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UNIQUENESS THEOREMS FOR HOLOMORPHIC CURVES

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A celebrated theorem of Nevanlinna ([9], [11], [12], see also [16]) asserts that if f and g are meromorphic functions on the entire plane and there are 5 values $a_1, \ldots, a_5$ for which $f^{-1}(a_i) = g^{-1}(a_i)$, not counting multiplicities, then f is identically equal to g. Theorem 2 of this paper is an analogue of Nevanlinna's theorem for holomorphic curves in 2-dimensional projective space. The 5 points a_i are replaced by 18 lines L_{ij}, which are required to be in a special configuration, which is never in general position. We take f, g: $\mathbf{C} \to \mathbf{CP}^2$ to be full holomorphic curves such that, for each line L_{ij}, the inverse images $f^{-1}(L_{ij})$ and $g^{-1}(L_{ij})$ are the same, counting multiplicities up to 2, and prove that f is identically equal to g. In other words, we distinguish between simple zeros and multiple zeros, but make no distinction between multiple zeros of different orders. Theorem 6 is a theorem of the same type for a somewhat more complicated configuration in which the lines may be in general position. The need to count multiplicities up to 2 comes from the ramification term in Cartan's version of the Second Main Theorem.

A theorem of this type has been published by H. Fujimoto [6, II, Theorem 1]. He considers lines in general position and proves a uniqueness theorem for the special case in which f and g do not pass through 3 of the lines at all. Other generalizations of Nevanlinna's theorem to higher-dimensional ranges have been published by S.J. Drouilhet ([4], [5]) and L.M. Smiley ([15], [17, Section 13]). In their work there is no need to count multiplicities up to 2, but the assumption on common values of f and g is that *every point* of some divisor has the same inverse image under f as under g.

Nevanlinna also proved that if f and g are meromorphic functions on the entire plane and there are 3 values a_1, a_2, a_3 for which $f^{-1}(a_i) = g^{-1}(a_i)$, counting multiplicities, then f is identically equal to g, unless f and g belong to a small family of exceptions ([11], [12]). Fujimoto [6] has obtained several generalizations of this theorem to holomorphic curves in $\mathbf{CP}^n$.

The Supplement at the end of the monograph by B.V. Shabat [14] contains a survey of work in this area.

Our arguments use the value-distribution theory of a holomorphic curve f: $\mathbf{C} \to \mathbf{CP}^n$. There are two approaches to this theory. Cartan ([2], [7], [10]) uses Wronskians to reduce to the 1-dimensional case, whereas Ahlfors ([1], [3], [14], [19])

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works directly with singular densities in $\mathbf{CP}^n$. In this paper we follow Cartan's approach, and in particular we rely on his treatment of ramification. Since Cartan's approach has only been worked out for the entire plane $\mathbf{C}$ as domain, we only consider holomorphic curves defined on $\mathbf{C}$.

To outline the ideas in this paper, let f, g: $\mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves. The assumption that, for certain lines L_{ij}, $f^{-1}(L_{ij})$ is the same as $g^{-1}(L_{ij})$, counting multiplicities up to 2, is used to bound the number of times f and g pass through the lines L_{ij}. A contradiction follows from the second main theorem of value-distribution theory. The application of the second main theorem is similar to that of Nevanlinna [12], but the estimation of the number of times f and g pass through L_{ij} is quite different, so we now explain it.

Define a holomorphic curve h: $\mathbf{C} \to \mathbf{CP}^{2*}$ by letting $h(z)$be the point in $\mathbf{CP}^{2*}$ corresponding to the line joining $f(z)$ and $g(z)$. This is defined except when $f(z) = g(z)$. We seek a bound on the number of times h passes through the points L^*_{ij} dual to certain lines L_{ij}. It turns out that the Nevanlinna–Cartan characteristic $T_h(r)$, defined later in the paper, can be estimated by $T_f(r) + T_g(r) + O(1)$. We can consider the line Λ joining any 2 of the L^*_{ij} and estimate the enumerative function for Λ in the usual way. The bound thus obtained is not good enough for our purpose, but if there exists a line through 3 of the L^*_{ij} we obtain a better bound. These considerations lead to the configuration of 15 lines described in the statement of Theorem 1.

It remains to discuss the points where $f(z) = g(z)$. It happens that the formula we use to bound $T_h(r)$ gives at the same time a bound for the total number of points where $f(z) = g(z)$, or rather for an enumerative function $N_c(r)$ that we define for these points. The common value $f(z) = g(z)$ may be the intersection of 2 or 3 of the lines L_{ij}, and so some consideration of multiplicities is needed.

Theorem 2 of this paper, the uniqueness theorem discussed above, is a simple consequence of Theorem 1. Theorems 3 and 4 are similar to Theorems 1 and 2 but concern a different type of configuration of lines. In Theorems 5 and 6 we treat certain configurations where the lines are in general position in the linear sense but subject to quadratic or cubic relations.

Let f: $\mathbf{C} \to \mathbf{CP}^n$ be holomorphic. If f is given in homogeneous coordinates by $(f_0, \dots, f_n)$, where $f_0, \dots, f_n$: $\mathbf{C} \to \mathbf{C}$ are holomorphic functions with no common zeros, we say that $(f_0, \dots, f_n)$ is a *reduced representation* of f.

Given any reduced representation $(f_0, \dots, f_n)$ of f, we define the *Nevanlinna–Cartan characteristic* of f to be

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \max_j \log\left|f_j(re^{i\theta})\right| d\theta - \max_j \log\left|f_j(0)\right|.$$

This definition does not depend on the choice of reduced representation.

We consider a hyperplane H in $\mathbf{CP}^n$ and also write H for the homogeneous linear form defining H. Assuming $H \circ f$ is not identically zero, let $\bar{\nu}_f(z,H)$ be the minimum of n and the multiplicity of the zero of $H \circ f$ at z. Then

$$\bar{n}_f(r,H) = \sum_{|z|\le r} \bar{\nu}_f(z,H)$$

is the number of zeros of $H \circ f$ in the closed disc $\bar{D}(0,r)$, counting multiplicities up to n. (We regard the point 0 as $\bar{D}(0,0)$.) The *enumerative function* is

$$\bar{N}_f(r,H) = \int_0^r \frac{\bar{n}_f(t,H) - \bar{n}_f(0,H)}{t}\,dt + \bar{n}_f(0,H)\log r.$$

We shall use the following form of the Nevanlinna inequality. The Nevanlinna–Cartan characteristic T_f can be used to bound the enumerative function defined by counting all multiplicities instead of multiplicities up to n, but we shall not need that function in the present paper.

Nevanlinna inequality ([2, p. 15], [7, formula (2.5)]). *If f is a holomorphic curve in $\mathbf{CP}^n$ and H is a hyperplane that does not contain the image of f, then*

(1) $$\bar{N}_f(r,H) \le T_f(r) + O(1).$$

We say that f is a *full* curve if the image of f is not contained in any proper linear subspace of $\mathbf{CP}^n$.

Second main theorem ([2, formula (3)], [7, Theorem 3.5], [10, p. 223]). *Let $f\colon \mathbf{C} \to \mathbf{CP}^n$ be a full holomorphic curve and let $H_1, \ldots, H_q$ be hyperplanes in general position. Then*

(2) $$(q-n-1)T_f(r) \le \sum_{i=1}^{q} \bar{N}_f(r,H_i) + O\big(\log rT_f(r)\big), \qquad \|$$

where the symbol $\|$ on the right indicates that (2) may fail for values of r in a set of finite measure.

In the next lemma we define a special enumerative function counting all multiplicities, but we shall only use it to estimate multiplicities up to 2.

Lemma 1. *Let $\varphi_0, \ldots, \varphi_n\colon \mathbf{C} \to \mathbf{C}$ be holomorphic functions, not identically zero. Let $\Phi\colon \mathbf{C} \to \mathbf{CP}^n$ be the holomorphic curve defined by $(\varphi_0, \ldots, \varphi_n)$, with analytic continuation across the common zeros of $\varphi_0, \ldots, \varphi_n$. Then*

$$T_\Phi(r) + N_c(r) = \frac{1}{2\pi}\int_0^{2\pi} \max_j \log\big|\varphi_j(re^{i\theta})\big|\,d\theta + O(1),$$

where

$$N_c(r) = \int_0^r \frac{n_c(t) - n_c(0)}{t}\,dt + n_c(0)\log r$$

and $n_c(r)$ is the number of common zeros of $\varphi_0, \ldots, \varphi_n$ in $\bar{D}(0,r)$, counting all multiplicities.

Proof. Let $(\chi_0, \dots, \chi_n)$ be a reduced representation of Φ, so that there exists a holomorphic function ψ, $\psi(0) \neq 0$, such that

$$\varphi_j = z^h \psi \chi_j, \qquad j = 0, \dots, n,$$

where $h = n_c(0)$. Then

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \max_j \log\left|\varphi_j(re^{i\theta})\right| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \max_j \log\left|\chi_j(re^{i\theta})\right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log\left|(re^{i\theta})^h \psi(re^{i\theta})\right| d\theta \\ &= T_\Phi(r) + \max_j \log\left|\chi_j(0)\right| + \log\left|\psi(0)\right| + \int_0^r \frac{n_c(t) - n_c(0)}{t}\, dt + n_c(0) \log r \end{aligned}$$

by Jensen's formula, since the zeros of $z^h \psi$ are precisely the common zeros of φ_0, $\dots$, φ_n. □

We shall use a construction from Grassmann algebra. If p, q are distinct points of $\mathbf{CP}^2$, the line through p and q is a point of the dual space $\mathbf{CP}^{2*}$ denoted by $p \wedge q$. If f, g: $\mathbf{C} \to \mathbf{CP}^2$ are holomorphic and f is not identically equal to g, then $f \wedge g$ is defined except on the discrete set where $f(z) = g(z)$. The singularities of $f \wedge g$ are removable and so we obtain a holomorphic curve $f \wedge g$: $\mathbf{C} \to \mathbf{CP}^{2*}$.

If (e_0, e_1, e_2) is a basis for $\mathbf{C}^3$, $(e_1 \wedge e_2, e_2 \wedge e_0, e_0 \wedge e_1)$ is a basis for $\mathbf{C}^{3*}$. With respect to these bases, if f is given by (f_0, f_1, f_2) in homogeneous coordinates and g is given by (g_0, g_1, g_2), then $f \wedge g$ is given by

$$(f_1 g_2 - f_2 g_1, f_2 g_0 - f_0 g_2, f_0 g_1 - f_1 g_0). \tag{3}$$

Even if f and g are given by reduced representations, the vector (3) may not be a reduced representation of $f \wedge g$.

If $p \in \mathbf{CP}^2$ and f: $\mathbf{C} \to \mathbf{CP}^2$ is holomorphic, the projection of f into the line polar to p will be denoted, by an abuse of language, by $p \wedge f$. This is an example of the *contracted curves* introduced by Ahlfors [1] and described in detail by Wu [19].

Lemma 2. *Let f, g: $\mathbf{C} \to \mathbf{CP}^2$ be holomorphic curves, f not identically equal to g. Let p be a point of $\mathbf{CP}^2$ and p^* be the line in $\mathbf{CP}^{2*}$ dual to p. The image of $f \wedge g$ lies in p^* if and only if $p \wedge f \equiv p \wedge g$.*

Proof. First assume

$$p^*(f \wedge g) \equiv 0. \tag{4}$$

Take a unitary basis (e_0, e_1, e_2) for $\mathbf{C}^3$ such that p is the point with coordinates $(0,0,1)$. Then p^* has coordinates $(0,0,1)$ with respect to the basis $(e_1 \wedge e_2, e_2 \wedge e_0, e_0 \wedge e_1)$ for $\mathbf{CP}^{2*}$. If f, g have representations (f_0, f_1, f_2), (g_0, g_1, g_2), equation (4) yields

$$f_0 g_1 - f_1 g_0 \equiv 0. \tag{5}$$

Now $p \wedge f$ and $p \wedge g$ are given in coordinates by (f_0, f_1) and (g_0, g_1), and (5) shows that these are identically equal as curves in $\mathbf{CP}^1$.

Conversely, if $p \wedge f \equiv p \wedge g$ then (5) holds, which implies that (4) holds. □

We now prove Theorem 1, the conclusion of which is a degeneracy condition of the form $p \wedge f \equiv p \wedge g$. Theorem 1 will be used to derive a uniqueness theorem as Theorem 2.

Theorem 1. *Let L_{ij}, $i = 1, \dots, 5$, $j = 1, 2, 3$, be 15 distinct lines in $\mathbf{CP}^2$ such that*

(1) for $i = 1, \dots, 5$, L_{i1}, L_{i2}, and L_{i3} have a common point p_i;

(2) the 10 lines L_{ij}, $i = 1, \dots, 5$, $j = 1, 2$ are in general position, and similarly for $j = 1, 3$ and $j = 2, 3$.

Let $f, g\colon \mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves such that for $i = 1, \dots, 5$, $j = 1, 2, 3$, $f^{-1}(L_{ij})$ is the same as $g^{-1}(L_{ij})$, counting multiplicities up to 2. Then $p_i \wedge f \equiv p_i \wedge g$ for some i in $1, \dots, 5$.

Proof. By hypothesis (2) of the Theorem, the lines L_{ij}, $i = 1, \dots, 5$, $j = 1, 2$, are in general position. By the second main theorem (2)

$$7T_f(r) \le \sum_{i=1}^{5} \sum_{j=1}^{2} \bar{N}_f(r, L_{ij}) + O\big(\log r T_f(r)\big). \qquad \|$$

There are similar inequalities for $j = 1, 3$ and $j = 2, 3$. Averaging these three inequalities, we obtain

$$\tfrac{21}{2} T_f(r) \le \sum_{i=1}^{5} \sum_{j=1}^{3} \bar{N}_f(r, L_{ij}) + O\big(\log r T_f(r)\big). \qquad \|$$

Adding this to the corresponding inequality for g, we obtain

$$\tfrac{21}{2}\big(T_f(r) + T_g(r)\big) \le \sum_{i=1}^{5} \sum_{j=1}^{3} \big(\bar{N}_f(r, L_{ij}) + \bar{N}_g(r, L_{ij})\big) + O\big(\log r T_f(r) T_g(r)\big), \quad \| \tag{6}$$

For any line H, let $n_0(r, H)$ be the number of points z in $\bar{D}(0, r)$ such that

$$H\big(f(z)\big) = H\big(g(z)\big) = 0,$$

counted twice if $H \circ f$ and $H \circ g$ both vanish to at least second order at z. Let

$$N_0(r,H) = \int_0^r \frac{n_0(t,H) - n_0(0,H)}{t}\,dt + n_0(0,H)\log r.$$

The assumption that $f^{-1}(L_{ij})$ and $g^{-1}(L_{ij})$ are the same, counting multiplicities up to 2, implies that

$$\bar{N}_f(r,L_{ij}) + \bar{N}_g(r,L_{ij}) = 2N_0(r,L_{ij}).$$

The inequality (6) becomes

$$\tfrac{21}{2}\big(T_f(r) + T_g(r)\big) \le 2\sum_{i=1}^{5}\sum_{j=1}^{3} N_0(r,L_{ij}) + O\big(\log r T_f(r) T_g(r)\big). \qquad (7) \quad \|$$

We now assume that, for $i = 1, \ldots, 5$, $p_i \wedge f$ is not identically equal to $p_i \wedge g$, and proceed to derive a contradiction from (7). The method is to estimate the right-hand side of (7) in terms of $T_f(r) + T_g(r)$.

Define a holomorphic curve $h\colon \mathbf{C} \to \mathbf{CP}^{2*}$ by setting $h(z) = f \wedge g(z)$. If (f_0, f_1, f_2) is a reduced representation for f and (g_0, g_1, g_2) is a reduced representation for g, then h is given in homogeneous coordinates by

$$(f_1g_2 - f_2g_1, f_2g_0 - f_0g_2, f_0g_1 - f_1g_0). \qquad (3)$$

This is not in general a reduced representation, since the coordinates may have common zeros. We define

$$\Theta(r) = \frac{1}{2\pi}\int_0^{2\pi} \max_{j,k} \log\big|(f_jg_k - f_kg_j)(re^{i\theta})\big|\,d\theta.$$

Lemma 1 gives

$$T_h(r) + N_c(r) = \Theta(r) + O(1), \qquad (8)$$

where, as in the statement of Lemma 1, $N_c(r)$ is the enumerative function for the common zeros of the components (3) of h.

To estimate $\Theta(r)$, we remark that

$$\begin{aligned}\max_{j,k}\log|f_jg_k - f_kg_j| &\le \max_{j,k}\log\big(|f_jg_k| + |f_kg_j|\big) \le \max_{j,k}\log(2|f_jg_k|)\\ &= \log 2 + \max_j \log|f_j| + \max_k \log|g_k|.\end{aligned}$$

Therefore

$$\Theta(r) \le T_f(r) + T_g(r) + O(1). \qquad (9)$$

Combining (8) and (9), we have

$$T_h(r) + N_c(r) \le T_f(r) + T_g(r) + O(1). \tag{10}$$

We now distinguish 2 contributions to $N_0(r, L_{ij})$. Let $n_e(r, L_{ij})$ be the number of points $z \in \bar{D}(0,r)$ such that $L_{ij}(f(z)) = L_{ij}(g(z)) = 0$, counting multiplicities up to 2, *and also* $f(z) = g(z)$. Let

$$N_e(r, L_{ij}) = \int_0^r \frac{n_e(t, L_{ij}) - n_e(0, L_{ij})}{t}\, dt + n_e(0, L_{ij}) \log r$$

and

$$N_u(r, L_{ij}) = N_0(r, L_{ij}) - N_e(r, L_{ij}). \tag{11}$$

We begin by estimating $N_e(r, L_{ij})$. For a point $z \in \bar{D}(0,r)$ we write $\nu_e(z, L_{ij})$ for the contribution that z makes to $n_e(r, L_{ij})$. Thus

$$n_e(r, L_{ij}) = \sum_{|z| \le r} \nu_e(z, L_{ij}).$$

We write $\nu_c(z)$ for the contribution that z makes to $n_c(r)$. We distinguish 4 cases according to the type of ramification.

Case 1. If z is not a branch point of f or g and none of the lines L_{ij} is tangent to f at $f(z)$, then, since at most 3 of the L_{ij} pass through any point of $\mathbf{CP}^2$,

$$\sum_{i,j} \nu_e(z, L_{ij}) \le 3\nu_c(z).$$

Case 2. If z is not a branch point of f or g and one of the lines, say L_{ab}, is tangent to f at $f(z)$, then by hypothesis L_{ab} must also be tangent to g at $f(z)$. At most 2 others of the lines L_{ij} can pass through $f(z)$, and f and g must intersect them with multiplicity 1, so that in this case

$$\sum_{i,j} \nu_e(z, L_{ij}) \le 4\nu_c(z).$$

Case 3. If f has a branch point at z and g has not, then f intersects any line through $f(z)$ with multiplicity at least 2, and the only line through $f(z)$ that g intersects at that point with multiplicity greater than 1 is the tangent to g. Similarly if g has a branch point at z and f has not. Therefore in this case

$$\sum_{i,j} \nu_e(z, L_{ij}) \le 2\nu_c(z).$$

Case 4. If both f and g have a branch point at z, each of them intersects every line through $f(z)$ with multiplicity at least 2. In this case the components (3) of h have a common zero of multiplicity at least 2. Since at most 3 of the lines L_{ij} pass through any point of $\mathbf{CP}^2$,

$$\sum_{i,j} \nu_e(z, L_{ij}) \leq 3\nu_c(z).$$

To summarize, in all four cases when $f(z) = g(z)$ we have

$$\sum_{i,j} \nu_e(z, L_{ij}) \leq 4\nu_c(z).$$

This yields the estimate

$$\sum_{i=1}^{5}\sum_{j=1}^{3} N_e(r, L_{ij}) \leq 4N_c(r). \tag{12}$$

Now we estimate $N_u(r, L_{ij})$. This is where we use the hypothesis (1) of the Theorem that, for $i = 1, \ldots, 5$, L_{i1}, L_{i2} and L_{i3} have a common point p_i. The dual of p_i is a line p_i^* in $\mathbf{CP}^{2*}$, and on p_i^* there are three points L_{i1}^*, L_{i2}^* and L_{i3}^*.

For any z such that $f(z) \neq g(z)$, the point $h(z) = f \wedge g(z) \in \mathbf{CP}^{2*}$ is dual to the line through $f(z)$ and $g(z)$. When $f(z) \neq g(z)$ and $L_{ij}(f(z)) = L_{ij}(g(z)) = 0$, $h(z)$ is at the point L_{ij}^*. If further $L_{ij} \circ f$ and $L_{ij} \circ g$ vanish to second order at z, h has a branch point at z.

Recall that our assumption for *reductio ad absurdum* is that, for $i = 1, \ldots, 5$, $p_i \wedge f$ is not identically equal to $p_i \wedge g$. By Lemma 2, this implies that the image of $f \wedge g$ is not contained in the line p_i^*. Since L_{ij}^* lies on p_i^*, we have

$$\sum_{j=1}^{3} N_u(r, L_{ij}) \leq \bar{N}_h(r, p_i^*) \leq T_h(r) + O(1)$$

by the Nevanlinna inequality (1), and hence

$$\sum_{i=1}^{5}\sum_{j=1}^{3} N_u(r, L_{ij}) \leq 5T_h(r) + O(1). \tag{13}$$

Applying successively (11), (12), (13) and (10), we have

$$\begin{aligned}\sum_{i=1}^{5}\sum_{j=1}^{3} N_0(r, L_{ij}) &= \sum_{i=1}^{5}\sum_{j=1}^{3} N_e(r, L_{ij}) + \sum_{i=1}^{5}\sum_{j=1}^{3} N_u(r, L_{ij}) \\ &\leq 4N_c(r) + 5T_h(r) + O(1) \leq 5\big(T_f(r) + T_g(r)\big) + O(1).\end{aligned} \tag{14}$$

This inequality (14) establishes that the sum on the right-hand side of (6) can be estimated by $10(T_f(r)+T_g(r))$. With this estimate (6) becomes

$$T_f(r)+T_g(r)=O(\log rT_f(r)T_g(r)). \tag{15}$$

We now pursue a standard argument to derive a contradiction to the assumption that, for $i=1, \ldots, 5$, $p_i \wedge f$ is not identically equal to $p_i \wedge g$. The estimate (15) implies that T_f and T_g are $O(\log r)$. Therefore f and g are rational, for the same reason as in the case of maps into $\mathbf{CP}^1$ [12, Paragraph 21]. Now, for rational functions, the error term in the second main theorem (2) is in fact $O(1)$ [9, proof of Theorem 2.3(a)]. Our inequalities therefore give

$$T_f(r)+T_g(r)=O(1),$$

which implies that f and g are constant. □

Theorem 2. *Let L_{ij}, $i=1,\ldots,6$, $j=1,2,3$, be 18 distinct lines in $\mathbf{CP}^2$ such that*

(1) *for $i=1,\ldots,6$, L_{i1}, L_{i2} and L_{i3} have a common point p_i;*
(2) *the 12 lines L_{ij}, $i=1,\ldots,6$, $j=1,2$, are in general position, and similarly for $j=1,3$ and $j=2,3$.*

Let $f,g\colon \mathbf{C}\to\mathbf{CP}^2$ be full holomorphic curves such that for $i=1,\ldots,6$, $j=1,2,3$, $f^{-1}(L_{ij})$ is the same as $g^{-1}(L_{ij})$, counting multiplicities up to 2. Then $f\equiv g$.

Proof. Apply Theorem 1 to the 15 lines L_{ij}, $i=1,\ldots,5$, $j=1,2,3$, to conclude that for some a we have $p_a\wedge f\equiv p_a\wedge g$. Now apply Theorem 1 to the 15 lines L_{ij} with $i\neq a$ to obtain $b\neq a$ with $p_b\wedge f\equiv p_b\wedge g$.

Suppose that f is not identically equal to g, so that the curve $f\wedge g$ is defined. By Lemma 2, the image of $f\wedge g$ is the point $L^*=p_a^*\cap p_b^*$. This point L^* is dual to a line L in $\mathbf{CP}^2$ and the image of f must lie in L, contradicting the assumption that f is a full curve. □

We now consider to what extent it is possible to vary the configuration of 15 lines in Theorem 1. Let L_{ij}, $i=1,\ldots,A$, $j=1,\ldots,B$, be AB distinct lines such that

(1) for $i=1,\ldots,A$, the lines $L_{i1},\ldots,L_{iB}$ have a common point p_i;
(2) for b in $1,\ldots,B$, the lines L_{ij}, $i=1,\ldots,A$, $j=b,b+1$, with the convention that $B+1$ stands for 1, are in general position.

We can attempt to follow the proof of Theorem 1. Corresponding to inequality (6) we have

$$\tfrac12 B(2A-3)(T_f(r)+T_g(r))\le\sum_{i=1}^{A}\sum_{j=1}^{B}(\bar N_f(r,L_{ij})+\bar N_g(r,L_{ij}))+O(\log rT_f(r)T_g(r)).$$

Corresponding to (12) we have

$$\sum_{i=1}^{A}\sum_{j=1}^{B} N_e(r, L_{ij}) \leq (B+1)N_c(r)$$

and corresponding to (13) we have

$$\sum_{i=1}^{A}\sum_{j=1}^{B} N_u(r, L_{ij}) \leq AT_h(r) + O(1).$$

The conclusion will follow if

$$\tfrac{1}{2}B(2A-3) > 2\max(A, B+1). \tag{16}$$

The solutions of (16) with $A > 0$ and $B > 0$ are the pairs (A, B) satisfying $A \geq 4$, $B \geq 5$ or $A \geq 5$, $B \geq 3$. Theorem 1 is the case $A = 5$, $B = 3$. The same argument with $A = 4$, $B = 5$ proves the corresponding proposition for a certain configuration of 20 lines. Corresponding to Theorem 2 there is a proposition about a configuration of 25 lines. We can obtain uniqueness theorems for some other configurations of lines by using a version of the second main theorem due to E.I. Nochka [13]. The corresponding theorem in the Ahlfors theory is due to C.-H. Sung [18].

Second main theorem (Nochka's version). *Let $f\colon \mathbf{C} \to \mathbf{CP}^n$ be a holomorphic curve such that $f(\mathbf{C})$ spans a k-dimensional linear subspace of $\mathbf{CP}^n$. Let $H_1, \ldots, H_q$ be hyperplanes in general position in $\mathbf{CP}^n$, such that, for $i = 1, \ldots, q$, H_i does not contain $f(\mathbf{C})$. Then*

$$(q - 2n + k - 1)T_f(r) \leq \sum_{i=1}^{q} \bar{N}_f(r, H_i) + O(\log rT_f(r)), \qquad \|$$

where $\bar{N}_f(r, H_i)$ is the enumerative function defined by counting multiplicities up to k.

From a different point of view, Nochka's theorem may be regarded as a theorem on holomorphic curves in $\mathbf{CP}^k$ in relation to configurations of hyperplanes that fail to be in general position to a bounded extent. This is the view that we shall take in Theorems 3 and 4.

Theorem 3. *Let $A = 5$, $B \geq 4$ or $A \geq 6$, $B \geq 3$. Let L_{ij}, $i = 1, \ldots, A$, $j = 1, \ldots, B$, be AB distinct lines in $\mathbf{CP}^2$ such that*
(1) for $i = 1, \ldots, A$, the lines $L_{i1}, \ldots, L_{iB}$ have a common point p_i;
(2) at most B of the L_{ij} pass through any point of $\mathbf{CP}^2$.
Let $f, g\colon \mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves such that, for $i = 1, \ldots, A$, $j = 1, \ldots, B$, $f^{-1}(L_{ij})$ is the same as $g^{-1}(L_{ij})$, counting multiplicities up to 2. Then $p_i \wedge f \equiv p_i \wedge g$ for some i in $1, \ldots, A$.

Remark. The possible values of A and B are obtained by reasoning similar to the discussion following Theorem 2.

Proof. The only difference from Theorem 1 is that we use Nochka's version of the second main theorem. Regard $\mathbf{CP}^2$ as a linear subspace of $\mathbf{CP}^B$. By a standard general position argument, there exist hyperplanes H_{ij}, $i = 1, \dots, A$, $j = 1, \dots, B$, in general position in $\mathbf{CP}^B$, such that $L_{ij} = H_{ij} \cap \mathbf{CP}^2$ for all i and j. Nochka's version of the second main theorem gives

$$\begin{aligned}(AB - 2B + 1)&\big(T_f(r) + T_g(r)\big) \\ &\leq \sum_{i=1}^{A}\sum_{j=1}^{B}\big(\bar{N}_f(r, L_{ij}) + \bar{N}_g(r, L_{ij})\big) + O\big(\log rT_f(r)T_g(r)\big), \qquad \|\end{aligned}$$

corresponding to (6) in the proof of Theorem 1. Now the argument proceeds as before. □

Theorem 4. *Let $A = 6$, $B \geq 4$ or $A \geq 7$, $B \geq 3$. Let L_{ij}, $i = 1, \dots, A$, $j = 1, \dots, B$, be AB distinct lines in $\mathbf{CP}^2$ such that*
(1) *for $i = 1, \dots, A$, the lines L_{i1}, ..., L_{iB} have a common point p_i;*
(2) *for a in $1, \dots, A$, if S is the set of lines L_{ij} such that $i \neq a$, then at most B of the lines in S pass through any point of $\mathbf{CP}^2$.*
Let $f, g\colon \mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves such that $f^{-1}(L_{ij})$ is the same as $g^{-1}(L_{ij})$, counting multiplicities up to 2. Then $f \equiv g$.

Proof. This follows from Theorem 3 in the same way as Theorem 2 follows from Theorem 1. □

In each of Theorems 1 to 4 it is an essential condition that, for a fixed index i, the lines L_{ij} have a common point p_i. This linear relation among the lines L_{ij} is used to obtain an estimate for the terms $N_u(r)$ in the form of the inequality (13). In Theorem 5 we replace this with a quadratic condition; in geometrical language, the lines L_i are assumed to lie on a *line conic* $C \subset \mathbf{CP}^{2*}$. The lines L_i of Theorem 5 are thus in general position. The conclusion of Theorem 5 is that f and g satisfy a certain algebraic identity that is quadratic in each; this is not so simple as for Theorem 1, but it is still the case that 2 such conditions imply $f \equiv g$.

If C is a curve in $\mathbf{CP}^{2*}$ we shall also write C for the homogeneous form defining C.

Theorem 5. *Let C be a non-degenerate curve of degree 2 in $\mathbf{CP}^{2*}$. Let L_1, ..., L_{10} be distinct lines on C. Let $f, g\colon \mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves such that, for $i = 1, \dots, 10$, $f^{-1}(L_i) = g^{-1}(L_i)$, counting multiplicities up to 2. Then $C(f \wedge g) \equiv 0$.*

Proof. We introduce the enumerative functions $N_0(r, L_i)$, $N_e(r, L_i)$ and $N_u(r, L_i)$, satisfying

$$N_0(r, L_i) = N_e(r, L_i) + N_u(r, L_i), \tag{17}$$

as in the proof of Theorem 1. The lines L_i are in general position, and so by the second main theorem (2)

$$7\big(T_f(r) + T_g(r)\big) \leq 2\sum_{i=1}^{10} N_0(r, L_i) + O\big(\log rT_f(r)T_g(r)\big), \qquad \| \tag{18}$$

corresponding to (7) in the proof of Theorem 1. Corresponding to (12) we have

$$\sum_{i=1}^{10} N_e(r, L_i) \leq 3N_c(r), \tag{19}$$

since at most 2 of the lines L_i pass through any point of $\mathbf{CP}^2$. We now wish to estimate the terms $N_u(r, L_i)$. If (x_0, x_1, x_2) are homogeneous coordinates on $\mathbf{CP}^{2*}$, the *Veronese embedding* V: $\mathbf{CP}^{2*} \to \mathbf{CP}^5$ is defined by

$$V(x_0, x_1, x_2) = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2). \tag{20}$$

The image under V of C is the section of $V(\mathbf{CP}^{2*})$ by a hyperplane H. We have $C = H \circ V$ as forms on $\mathbf{CP}^{2*}$. Suppose, to obtain a contradiction, that $C \circ (f \wedge g)$ is not identically zero. Write $h = f \wedge g$. For $i = 1, \ldots, 10$, $V(L_i^*)$ lies on $V(C)$, and so by the Nevanlinna inequality (1) we have

$$\sum_{i=1}^{10} N_u(r, L_i) \leq \bar{N}_{V\circ h}(r, H) \leq T_{V\circ h}(r) + O(1). \tag{21}$$

The formula (20) for the Veronese embedding V gives

$$T_{V\circ h}(r) = 2T_h(r). \tag{22}$$

Combining (21) and (22) we have

$$\sum_{i=1}^{10} N_u(r, L_i) \leq 2T_h(r) + O(1), \tag{23}$$

which corresponds to (13) in the proof of Theorem 1. Applying successively (17), (19), (23) and (10), we have

$$\begin{aligned}\sum_{i=1}^{10} N_0(r, L_i) &= \sum_{i=1}^{10} N_e(r, L_i) + \sum_{i=1}^{10} N_u(r, L_i) \\ &\leq 3N_c(r) + 2T_h(r) + O(1) \leq 3\big(T_f(r) + T_g(r)\big) + O(1).\end{aligned}$$

Hence the sum on the right-hand side of (18) can be estimated by $6\big(T_f(r)+T_g(r)\big)$. A contradiction follows as in the proof of Theorem 1. □

The method of Theorem 5 applies to curves in $\mathbf{CP}^{2*}$ of any degree. In particular, if we replace the conic C with an irreducible cubic, then, provided no three of the lines L_i are concurrent, we have

$$\sum_{i=1}^{10} N_u(r, L_i) \le 3T_h(r) + O(1),$$

corresponding to (23), and the rest of the proof remains the same. For curves C of degree 4 or higher, the number of lines L_i has to be more than 10.

By taking two curves C_1, $C_2 \subset \mathbf{CP}^{2*}$, we may obtain a theorem with the conclusion that $f \equiv g$. For simplicity we consider only curves of degree 2 or 3.

Theorem 6. *Let C_1, C_2 be distinct non-degenerate curves of degree 2 or 3 in $\mathbf{CP}^{2*}$. Let $L_{1,1}, \dots, L_{10,1}$ be distinct lines on C_1, no three of them concurrent, and let $L_{1,2}, \dots, L_{10,2}$ be distinct lines on C_2, no three of them concurrent. Let $f, g\colon \mathbf{C} \to \mathbf{CP}^2$ be full holomorphic curves such that, for $i = 1, \dots, 10$, $j = 1, 2$, $f^{-1}(L_{ij}) = g^{-1}(L_{ij})$, counting multiplicities up to 2. Then $f \equiv g$.*

Proof. We have observed above that Theorem 5 remains true if C is of degree 3. Applying Theorem 5 we obtain $C_1(f \wedge g) \equiv 0$ and $C_2(f \wedge g) \equiv 0$. Therefore the image of $f \wedge g$ lies in $C_1 \cap C_2$ and so $f \wedge g$ is a constant curve. As in the proof of Theorem 2, this contradicts the assumption that f is a full curve. □

The hypotheses of Theorem 6 allow some of the lines $L_{i,1}$ to coincide with some of the lines $L_{i,2}$. If this happens there are less than 20 lines in the configuration. For example, if C_1 and C_2 are cubic curves that intersect in nine points $P_1, \dots, P_9$, we may take $L_{i,1}^* = L_{i,2}^* = P_i$ for $i = 1, \dots, 9$. A suitable choice of $L_{10,1}$ and $L_{10,2}$ gives a configuration of 11 lines in general position that satisfies the hypotheses of Theorem 6.

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