conclusions as M.L. Nikolaev [4] in solving the two-choice problem we observe that the structure of the optimal stopping rule is as follows. In the following considerations the index $j \in \mathbf{N}$ refers to the jth applicant in the order of presentation. Stopping indices $j_1, j_2, j_3 \in [1,N] \cap \mathbf{N}$, $j_1 \le j_2 \le j_3$, determine intervals $\delta_1 = [j_1, j_2 - 1]$, $\delta_2 = [j_2, j_3 - 1]$, $\delta_3 = [j_3, N]$ and the optimal stopping rule as follows: from interval δ_1 accept only first candidates, from interval δ_2 accept second candidates and from δ_3 accept third candidates. All this is to be done with the restriction that just three choices are made.

Note that the optimal stopping rule for the general k-choice problem is of the same form: k stopping indices $j_1, j_2, \ldots, j_k$ determine intervals $\delta_1 = [j_1, j_2 - 1]$, $\delta_2 = [j_2, j_3 - 1]$, $\ldots$, $\delta_k = [j_k, N]$, and the best strategy is to accept only ith candidates from the interval δ_i, $i \in \{1, 2, \ldots, k\}$, with the restriction that exactly k choices are made. The ith candidate being the one who is at least the ith best candidate among applicants presented so far. Concerning the structure of the optimal stopping rule in the k-choice problem, we also recall the natural requirement that every chosen candidate must be better than the best among the earlier rejected applicants.

3. Finding the probability of success by using NFD-method

The main goal of this section is to express the probability of winning by using the stopping rule defined above. The strategy is uniquely determined by the vector (j_1, j_2, j_3). A successful three-choice consists of the events, denoted by $A(u,v,w)$, where $u,v,w \in \{0,1,2,3\}$ and $u+v+w=3$. Here $A(u,v,w) = \{u$ 1st candidates from the interval δ_1, v 2nd candidates from δ_2 and, respectively, w 3rd candidates from δ_3 are (successfully) chosen$\}$. There are exactly ten events $A(u,v,w)$ satisfying the condition $u+v+w=3$.

Let us, for example, determine the probability $\mathbf{P}(A(3,0,0))$. In practice, this can be done quite easily by using the nonfeasible domain method (NFD), i.e., by describing the event $A(3,0,0)$ by a block diagram (see Figure 1).

The indices below the horizontal axis denote the ordinal numbers of the candidates in their order of presentation. The coordinates of the vertical axis denote the relative values of the candidates in the eyes of the decision-maker. Here $*$ denotes a chosen applicant, and the value of the best candidate among the $j_1 - 1$ first presented applicants is denoted by ♦. Boxes NFD indicate nonfeasible domains having the restriction that none of the applicants is allowed to take value in these boxes if $A(3,0,0)$ is to happen. Actually, this is equivalent to the event $A(3,0,0)$, i.e., $A(3,0,0)$ happens only with the restrictions of the NFD-diagram below.

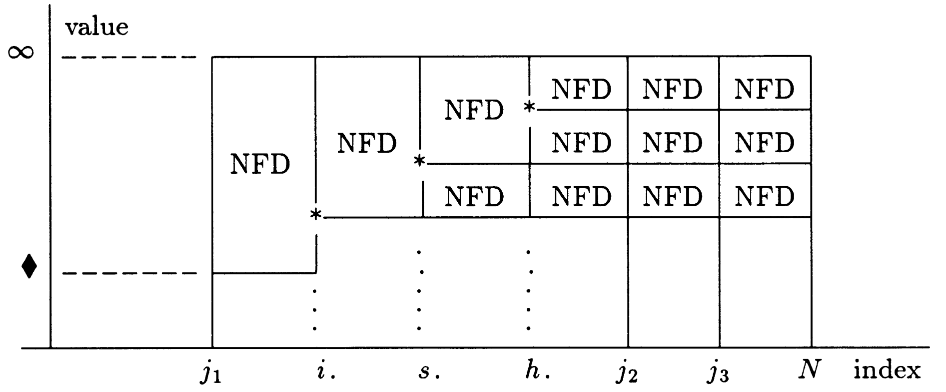

Figure 1. The event $A(3,0,0)$ described by a block diagram.

The solution of the classical problem indicates that the ith presented applicant is the best candidate in the relative ranking of the observed applicants with probability $1/i$, which implies that

$$(*)\qquad \mathbf{P}\{\text{the } i\text{th presented applicant is not among the } d \text{ best applicants (so far)}\} = 1 - d/i.$$

By (∗) and the NFD-diagram (Figure 1) we find that

$$\begin{aligned}
\mathbf{P}\big(A(3,0,0)\big) &= \big[(j_2-3)(j_2-2)(j_2-1)/(N-2)(N-1)N\big]\cdot \\
&\cdot \sum_{i,s,h=j_1, i<s<h}^{j_2-1} \big[(j_1-1)/j_1\big]\big[j_1/(j_1+1)\big]\cdots \\
&\cdots\big[(i-3)/(i-2)\big]\big[(i-2)/(i-1)\big](1/i)\big[i/(i+1)\big]\big[(i+1)/(i+2)\big]\cdots \\
&\cdots\big[(s-3)/(s-2)\big]\big[(s-2)/(s-1)\big](1/s)\big[(s-1)/(s+1)\big]\big[s/(s+2)\big]\cdots \\
&\cdots\big[(h-4)/(h-2)\big]\big[(h-3)/(h-1)\big](1/h)\big[(h-2)/(h+1)\big]\big[(h-1)/(h+2)\big]\cdots \\
&\cdots\big[(j_2-6)/(j_2-3)\big]\big[(j_2-5)/(j_2-2)\big]\big[(j_2-4)/(j_2-1)\big] \\
&= \big\{(j_1-1)/(N-2)(N-1)N\big\} \sum_{i,s,h=j_1, i<s<h}^{j_2-1} 1/(i-1).
\end{aligned}$$

Here $\prod_{i=j_2}^{N}(1-3/i) = (j_2-3)(j_2-2)(j_2-1)/(N-2)(N-1)N$ is the probability that no applicants among the three best candidates so far belong to the interval $[j_2, N]$. Hence we obtain, by suitable approximations,

$$\mathbf{P}\big(A(3,0,0)\big) \approx \frac{j_1}{N^3} \sum_{i,s,h=j_1, i<s<h}^{j_2} \frac{1}{i}$$

which will still be asymptotically exact. Using the well-known logarithm and integral approximations for harmonic sums we finally note (with $\alpha = \lim_{N\to\infty} j_1/N$, $\beta = \lim_{N\to\infty} j_2/N$ and $\tau = \lim_{N\to\infty} j_3/N$) that

$$\lim_{N\to\infty} \mathbf{P}\big(A(3,0,0)\big) = \tfrac{1}{2}\alpha\beta^2 \ln(\beta/\alpha) - \tfrac{1}{4}\alpha^3 - \tfrac{3}{4}\alpha\beta^2 + \alpha^2\beta.$$

With help of $(*)$, the NFD-method and logarithm/integral approximations we finally obtain the following expressions for probabilities which we need in computing the probability of choosing the three best candidates:

$$\mathbf{P}\big(A(0,3,0)\big) \approx \frac{j_1}{N^3} \sum_{i,s,h=j_2,i<s<h}^{j_3} \frac{1}{s}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(0,3,0)\big) = -\alpha\beta\tau \ln(\tau/\beta) + \tfrac{1}{2}\alpha\tau^2 - \tfrac{1}{2}\alpha\beta^2.$$

We have to multiply this expression by four because the respective block diagram allows in principle four different orders of values to the three chosen candidates. Respectively, we obtain

$$\mathbf{P}\big(A(0,0,3)\big) \approx \frac{j_1}{N^3} \sum_{i,s,h=j_3,i<s<h}^{N} \frac{1}{h}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(0,0,3)\big) = \tfrac{1}{4}\alpha - \alpha\tau + \tfrac{3}{4}\alpha\tau^2 - \tfrac{1}{2}\alpha\tau^2 \ln(\tau).$$

The respective block diagram allows six different orders of values to the chosen candidates, so we take this probability into account multiplied by six. In the same way,

$$\mathbf{P}\big(A(1,1,1)\big) \approx \frac{j_1 j_2 j_3}{N^3} \sum_{i=j_1}^{j_2} \frac{1}{i} \sum_{s=j_2}^{j_3} \frac{1}{s} \sum_{h=j_3}^{N} \frac{1}{h}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(1,1,1)\big) = -\alpha\beta\tau \ln(\tau) \ln(\beta/\alpha) \ln(\tau/\beta)$$

which has to be multiplied by six;

$$\mathbf{P}\big(A(1,2,0)\big) \approx \frac{j_1 j_2}{N^3} \sum_{i=j_1}^{j_2} \frac{1}{i} \sum_{s,h=j_2,s<h}^{j_3} \frac{1}{s}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(1,2,0)\big) = \alpha\beta\tau\ln(\beta/\alpha)\ln(\tau/\beta) - \alpha\beta\tau\ln(\beta/\alpha) + \alpha\beta^2\ln(\beta/\alpha)$$

which has to be multiplied by four;

$$\mathbf{P}\big(A(1,0,2)\big) \approx \frac{j_1 j_2}{N^3}\sum_{i=j_1}^{j_2}\frac{1}{i}\sum_{s,h=j_3,s<h}^{N}\frac{1}{h}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(1,0,2)\big) = \alpha\beta\tau\ln(\tau)\ln(\beta/\alpha) + \alpha\beta\ln(\beta/\alpha) - \alpha\beta\tau\ln(\beta/\alpha)$$

which has to be multiplied by six;

$$\mathbf{P}\big(A(0,1,2)\big) \approx \frac{j_1}{N^3}(j_3 - j_2)\sum_{s,h=j_3,s<h}^{N}\frac{1}{h}$$

having the limit

$$\lim_{N\to\infty} \mathbf{P}\big(A(0,1,2)\big) = \alpha\tau - \alpha\beta - \alpha\tau^2 + \alpha\beta\tau + \alpha\tau^2\ln(\tau) - \alpha\beta\tau\ln(\tau)$$

which has to be multiplied by six;

$$\mathbf{P}\big(A(2,1,0)\big) \approx \frac{j_1}{N^3}(j_3 - j_2)\sum_{i,s=j_1,i<s}^{j_2}\frac{1}{i}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(2,1,0)\big) = \alpha\beta\tau\ln(\beta/\alpha) - \alpha\beta^2\ln(\beta/\alpha) + \alpha^2\tau - \alpha^2\beta - \alpha\beta\tau + \alpha\beta^2$$

which has to be multiplied by two;

$$\mathbf{P}\big(A(2,0,1)\big) \approx \frac{j_1 j_3}{N^3}\sum_{i,s=j_1,i<s}^{j_2}\frac{1}{i}\sum_{h=j_3}^{N}\frac{1}{h}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(2,0,1)\big) = -\alpha\beta\tau\ln(\tau)\ln(\beta/\alpha) - \alpha^2\tau\ln(\tau) + \alpha\beta\tau\ln(\tau)$$

which has to be multiplied by three; and

$$\mathbf{P}\big(A(0,2,1)\big) \approx \frac{j_1 j_3}{N^3}\sum_{h=j_3}^{N}\frac{1}{h}\sum_{i,s=j_2,i<s}^{j_3}\frac{1}{s}$$

implying

$$\lim_{N\to\infty} \mathbf{P}\big(A(0,2,1)\big) = \alpha\beta\tau\ln(\tau)\ln(\tau/\beta) - \alpha\tau^2\ln(\tau) + \alpha\beta\tau\ln(\tau)$$

which has to be multiplied by six.

Combining these ten results we can finally form the probability function $\mathbf{P}\colon [0,1]^3 \to [0,1]$ for a successful three-choice. Going to the limit, as $N \to \infty$, we obtain

$$\begin{aligned}
\mathbf{P}(\alpha,\beta,\tau) &= \mathbf{P}\big(A(3,0,0)\big) + 4\mathbf{P}\big(A(0,3,0)\big) + 6\mathbf{P}\big(A(0,0,3)\big) + 6\mathbf{P}\big(A(1,1,1)\big) \\
&\quad + 4\mathbf{P}\big(A(1,2,0)\big) + 6\mathbf{P}\big(A(1,0,2)\big) + 6\mathbf{P}\big(A(0,1,2)\big) \\
&\quad + 2\mathbf{P}\big(A(2,1,0)\big) + 3\mathbf{P}\big(A(2,0,1)\big) + 6\mathbf{P}\big(A(0,2,1)\big) \\
&= \tfrac{3}{2}\alpha - \tfrac{1}{4}\alpha^3 - 6\alpha\beta + 4\alpha\beta\tau - \tfrac{3}{4}\alpha\beta^2 + \tfrac{1}{2}\alpha\tau^2 - \alpha^2\beta \\
&\quad + 2\alpha^2\tau + 6\alpha\beta\ln(\beta/\alpha) + \tfrac{5}{2}\alpha\beta^2\ln(\beta/\alpha) - 8\alpha\beta\tau\ln(\beta/\alpha) \\
&\quad + 3\alpha\beta\tau\ln(\tau)\ln(\beta/\alpha) + 4\alpha\beta\tau\ln(\beta/\alpha)\ln(\tau/\beta) + 6\alpha\beta\tau\ln(\tau)\ln(\tau/\beta) \\
&\quad - 6\alpha\beta\tau\ln(\tau)\ln(\beta/\alpha)\ln(\tau/\beta) - 4\alpha\beta\tau\ln(\tau/\beta) - 3\alpha^2\tau\ln(\tau) \\
&\quad - 3\alpha\tau^2\ln(\tau) + 3\alpha\beta\tau\ln(\tau).
\end{aligned}$$

This function has to be maximized under the condition $0 < \alpha \le \beta \le \tau \le 1$. We have carried it out by computer using the SQP-method (EMP—Expert System for Mathematical Programming). This gave for the maximum probability of winning by this policy the asymptotic value

$$\lim_{N\to\infty} \mathbf{P}(\text{win}) = \mathbf{P}(\alpha^*,\beta^*,\tau^*) \approx 0.1605$$

obtained at the point $\alpha^* \approx 0.1712$, $\beta^* \approx 0.4053$, $\tau^* \approx 0.7165$.

Note that $\tau^* = e^{-1/3}$ which is easy to show by computing $d\mathbf{P}/\delta\tau$ at the point $\tau^* = e^{-1/3}$. We observe that $d\mathbf{P}/d\tau = 0$ at this point independently of values of variables α and β. Starting from equations $d\mathbf{P}/d\alpha = 0$ and $d\mathbf{P}/d\beta = 0$ at the point $(\alpha^*,\beta^*,e^{-1/3})$ one can also obtain the solution $\alpha^* \approx 0.1712$, $\beta^* \approx 0.4053$.

Summing up, we have determined the stopping indices $j_1 \approx \alpha^* N \approx 0.1712N$, $j_2 \approx \beta^* N \approx 0.4053N$, $j_3 \approx \tau^* N \approx 0.7165N$. The optimal stopping rule has the property $\mathbf{P}(\text{win}) \approx 0.1605$. Although the result is exact only when $N \to \infty$, it works also well enough for small values of the number of candidates N.

Remarks. Comparing the result $\tau^* = e^{-1/3}$ with the corresponding result $\beta^* = e^{-1/2}$ in the two-choice problem (Nikolaev [4], Tamaki [5]) suggests strongly that the largest stopping parameter $\lim_{N\to\infty} j_k/N$ in the general k-choice problem could be $r^*(k) = e^{-1/k}$, $k \in \{1,2,3,\ldots\}$. Thus $r^*(k)$ tends to 1 as $k \to \infty$. The convergence towards 1 is rather slow as expected.

Using the NFD-method it is possible to find the respective probability of choosing the k best candidates. Maximizing this, we can fix the optimal stopping rule in the general k-choice problem.

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