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# TRANSFINITE EXTENT

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## 1. Introduction

In this paper, we shall consider a set function $e(E)$ for compact sets in the complex plane $\mathbf{C}$, that we shall call the transfinite extent of $E$. It is closely connected with the transfinite diameter $d(E)$, which was introduced by M. Fekete [2].

F. Leja [5] generalized the transfinite diameter to an écart $v(E,\varphi)$ of $E$ with respect to a generating function $\varphi$. Here $\varphi$ is a continuous, nonnegative, symmetric function $\varphi\colon M^m \to \mathbf{R}$ of $m \geq 2$ variables on a metric space $(M,\varrho)$, satisfying the additional condition that $\varphi(p_1,\dots,p_m) = 0$ if $p_j = p_k$ for some $j \neq k$.

Put, for any finite subset $\{p_1,\dots,p_n\} \subseteq M$, $n \geq m$,

$$V(p_1,\dots,p_n) = \prod_{1 \leq j_1 < \cdots < j_m \leq n} \varphi(p_{j_1},\dots,p_{j_m}),$$

and let

$$V_n(E) = \max_{p_j \in E} V(p_1,\dots,p_n).$$

Then Leja [5] showed that

$$V_{n+1}(E)^{1/\binom{n+1}{m}} \leq V_n(E)^{1/\binom{n}{m}}$$

and so

$$v(E,\varphi) = \lim_{n\to\infty} V_n(E)^{1/\binom{n}{m}}$$

exists.

If $M = \mathbf{C}$, $\varrho(z_1,z_2) = |z_1 - z_2|$, and $\varphi = \varrho$, then $v(E,\varphi) = d(E)$, the transfinite diameter of $E$. If, in the same space, $\varphi$ is chosen to be the area of the triangle $Oz_1z_2$, then $v(E,\varphi)$ is the original écart of Leja [5, 6]. It is connected with convergence questions for homogenous polynomials of two real variables.

The *transfinite extent* is defined by choosing $\varphi(z_1,z_2,z_3)$ to be the area of the triangle in $\mathbf{C}$ spanned by the points $z_1,z_2,z_3$, and by putting

$$e(E) = v(E,\varphi).$$

It exists for any compact set $E \subset \mathbf{C}$.

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## 2. Elementary properties

**Proposition 1.** *Let $E$ and $F$ be compact sets in $\mathbf{C}$.*
(a) *$E \subseteq F$ implies $e(E) \leq e(F)$.*
(b) *$e(rE) = r^2 e(E)$ for any $r \in \mathbf{R}$.*
(c) *$e(fE) = e(E)$ for any area-preserving affine map $f\colon \mathbf{C} \to \mathbf{C}$.*
(d) *$e(L) = 0$ for any line segment $L \subset \mathbf{C}$.*

*Proof.* All these properties are obvious.

In the following, $[z_1, z_2, z_3]$ will denote the triangle spanned by $z_1$, $z_2$, $z_3$, and $\big|[z_1, z_2, z_3]\big|$ its area. We have

$$V(z_1, \dots, z_n) = \prod_{1 \leq j_1 < j_2 < j_3 \leq n} \big|[z_{j_1}, z_{j_2}, z_{j_3}]\big|.$$

Then

$$e_n(E) = V_n(E)^{1/\binom{n}{3}}$$

is the $n$-*extent* of $E$, and $e_n(E) \to e(E)$ as $n \to \infty$. We will also define

$$U(z_1, \dots, z_n) = \prod_{1 \leq j_1 < j_2 \leq n} |z_{j_1} - z_{j_2}|$$

and

$$U_n(E) = \max_{z_j \in E} U(z_1, \dots, z_n).$$

Then

$$d_n(E) = U_n(E)^{1/\binom{n}{2}}$$

is the $n$-diameter of $E$, and $d_n(E) \to d(E)$ as $n \to \infty$, where $d(E)$ is the transfinite diameter if $E$. Let $\mathbf{D} = \{z : |z| < 1\}$.

**Proposition 2.** *For compact sets $E \subseteq \partial\mathbf{D}$, we have $e_n(E) = \frac{1}{4} d_n(E)^3$ and $e(E) = \frac{1}{4} d(E)^3$.*

*Proof.* We use the relation

$$\big|[z_1, z_2, z_3]\big| = \tfrac{1}{4} |z_1 - z_2||z_1 - z_3||z_2 - z_3|$$

for $z_1$, $z_2$, $z_3 \in \partial\mathbf{D}$. The reason for this is that the radius of the circumscribed circle of a triangle equals the product of the lengths of its sides, divided by four times its area. Then we have

$$\begin{aligned} V(z_1, \dots, z_n) &= 4^{-\binom{n}{3}} \prod_{1 \leq j_1 < j_2 < j_3 \leq n} |z_{j_1} - z_{j_2}||z_{j_1} - z_{j_3}||z_{j_2} - z_{j_3}| \\ &= 4^{-\binom{n}{3}} \Big( \prod_{1 \leq j_1 < j_2 \leq n} |z_{j_1} - z_{j_2}| \Big)^{n-2} = 4^{-\binom{n}{3}} U(z_1, \dots, z_n)^{n-2}. \end{aligned}$$

From this it follows that

$$V_n(E) = 4^{-\binom{n}{3}} U_n(E)^{n-2}$$

and so

$$e_n(E) = \tfrac{1}{4} d_n(E)^3,$$

which leads to the assertions above.

**Examples.** Since the transfinite diameter of the unit circle is 1, it follows that $e(\partial\mathbf{D}) = \frac{1}{4}$. Moreover, since the transfinite diameter of an arc on $\partial\mathbf{D}$ of length $l$ is $\sin(l/4)$, it follows that its transfinite extent is $\frac{1}{4}\sin^3(l/4)$. Furthermore, by using Proposition 1bc, it is possible to calculate the transfinite extent of any arc of any ellipse.

For any compact set $E \subseteq \partial\mathbf{D}$, we may estimate

$$e_n(E) = \tfrac{1}{4} d_n(E)^3 \le \tfrac{1}{4} d_n(\partial\mathbf{D}) d_n(E)^2.$$

Schur [12, p. 385] credits to Pólya the observation that the maximum of the product $U(z_1, \dots, z_n)$ for points $z_1, \dots, z_n \in \overline{\mathbf{D}}$ is $n^{n/2}$ and that it occurs for equally spaced points on $\partial\mathbf{D}$. It follows that

$$d_n(\partial\mathbf{D}) = U\big(1, e^{2\pi i/n}, \dots, e^{2\pi i(n-1)/n}\big)^{1/\binom{n}{2}} = n^{1/(n-1)},$$

and so

$$e_n(E) \le \tfrac{1}{4} n^{1/(n-1)} d_n(E)^2.$$

This inequality is invariant under translations and dilations of the set $E$. Thus we have proved the following

**Proposition 3.** *If $E$ is a compact set lying on some circle, then $e_n(E) \le \frac{1}{4} n^{1/(n-1)} d_n(E)^2$ and $e(E) \le \frac{1}{4} d(E)^2$.*

It is quite possible, but not proved, that the inequalities of Proposition 3 remain valid for arbitrary compact sets $E$. However, for arbitrary compact sets we have the following estimates.

**Proposition 4.** *For any compact set $E \subset \mathbf{C}$, we have $e_n(E) \le \frac{\sqrt{3}}{4} d_n(E)^2$ and $e(E) \le \frac{\sqrt{3}}{4} d(E)^2$.*

*Proof.* Given a triangle with side lengths $a$, $b$, $c$ and $\theta$ the angle opposite $a$, we have

$$\frac{A^3}{(abc)^2} = \frac{1}{8} \frac{\sin^3\theta}{\frac{b}{c} + \frac{c}{b} - 2\cos\theta}$$

where $A$ is the area of the triangle, using the law of cosines. Since $b/c + c/b \geq 2$, this yields

$$\frac{A^3}{(abc)^2} \leq \frac{1}{16}(\sin\theta)(1+\cos\theta) \leq \frac{3\sqrt{3}}{64},$$

and so $A \leq \frac{\sqrt{3}}{4}(abc)^{2/3}$ for any triangle, with equality only for equilateral triangles.

Now let $\xi_1, \ldots, \xi_n \in E$ be such that

$$V(\xi_1, \ldots, \xi_n) = V_n(E).$$

Then

$$\begin{aligned} V(\xi_1, \ldots, \xi_n) &\leq \left(\frac{\sqrt{3}}{4}\right)^{\binom{n}{3}} \prod_{1\leq j_1<j_2<j_3\leq n} \left(|\xi_{j_1} - \xi_{j_2}||\xi_{j_1} - \xi_{j_3}||\xi_{j_2} - \xi_{j_3}|\right)^{2/3} \\ &= \left(\frac{\sqrt{3}}{4}\right)^{\binom{n}{3}} \left(\prod_{1\leq j_1<j_2\leq n} |\xi_{j_1} - \xi_{j_2}|\right)^{2(n-2)/3} \\ &\leq \left(\frac{\sqrt{3}}{4}\right)^{\binom{n}{3}} U_n(E)^{2(n-2)/3} \end{aligned}$$

which yields the desired inequalities by taking the $\binom{n}{3}$-root on both sides and letting $n \to \infty$.

**Remark.** In the special case $n = 3$, the inequality $e_3(E) \leq \frac{\sqrt{3}}{4} d_3(E)^2$ is sharp when $E$ is any equilateral triangle $T$ or when $E$ is any compact subset of $T$ that contains its vertices.

## 3. Null-sets

We consider compact null-sets $E$ for the transfinite extent, i.e., $e(E) = 0$. From Proposition 4 it is clear that any null-set for the transfinite diameter is a null-set for the transfinite extent. And from Proposition 2 we see that on the unit circle $\partial\mathbf{D}$ the null-sets for transfinite extent and transfinite diameter actually coincide. It is clear, though, that there are null-sets for the transfinite extent that are not null-sets for the transfinite diameter: for instance, all line segments.

**Proposition 5.** *A compact null-set for transfinite extent has zero area.*

*Proof.* Let $E$ be compact with positive area, and let $C(r)$ be the circle $|z| = r$. Since $E \cap C(r)$ is closed in $C(r)$, the linear measure of $E \cap C(r)$ exists; we will denote it by $l(r)$. By Fubini's theorem

$$\int_0^\infty l(r)\,dr = \text{Area}(E) > 0,$$

and so there exists an $r_0 > 0$ with $l(r_0) > 0$. Then

$$e(E) \geq e\big(E \cap C(r_0)\big) = \frac{1}{4r_0} d\big(E \cap C(r_0)\big)^3 > 0$$

since a set of positive length has positive capacity.

**Proposition 6.** *Let $E_1$ and $E_2$ be compact sets in $\mathbf{C}$, and let the area of any triangle spanned by $E = E_1 \cup E_2$ be bounded above by $A > 0$. Then*

$$h\left(\frac{e(E)}{A}\right) \leq h\left(\frac{e(E_1)}{A}\right) + h\left(\frac{e(E_2)}{A}\right)$$

*where $h(x) = \big(\log(1/x)\big)^{-1/2}$.*

*Proof.* Let $V(z_1, \dots, z_n)$ attain its maximum $V_n(E)$ on $E$ at $\xi_1, \dots, \xi_n$. Let $k$ of the points $\xi_j$ lie in $E_1$, the other $n-k$ in $E_2$. Using the estimate

$$\big|[\xi_{j_1}, \xi_{j_2}, \xi_{j_3}]\big| \leq A$$

when not all of $\xi_{j_1}, \xi_{j_2}, \xi_{j_3}$ lie in $E_1$ or lie in $E_2$, we get

$$V(\xi_1, \dots, \xi_n) \leq V_k(E_1) V_{n-k}(E_2) A^{\binom{k}{1}\binom{n-k}{2} + \binom{k}{2}\binom{n-k}{1}}.$$

We take the logarithm on both sides and divide by $\binom{n}{3}$ to get

$$\log e_n(E) \leq \frac{\binom{k}{3}}{\binom{n}{3}} \log e_k(E_1) + \frac{\binom{n-k}{3}}{\binom{n}{3}} \log e_{n-k}(E_2) + \frac{\log A}{\binom{n}{3}} \left(\binom{k}{1}\binom{n-k}{2} + \binom{k}{2}\binom{n-k}{1}\right).$$

If we let $n \to \infty$ through a suitable subsequence, then $k/n \to \lambda$ $(0 \leq \lambda \leq 1)$ and we obtain

$$\log e(E) \leq \lambda^3 \log e(E_1) + (1-\lambda)^3 \log e(E_2) + \big(1 - (1-\lambda)^3 - \lambda^3\big) \log A$$

or

$$\log \frac{A}{e(E)} \geq \lambda^3 \log \frac{A}{e(E_1)} + (1-\lambda)^3 \log \frac{A}{e(E_2)}.$$

The right-hand side attains its maximum as a function of $\lambda$ when

$$\lambda^2 \log \frac{A}{e(E_1)} = (1-\lambda)^2 \log \frac{A}{e(E_2)}.$$

Substitution yields the desired inequality.

Proposition 6 is an analogue of a result that seems to have been first proved, but not published, by Fekete for the transfinite diameter. The proof above closely follows one given by Pommerenke [8, Theorem 11.4].

From Proposition 6 one can easily conclude that if $E_1$, $E_2$, $E_3$, ... are compact null-sets for transfinite extent, and $E = E_1 \cup E_2 \cup E_3 \cup \cdots$ is compact, then $E$ is a null-set for transfinite extent.

By a standard technique of potential theory, see Carleson [1] or Pommerenke [8], $e(E)$ can be extended to an outer capacity $e^*(E)$, and the requirement that $E$ be compact could be removed from the statements of most of our theorems.

Next, let $h$ be a measure function, i. e., $h(x)$ is defined and continuous for $x \geq 0$, $h(0) = 0$, and $h(x)$ is increasing. We define a measure $\Omega_h(E)$ for compact sets $E \subset \mathbf{C}$ as follows:

$$\Omega_h(E) = \lim_{\varepsilon \to 0} \inf_{E \subset \cup G_j} \sum_j (h \circ g)\big(\mathrm{Area}\,(G_j)\big)$$

where $g(x) = (x/\pi)^{1/2}$, and the infimum is taken over all finite coverings of $E$ by ellipses $G_j$ with $\mathrm{Area}(G_j) \leq \varepsilon$. The classical Hausdorff measure is given by ($E$ compact)

$$\Lambda_h(E) = \lim_{\varepsilon \to 0} \inf_{E \subset \cup \Delta_j} \sum_j (h \circ g)\big(\mathrm{Area}\,(\Delta_j)\big)$$

where the infimum is taken over all finite coverings of $E$ by disks $\Delta_j$ with $\mathrm{Area}(\Delta_j) \leq \varepsilon$. Since ellipses include disks, it is clear that

$$\Omega_h(E) \leq \Lambda_h(E).$$

Null-sets for the measure $\Omega_h$ are connected with null-sets for transfinite extent as follows.

**Proposition 7.** *Let* $h(x) = \big(\log \frac{1}{x}\big)^{-1/2}$. *Then* $\Omega_h(E) = 0$ *implies* $e(E) = 0$ *for compact sets* $E \subset \mathbf{C}$.

*Proof.* Let the area of any triangle spanned by $E$ be bounded above by $A > 0$. Let $G_j$ be a finite covering of $E$ by ellipses with $\mathrm{Area}(G_j) \leq \varepsilon$. Put $E_j = E \cap G_j$, and assume

$$\varepsilon \leq \frac{\pi A^2}{e(\bar{\mathbf{D}})^2}.$$

Then, using Proposition 6 and Proposition 1, we have

$$\begin{aligned} h\left(\frac{e(E)}{A}\right) &\leq \sum_j h\left(\frac{e(E_j)}{A}\right) \leq \sum_j h\left(\frac{e(G_j)}{A}\right) \\ &= \sum_j h\left(\frac{e(\bar{\mathbf{D}})}{\pi A}\mathrm{Area}\,(G_j)\right) \leq \sum_j (h \circ g)\big(\mathrm{Area}\,(G_j)\big). \end{aligned}$$

Thus

$$h\left(\frac{e(E)}{A}\right) \leq \Omega_h(E)$$

and so the assertion above follows.

We note that for $h(x) = \left(\log \frac{1}{x}\right)^{-1/2}$ the measure $\Omega_h$ has more null-sets than the measure $\Lambda_h$: for instance,

$$\Omega_h([0,1]) = 0, \qquad \Lambda_h([0,1]) = +\infty.$$

Proposition 7 is an analogue of a result about harmonic measure and hence capacity due to Lindeberg [7].

## 4. A connection with curvature

**Proposition 8.** *Let $\Gamma$ be a $C^2$ arc in the plane, $z_0$ an interior point of the arc, and $\kappa$ the unsigned curvature of $\Gamma$ at $z_0$. Then*

$$\kappa = 32 \lim_{\varepsilon \to 0} \frac{e(\Gamma \cap D_\varepsilon)}{\varepsilon^3}$$

*where $D_\varepsilon$ is the closed disk with center $z_0$ and radius $\varepsilon$.*

*Proof.* We may without loss of generality assume that $z_0 = 0$ and that the tangent to $\Gamma$ at the origin is the $x$-axis. Then $\Gamma$ is the graph of a function $y = f(x)$ in a sufficiently small neighborhood of $z = 0$.

We first consider the case $\kappa = 0$. If $\Gamma$ reduces to a line segment near $z = 0$, the assertion to be proved is obvious. If not, then

$$h(u) = \max_{-u \leq x \leq u} |f(x)| > 0$$

for $u > 0$. Since $\kappa = 0$, the function $f(x) = o(x^2)$ near the origin, and so $h(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$. Furthermore, for small $\varepsilon$ we have $\Gamma \cap D_\varepsilon \subseteq R_\varepsilon$ where $R_\varepsilon$ is the axes-parallel rectangle centered at $z = 0$ of length $2\varepsilon$ and height $2h(\varepsilon)$. Now

$$e(\Gamma \cap D_\varepsilon) \leq e(R_\varepsilon) = \varepsilon h(\varepsilon) e(R_0)$$

where $R_0$ is the square of side length $2$. Thus

$$\lim_{\varepsilon \to 0} \frac{e(\Gamma \cap D_\varepsilon)}{\varepsilon^3} \leq \lim_{\varepsilon \to 0} \frac{\varepsilon h(\varepsilon) e(R_0)}{\varepsilon^3} = 0$$

and the asserted equality has been established for the case $\kappa = 0$.

We now suppose that $\kappa > 0$. We may without loss of generality assume that the circle of curvature $C$ of $\Gamma$ at $z = 0$ lies above the $x$-axis. Clearly $C$ is the graph of a function $y = g(x)$ near $z = 0$.

We have

$$f(x) = \tfrac{1}{2}\kappa x^2 + j(x) \qquad \text{and} \qquad g(x) = \tfrac{1}{2}\kappa x^2 + k(x)$$

near $x = 0$, with $j(x) = o(x^2)$ and $k(x) = o(x^2)$ as $x \to 0$.

Consider an arbitrary triple $a < b < c$ of points in $[-\varepsilon, \varepsilon]$, with $\varepsilon$ so small that $\Gamma$ and $C$ are graphs above $[-\varepsilon, \varepsilon]$. To this triple there corresponds a triangle with vertices lying on $\Gamma$, namely the triangle spanned by $(a, f(a))$, $(b, f(b))$ and $(c, f(c))$, and a triangle with vertices lying on $C$, namely the triangle spanned by $(a, g(a))$, $(b, g(b))$ and $(c, g(c))$. So by orthogonal projection from the real axis, we have a bijective correspondence between triangles lying above $[-\varepsilon, \varepsilon]$ with vertices on $\Gamma$, and triangles lying above $[-\varepsilon, \varepsilon]$ with vertices on $C$.

Let $A$ be the area of the triangle with vertices $(a, f(a))$, $(b, f(b))$, $(c, f(c))$ and $A^*$ the area of the triangle with vertices $(a, g(a))$, $(b, g(b))$, $(c, g(c))$. Since

$$A = \left| \frac{f(a) + f(b)}{2}(b - a) + \frac{f(b) + f(c)}{2}(c - b) - \frac{f(a) + f(c)}{2}(c - a) \right|,$$

$$A^* = \left| \frac{g(a) + g(b)}{2}(b - a) + \frac{g(b) + g(c)}{2}(c - b) - \frac{g(a) + g(c)}{2}(c - a) \right|,$$

we have

$$\frac{A^*}{A} = \left| \frac{\frac{g(a)-g(b)}{a-b} - \frac{g(c)-g(b)}{c-b}}{\frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(b)}{c-b}} \right|.$$

If we apply the generalized mean value theorem to the expression inside the absolute value brackets, we obtain

$$\frac{A^*}{A} = \left| \frac{(\zeta - b)g'(\zeta) - \big(g(\zeta) - g(b)\big)}{(\zeta - b)f'(\zeta) - \big(f(\zeta) - f(b)\big)} \right|$$

where $a < \zeta < c$. Now apply the generalized mean value theorem again, on the interval from $b$ to $\zeta$, to conclude that

$$\frac{A^*}{A} = \left| \frac{g''(\xi)}{f''(\xi)} \right|.$$

Since $f$, $g \in C^2$ and $f''(0) = g''(0) = \kappa > 0$, it follows that

$$A^* = A\big(1 + o(1)\big)$$

for $\varepsilon \to 0$, uniformly with respect to $a$, $b$, $c$. Therefore

$$\frac{A^*}{\varepsilon^3} = \frac{A}{\varepsilon^3}\big(1 + o(1)\big)$$

and it follows that

$$\lim_{\varepsilon\to 0}\frac{e(\Gamma\cap D_\varepsilon)}{\varepsilon^3}=\lim_{\varepsilon\to 0}\frac{e(C\cap D_\varepsilon)}{\varepsilon^3}.$$

Now

$$e(C\cap D_\varepsilon)=\kappa^{-2}e(\partial\mathbf{D}\cap D_{\varepsilon\kappa})=\frac{\kappa^{-2}}{4}d(\partial\mathbf{D}\cap D_{\varepsilon\kappa})^3=\frac{\kappa^{-2}}{4}\sin^3\left(\frac{L_{\varepsilon\kappa}}{4}\right)$$

where $L_{\varepsilon\kappa}$ is the length of the arc on $\partial\mathbf{D}$ cut out by a circle of radius $\varepsilon\kappa$. We have

$$L_{\varepsilon\kappa}=4\arcsin\left(\tfrac{1}{2}\varepsilon\kappa\right)$$

and thus

$$e(C\cap D_\varepsilon)=\frac{\kappa\varepsilon^3}{32}.$$

So

$$\lim_{\varepsilon\to 0}\frac{e(\Gamma\cap D_\varepsilon)}{\varepsilon^3}=\lim_{\varepsilon\to 0}\frac{e(C\cap D_\varepsilon)}{\varepsilon^3}=\lim_{\varepsilon\to 0}\frac{\kappa\varepsilon^3}{32\varepsilon^3}=\frac{\kappa}{32}$$

and thus the asserted equality is true.

Proposition 8 suggests a definition for generalized (unsigned) curvature of a compact set $\Gamma$ at a point $z_0\in\Gamma$:

$$\kappa(\Gamma;z_0)=32\lim_{\varepsilon\to\infty}\frac{e(\Gamma\cap D_\varepsilon)}{\varepsilon^3}$$

where $D_\varepsilon$ is the closed disk with center $z_0$ and radius $\varepsilon$.

It is clear that $\kappa(\Gamma;z_0)$ may easily fail to exist, though by replacing $e()$ by outer transfinite extent $e^*()$, and limes by limes superior, in the definition for $\kappa(\Gamma;z_0)$, we may obtain a generalized curvature $\kappa^*(\Gamma;z_0)$ that exists for any point $z_0$ of any plane set $\Gamma$, and satisfies $0\le\kappa^*(\Gamma;z_0)\le\infty$.

**Proposition 9.** *Let $\Gamma$ be a compact set with positive area. Then $k(\Gamma;z_0)=\infty$ for almost all points $z_0\in\Gamma$.*

*Proof.* Let $z_0$ be a point of density of $\Gamma$. Then there exists some $\varepsilon_0>0$ such that if $\varepsilon\le\varepsilon_0$, then

$$\text{Area}\,(\Gamma\cap D_\varepsilon)>\frac{\pi}{2}\varepsilon^2$$

where $D_\varepsilon$ is the closed disk with center $z_0$ and radius $\varepsilon$. We have

$$\int_0^\varepsilon l(r)\,dr=\text{Area}\,(\Gamma\cap D_\varepsilon)>\frac{\pi}{2}\varepsilon^2$$

where $l(r)$ is the length of $\Gamma\cap C_r$, and $C_r$ is the circle around $z_0$ of radius $r$. Now

$$\int_0^{\varepsilon/2} l(r)\,dr\le\int_0^{\varepsilon/2}2\pi r\,dr=\frac{\pi}{4}\varepsilon^2$$

and so

$$\int_{\varepsilon/2}^{\varepsilon} l(r)\,dr > \frac{\pi}{4}\varepsilon^2.$$

Now assume that for $\varepsilon/2 \le r \le \varepsilon$, we have $l(r) \le kr$. Then we obtain

$$\frac{\pi}{4}\varepsilon^2 < \int_{\varepsilon/2}^{\varepsilon} l(r)\,dr \le \int_{\varepsilon/2}^{\varepsilon} kr\,dr = \frac{3}{8}k\varepsilon^2$$

and thus $k > 2\pi/3$. So we see that there exists some $r_0$, $\varepsilon/2 \le r_0 \le \varepsilon$, such that $l(r_0) \ge 2\pi r_0/3$. Thus

$$\begin{aligned} e(\Gamma \cap D_\varepsilon) \ge e(\Gamma \cap C_{r_0}) &= \frac{1}{4r_0} d(\Gamma \cap C_{r_0})^3 \\ \frac{r_0^2}{4} d\big(r_0^{-1}(\Gamma \cap C_{r_0})\big)^3 &\ge \frac{r_0^2}{4}\sin\big(\frac{\pi}{6}\big)^3 \ge \frac{\varepsilon^2}{128} \end{aligned}$$

and so

$$\kappa(\Gamma; z_0) = 32 \lim_{\varepsilon\to 0} \frac{e(\Gamma \cap D_\varepsilon)}{\varepsilon^3} = \infty.$$

## 5. The $n$-extent problem

The $n$-extent problem ($n \ge 3$) is the extremal problem

$$\sup_\Gamma e_n(\Gamma)$$

where the supremum is taken over all continua $\Gamma$ of capacity 1. In this section we shall see that extremal continua for the 3-extent problem are symmetric three-pointed stars, and so they coincide with the extremal continua for the 3-diameter problem (cf. [3, 9, 11]). Therefore it is somewhat of a surprise when we show that extremal continua for the 4-extent and 4-diameter problems are different.

We have

$$\sup_\Gamma e_n(\Gamma) = \sup_{f\in\Sigma} e_n\big(\mathbf{C} \setminus f(|\zeta| > 1)\big)$$

where $\Sigma$ denotes the familiar class of normalized univalent functions $f(\zeta) = \zeta + \sum_{k=0}^{\infty} b_k \zeta^{-k}$ in $|\zeta| > 1$. Since $\Sigma$ is compact modulo translations, it follows that the supremum is always assumed.

Now suppose that $\Gamma$ is extremal for the $n$-extent problem. Then there exist points $z_1, \ldots, z_n \in \Gamma$ such that

$$e_n(\Gamma) = \Bigg( \prod_{1\le j_1<j_2<j_3\le n} \big|[z_{j_1}, z_{j_2}, z_{j_3}]\big| \Bigg)^{1/\binom{n}{3}}.$$

In order to compute areas, we shall use the formula

$$\left|[z_{j_1}, z_{j_2}, z_{j_3}]\right| = \tfrac{1}{2}\left|\mathrm{Im}\left\{(z_{j_1} - z_{j_3})(\bar{z}_{j_2} - \bar{z}_{j_3})\right\}\right|.$$

Thus it will be convenient to denote

$$A_{j_1 j_2 j_3} = \mathrm{Im}\{B_{j_1 j_2 j_3}\} \quad \text{where} \quad B_{j_1 j_2 j_3} = (z_{j_1} - z_{j_3})(\bar{z}_{j_2} - \bar{z}_{j_3}),$$

so that $\frac{1}{2}|A_{j_1 j_2 j_3}|$ is the area of the triangle $[z_{j_1}, z_{j_2}, z_{j_3}]$.

Since $e_n(\Gamma) > 0$ for an extremal $\Gamma$, all $A_{j_1 j_2 j_3}$ are non-zero and we may replace the functional $e_n(\Gamma)$ by the equivalent functional

$$\mathrm{Re} \sum_{1 \le j_1 < j_2 < j_3 \le n} \log A_{j_1 j_2 j_3}.$$

If we perform a Schiffer boundary variation (cf. [10]) of the form

$$w^* = w + \frac{\varepsilon}{w - z} + o(\varepsilon)$$

within $\Sigma$, it induces a variation

$$z_j^* = z_j + \frac{\varepsilon}{z_j - z} + o(\varepsilon)$$

of the $z_j$'s and thus a variation $A^*_{j_1 j_2 j_3}$ of the $A_{j_1 j_2 j_3}$'s. A calculation shows that

$$\mathrm{Re}\log A^*_{j_1 j_2 j_3} = \mathrm{Re}\log A_{j_1 j_2 j_3} - \frac{1}{A_{j_1 j_2 j_3}} \mathrm{Im}\left\{\varepsilon\left(\frac{B_{j_1 j_2 j_3}}{z_{j_1} - z} - \frac{\bar{B}_{j_1 j_2 j_3}}{z_{j_2} - z}\right\} \frac{1}{z_{j_3} - z}\right) + o(\varepsilon).$$

Thus Schiffer's fundamental lemma [10] leads to the differential equation

$$\sum_{1 \le j_1 < j_2 < j_3 \le n} \frac{i}{A_{j_1 j_2 j_3}} \left(\frac{B_{j_1 j_2 j_3}}{z_{j_1} - z} - \frac{\bar{B}_{j_1 j_2 j_3}}{z_{j_2} - z}\right) \frac{dz^2}{z_{j_3} - z} > 0$$

for the $n$-extent problem. That is, an extremal continuum $\Gamma$ for the $n$-extent problem consists of analytic arcs satisfying this differential equation.

We now consider the 3-extent problem. Then the differential equation has just one term, and by permuting $z_1$, $z_2$, $z_3$ we may assume that $A_{123} > 0$. Thus the equation takes the form

$$i\frac{(B - \bar{B})z + \bar{B}z_1 - Bz_2}{(z - z_1)(z - z_2)(z - z_3)}\, dz^2 > 0$$

where $B = B_{123}$. The quadratic differential appears to have three simple poles $z_1$, $z_2$, $z_3$. None of them is removable. For if $z_1$ were removable, we would have

$$(B - \bar{B})z_1 + \bar{B}z_1 - Bz_2 = 0$$

or $B(z_1 - z_2) = 0$, which is impossible since $\operatorname{Im}\{B\} = A_{123} > 0$ and $z_1 \neq z_2$. The same reasoning shows that $z_2$ is not removable. If $z_3$ were removable, we would have

$$(B - \bar{B})z_3 + \bar{B}z_1 - Bz_2 = 0$$

and this leads to the conclusion that $z_1 = z_2$, which is impossible.

Since $\Gamma$ is a continuum, the trajectory arcs from $z_1$, $z_2$, $z_3$ must join up at some point, and this point must be a zero of the quadratic differential; thus it must be a zero of the numerator $(B - \bar{B})z + \bar{B}z_1 - Bz_2$. By a translation, we may arrange for this point to be origin. Thus $\bar{B}z_1 - Bz_2 = 0$, and since $B \neq 0$, it follows that $|z_1| = |z_2|$. By interchanging the role of $z_2$ and $z_3$, say, we find that

$$|z_1| = |z_2| = |z_3|.$$

Since $B - \bar{B} = 2iA_{123}$, the equation for the 3-extent problem finally takes the form

$$Q_3(z)\,dz^2 > 0 \quad \text{where} \quad Q_3(z) = \frac{-z}{(z - z_1)(z - z_2)(z - z_3)}.$$

This is the same differential equation as for the 3-diameter problem, but we have arrived at it through a different choice of accessory parameters.

By rotation, we may assume that $z_1 > 0$. Then, following Kuz'mina [4, p. 92], there is by Lemma 1.2 of [4] a point $z_0 \in (0, z_1)$ such that $Q(z_0) > 0$. This implies $(z_2 - z_0)(z_3 - z_0) > 0$. Thus $z_2$ and $z_3$ lie on conjugate rays issuing from the real point $z_0$ inside the circle $|z| = r$ on which $z_1$, $z_2$, $z_3$ lie. As a consequence, $z_2$ and $z_3$ are complex conjugates. Now it follows that the trajectory joining 0 to $z_1$ is a straight line segment. Similar arguments with respect to $z_2$ and $z_3$ imply that trajectories from the origin to these points are also line segments. Finally, since the origin is a simple zero, these segments emanate at equal angles, and since the points $z_1$, $z_2$, $z_3$ are simple poles, the segments terminate there. Thus we obtain the following.

**Proposition 10.** *The extremal continua for the* 3*-extent problem are symmetric three-pointed stars.*

The functions in $\Sigma$ that map onto the complement of symmetric three-pointed stars are translations and rotations of $f(\zeta) = \zeta(1 + \zeta^{-3})^{2/3}$. Its omitted set

$\Gamma = \mathbf{C} \setminus f(|\zeta| > 1)$ is the star with tips at the points $z_k = 2^{2/3}e^{2\pi i(k-1)/3}$, $1 \le k \le 3$. The triangle with these vertices has area $3^{3/2}/2^{2/3}$. Thus

$$e_3(E) \le \frac{3^{3/2}}{2^{2/3}}$$

is a sharp inequality for all continua $E$ with capacity equal to one. In fact, we are led to the same result by combining the solution $d_3(E) \le 3^{1/2}2^{2/3}$ to the corresponding 3-diameter problem [3, 9] with Proposition 4.

In contrast to the 3-extent and the 3-diameter problems, we shall now show that the extremal continua, and hence solutions, for the 4-extent and 4-diameter problems are different. Assume, to the contrary, that $\Gamma$ is a common extremal for the two problems. Then $\Gamma$ satisfies the differential equations $Q_4(z)\,dz^2 > 0$ for the 4-extent problem and $R_4(z)\,dz^2 > 0$ for the 4-diameter problem [3, 4, 9], where

$$Q_4(z) = \sum_{1 \le j_1 < j_2 < j_3 \le 4} \frac{i}{A_{j_1 j_2 j_3}} \left( \frac{B_{j_1 j_2 j_3}}{z_{j_1} - z} - \frac{\bar{B}_{j_1 j_2 j_3}}{z_{j_2} - z} \right) \frac{1}{z_{j_3} - z},$$

$$R_4(z) = \sum_{1 \le j_1 < j_2 \le 4} \frac{-1}{(z_{j_1} - z)(z_{j_2} - z)}.$$

Since $\Gamma$ satisfies both equations, it follows that the quotient $Q_4(z)/R_4(z)$ is real and positive along $\Gamma$. In particular,

$$q = \lim_{z \to z_1} (z - z_1)Q_4(z) = \sum_{2 \le j < k \le 4} \frac{iB_{1jk}}{A_{1jk}(z_k - z_1)} = \sum_{2 \le j < k \le 4} \frac{i(\bar{z}_k - \bar{z}_j)}{A_{1jk}}$$

and

$$r = \lim_{z \to z_1} (z - z_1)R_4(z) = \sum_{2 \le j \le 4} \frac{-1}{z_j - z_1}$$

have the property that $q/r$ is real.

The extremal continua for the 4-diameter problem are known [4, Theorem 2.3] and, for example, their endpoints form the vertices of a rectangle. After a translation and rotation, we may assume that

$$z_1 = x + iy, \quad z_2 = x - iy, \quad z_3 = -x - iy, \quad z_4 = -x + iy$$

where $x > 0$ and $y > 0$. Then

$$A_{123} = A_{124} = A_{134} = 4xy$$

and so

$$q = i\left( \frac{-2x}{4xy} + \frac{-2x - 2iy}{4xy} + \frac{-2iy}{4xy} \right) = \frac{x + iy}{ixy},$$

$$r = -\left(\frac{1}{-2iy} + \frac{1}{-2x - 2iy} + \frac{1}{-2x}\right) = \tfrac{1}{2}\left(\frac{x+iy}{ixy} + \frac{1}{x+iy}\right).$$

Now

$$\frac{r}{q} = \tfrac{1}{2}\left(1 + \frac{ixy}{(x+iy)^2}\right)$$

must be real, and this is the case only if $(x+iy)^2$ is purely imaginary. In other words, it must be that $x = y$. But in Kuz'mina's solution [4, Theorem 2.3] to the 4-diameter problem the endpoints of the extremal continuum do not form the vertices of a square, and so we are finished. This yields the following.

**Proposition 11.** *No continuum can simultaneously maximize the* 4*-extent and the* 4*-diameter among continua of capacity one.*

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