Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 13, 1988, 355–361

Commentationes in honorem
Lars V. Ahlfors
LXXX annos nato

TEICHMÜLLER SPACES

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1. Introduction

In his ground-breaking papers [2] and [3] and the joint paper [5] with Lipman Bers, Lars Ahlfors set the theory of Teichmüller spaces on a firm foundation and opened the way to further developments that have continued to the present day. A trio of recent books ([15], [16], and [18]) have done much to make Teichmüller space theory accessible to the mathematical public.

This article, like the talk on which it is based, will concentrate on two recent papers, both of which rely crucially on fundamental ideas of Lars Ahlfors. We shall discuss them in Sections 3 and 5. Sections 2 and 4 are devoted to now-classical background material that can be found in the above-mentioned books.

2. The theorems of Teichmüller and Royden

Let the Riemann surface X be obtained from a compact Riemann surface of genus g by removing n (≥ 0) points. If $2g+n-2$ is positive, the Teichmüller space $T(g,n)$ is the set of equivalence classes of quasiconformal (qc) mappings of X onto Riemann surfaces Y. Here two such mappings $f_1\colon X\to Y_1$ and $f_2\colon X\to Y_2$ are equivalent if there is a conformal map $h\colon Y_1\to Y_2$ such that the qc map $f_2^{-1}\circ h\circ f_1$ of X onto itself is homotopic to the identity.

The Teichmüller metric on $T(g,n)$ is defined by setting

$$d([f_1],[f_2]) = \tfrac{1}{2}\log K \tag{1}$$

if K is the smallest number (≥ 1) such that there is a K-qc mapping $f\colon f_1(X)\to f_2(X)$ homotopic to the map $f_2\circ f_1^{-1}$. Teichmüller's theorem asserts that $T(g,n)$, with the Teichmüller metric, is a complete metric space homeomorphic to Euclidean space of dimension $2d=6g-6+2n$. In particular $T(g,n)$ is a manifold of (real) dimension $2d$. This manifold has a natural complex structure, as shown in 1957 by Ahlfors [3] when $n=0$ and by Bers [6] shortly thereafter for arbitrary n.

A deep and unexpected connection between the complex structure and Teichmüller metric on $T(g,n)$ was discovered in 1969 by Royden [19], who proved that

[1] This research was supported in part by a grant from the National Science Foundation.

doi:10.5186/aasfm.1988.1326

the Teichmüller metric is Kobayashi's hyperbolic metric on the complex manifold $T(g,n)$. In concrete terms Royden's theorem states that the Teichmüller distance $d([f_1],[f_2])$ is the minimum of the Poincaré distances $\varrho(z_1,z_2)$ over all holomorphic maps φ of the open unit disk into $T(g,n)$ with $\varphi(z_1)=[f_1]$ and $\varphi(z_2)=[f_2]$. Here the Poincaré distance

$$\varrho(z,w)=\tanh^{-1}\left|\frac{z-w}{1-\bar{z}w}\right|$$

is normalized to have curvature -4. The proof of Royden's theorem depends crucially on Ahlfors's generalization [1] of Schwarz's lemma. The most general form of Royden's theorem (for arbitrary Teichmüller spaces) can be found in Gardiner's book [15].

3. The λ-lemma

In the course of their study [17] of the dynamics of rational functions, Mañé, Sad, and Sullivan established an astonishing and useful result, which they called the λ-lemma. Let E be any subset of the complex plane $\mathbf{C}$, and let D be any connected open neighborhood of the origin in $\mathbf{C}$. By definition a *holomorphic motion* of E over D is a function $\varphi\colon D\times E\to\mathbf{C}$ with the three properties

(a) for any fixed λ in D, $\varphi(\lambda,\cdot)\colon E\to\mathbf{C}$ is injective,

(b) for any fixed z in E, $\varphi(\cdot,z)\colon D\to\mathbf{C}$ is holomorphic, and

(c) $\varphi(0,z)=z$ for all z in E.

(Note that continuity of $\varphi(\lambda,\cdot)$ is not assumed.) The amazing conclusion of the λ-lemma is that for every λ in D there is a qc mapping $w(z)$ of $\mathbf{C}$ onto itself such that $\varphi(\lambda,z)=w(z)$ for every z in E.

The recent paper [9] of Bers and Royden gives a sharp form of the λ-lemma. They show that if D is the open unit disk Δ the qc mapping $w(z)$ above can be taken to be K-qc, with

$$K=\frac{1+|\lambda|}{1-|\lambda|}. \tag{2}$$

The example $\varphi(\lambda,z)=z+\lambda\bar{z}$, $\varphi\colon\Delta\times\mathbf{C}\to\mathbf{C}$, shows that the bound (2) is best possible.

Because of equicontinuity properties of K-qc mappings it suffices to prove the Bers–Royden result when E is a finite set. The proof in that case, which we shall now sketch, is a simple and ingenious application of Royden's theorem about the Teichmüller metric. Let $E=\{0,1,z_1,\dots,z_n\}$ and let

$$V_n=\{\zeta\in(\mathbf{C}\setminus\{0,1\})^n\ ;\ \zeta_i\neq\zeta_j\quad\text{if}\quad i\neq j\}.$$

Any point ζ in V_n determines a Riemann surface $\mathbf{C}\setminus\{0,1,\zeta_1,\ldots,\zeta_n\}$, and there is an obvious map from the Teichmüller space $T(0,n+3)$ onto V_n. That map is a holomorphic universal covering.

A holomorphic motion $\varphi\colon \Delta\times E\to\mathbf{C}$ induces the holomorphic map $\Phi\colon \Delta\to V_n$ whose component functions are

$$\Phi_j(\lambda)=\frac{\varphi(\lambda,z_j)-\varphi(\lambda,0)}{\varphi(\lambda,1)-\varphi(\lambda,0)},\qquad 1\le j\le n.$$

Since Δ is simply connected, Φ lifts to a holomorphic map $\Psi\colon \Delta\to T(0,n+3)$. Fix any λ in Δ. By Royden's theorem the Teichmüller distance from $\Psi(0)$ to $\Psi(\lambda)$ is no greater than $\tanh^{-1}|\lambda|$. It follows at once from the definition (1) that there is a K-qc mapping w of $\mathbf{C}$ onto itself such that $w(z)=\varphi(\lambda,z)$ for all z in E and K is given by (2). That is the desired conclusion.

Using properties of $T(0,n+3)$, Bers and Royden go on to show that every holomorphic motion of E over Δ extends to a holomorphic motion of $\mathbf{C}$ over the open disk $\{\lambda;\,|\lambda|<1/3\}$, first for finite sets E, then for arbitrary sets. A similar result, with an unknown positive constant in place of $1/3$, had been obtained earlier by Sullivan and Thurston [20], using quite different methods. It is unknown whether the number $1/3$ can be replaced by a larger number, such as one.

The technique of interpreting a holomorphic motion of E over Δ as a holomorphic map from Δ to the Teichmüller space of $\mathbf{C}\setminus E$, as in the Bers–Royden argument above, is natural and powerful. A Cornell University graduate student, G. Lieb, is exploring the appropriate generalization of this technique from finite sets to arbitrary closed sets E in $\mathbf{C}$.

4. Quasifuchsian groups and the Bers map

The remainder of this article concerns the complex structure of infinite dimensional Teichmüller spaces. In this section we review the basic definitions. We begin as in Section 2 with a Riemann surface X obtained from a compact Riemann surface of genus g by deleting $n\ge 0$ points. Again we require $2g-2+n>0$. Choose a bounded quasidisk D (the image of Δ under a qc mapping of $\mathbf{C}$ onto itself) and a holomorphic universal covering map $\pi\colon D\to X$ such that the group Γ of cover transformations consists entirely of Möbius transformations.

Suppose f is a qc mapping of X onto a Riemann surface Y. By the Ahlfors–Bers theorem [5] there is a commutative diagram

(3)
$$\begin{array}{ccc} D & \xrightarrow{w} & D \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & Y \end{array}$$

such that both π and π' are holomorphic universal covering maps. The qc mapping w is uniquely determined by its complex dilatation

$$\mu(z) = w_{\bar{z}}(z)/w_z(z) \tag{4}$$

if we require it to fix three given boundary points of D; we write $w = w_\mu$. The function μ defined by (4) belongs to the open unit ball in $L^\infty(D, \mathbf{C})$. In addition μ is a *Beltrami differential* for Γ; it satisfies

$$(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \qquad \text{for all } \gamma \text{ in } \Gamma. \tag{5}$$

The set of all μ in $L^\infty(D, \mathbf{C})$ that satisfy (5) is a closed complex subspace; its open unit ball $M(D, \Gamma)$ is therefore a complex Banach manifold.

The diagram (3), with $w = w_\mu$, determines a map $\mu \mapsto [f]$ from $M(D, \Gamma)$ to the Teichmüller space $T(g, n)$. It is easy to see that μ and ν determine the same point of $T(g, n)$ if and only if $w_\mu = w_\nu$ on the boundary of D. When that happens we say that μ and ν are equivalent and write $\mu \sim \nu$.

A discrete group of Möbius transformations that leaves a bounded quasidisk D invariant is called a *quasifuchsian group.* For any such group Γ we can define $M(D, \Gamma)$ and the equivalence relation $\mu \sim \nu$ as above. The Teichmüller space $T(D, \Gamma)$ is defined as the set of equivalence classes in $M(D, \Gamma)$.

Lipman Bers [7] introduced an ingenious mapping of $T(D, \Gamma)$ onto a bounded region in a complex Banach space, defined in the following way. For each μ in $M(D, \Gamma)$ let w^μ be the unique qc mapping of the plane onto itself that solves the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{6}$$

in D and is conformal in the exterior D^* of D, with the behavior

$$w^\mu(z) = z + O(z^{-1})$$

at infinity. Let $S(w^\mu)$ be the Schwarzian derivative

$$S(w^\mu) = \frac{(w^\mu)'''}{(w^\mu)'} - \frac{3}{2}\left[\frac{(w^\mu)''}{(w^\mu)'}\right]^2$$

of the conformal map w^μ in D^*. The Banach space $B(D^*, \Gamma)$ of bounded quadratic differentials consists of the holomorphic functions φ on D^* such that

$$\|\varphi\| = \sup\left\{|\varphi(z)|\lambda(z)^{-2} \ ; \ z \in D^*\right\} < \infty,$$

and

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi \qquad \text{for all } \gamma \text{ in } \Gamma.$$

Here $ds = \lambda(z)\,|dz|$ is the Poincaré metric on $D^* \cup \{\infty\}$. Bers observed that $\Phi(\mu) = S(w^\mu)$ belongs to $B(D^*,\Gamma)$ if $\mu \in M(D,\Gamma)$, that Φ: $M(D,\Gamma) \to B(D^*,\Gamma)$ is a holomorphic map, and that $\Phi(\mu) = \Phi(\nu)$ if and only if $\mu \sim \nu$. In addition he proved that the image $\Phi(M(D,\Gamma))$, which we may identify with $T(D,\Gamma)$, is a bounded open set and that every point in the image lies in the domain of a holomorphic right inverse of Φ. The Bers map Φ thus establishes an isomorphism of complex manifolds between $T(D,\Gamma)$, viewed abstractly as a quotient space of $M(D,\Gamma)$, and the bounded open set $\Phi(M(D,\Gamma))$ in $B(D^*,\Gamma)$.

5. Inverting the Bers map

The crucial step in finding holomorphic right inverses to the Bers map is to find one in a neighborhood of zero in $B(D^*,\Gamma)$. Bers did this implicitly in [8], by inverting the derivative of Φ at zero and using the inverse function theorem. In some ways a direct construction is more satisfying, and Ahlfors, in his fundamental paper [4], gave one that works when $\Gamma = \{\mathrm{id}\}$. The Ahlfors construction proceeds as follows. Let $z \mapsto z^*$ be a qc reflection in ∂D. (That is, $z \mapsto z^*$ is a sense-reversing qc involution of the sphere, fixing every boundary point of D.) Given φ in $B(D^*,\Gamma)$, let u_1 and u_2 be the solutions in D^* of the differential equation

$$2u'' + \varphi u = 0$$

with the behavior

$$u_1(z) = z + O(z^{-1}) \qquad \text{and} \qquad u_2(z) = 1 + O(z^{-2})$$

at infinity. Define $w(z)$ on $D \cup D^*$ by

$$w(z) = u_1(z)/u_2(z) \qquad \text{for all } z \text{ in } D^*, \tag{7a}$$

$$w(z) = \frac{u_1(z^*) + (z - z^*)u_1'(z^*)}{u_2(z^*) + (z - z^*)u_2'(z^*)} \qquad \text{for all } z \text{ in } D. \tag{7b}$$

Ahlfors proved that if the reflection $z \mapsto z^*$ is suitably chosen and $\|\varphi\|$ is sufficiently small, then $w(z)$ has a global qc extension to **C** and equals w^μ, where $\mu = w_{\bar z}/w_z$ in D. It is easy to calculate μ from (7b) and to see that μ is a holomorphic function of φ. In addition, (7a) implies that

$$\Phi(\mu) = S(u_1/u_2) = \varphi.$$

The only difficulty with Ahlfors's contruction is that if Γ is nontrivial, μ will not satisfy the invariance condition (5) unless the qc reflection is very carefully chosen.

The author and S. Nag overcame that difficulty in the recent paper [14], whose contents we shall briefly describe.

Let $f\colon \Delta \to D$ and $g\colon \Delta \to D^* \cup \{\infty\}$ be conformal maps of Δ onto the complementary regions D and $D^* \cup \{\infty\}$. The mapping

$$h(z) = f^{-1}\big(g(z)^*\big) \qquad \text{for all } z \text{ in } \Delta$$

is a sense-reversing qc mapping of Δ onto itself with boundary values $f^{-1} \circ g$ on the unit circle S^1. As Ahlfors observed in [4], the problem of finding a suitable qc reflection $z \mapsto z^*$ in ∂D is equivalent to the problem of finding an appropriate sense-reversing qc mapping $h\colon \Delta \to \Delta$ with these boundary values $f^{-1} \circ g$. In [4], Ahlfors relied on the Beurling–Ahlfors method [10] to extend $f^{-1} \circ g$ from S^1 to Δ. In a recent paper [11], Douady and the author introduced a new method of extending homeomorphisms of S^1 to homeomorphisms of Δ. That method assigns to each homeomorphism $\varphi\colon S^1 \to S^1$ a barycentric extension $\operatorname{ex}(\varphi)\colon \Delta \to \Delta$ in such a way that if A and B are conformal maps of Δ onto itself, then

$$\operatorname{ex}(A \circ \varphi \circ B) = A \circ \operatorname{ex}(\varphi) \circ B. \tag{8}$$

Using the invariance property (8), the author and Nag show in [14] that the barycentric extension $h = \operatorname{ex}(f^{-1} \circ g)$ produces a qc reflection such that the Ahlfors construction yields a holomorphic local right inverse of $\Phi\colon M(D,\Gamma) \to T(D,\Gamma)$ for any quasifuchsian group Γ. For some other applications of the barycentric extension to the theory of quasiconformal mappings and Teichmüller spaces we refer the reader to the papers [11], [12], and [13].

References

[1] Ahlfors, L.V.: An extension of Schwarz's lemma. - Trans. Amer. Math. Soc. 43, 1938, 359–364.

[2] Ahlfors, L.V.: On quasiconformal mappings. - J. Analyse Math. 3, 1954, 1–58, 207–208.

[3] Ahlfors, L.V.: The complex analytic structure of the space of closed Riemann surfaces. - Analytic functions, Princeton University Press, Princeton, N.J., 1960, 45–66.

[4] Ahlfors, L.V.: Quasiconformal reflections. - Acta Math. 109, 1963, 291–301.

[5] Ahlfors, L.V., and L. Bers: Riemann's mapping theorem for variable metrics. - Ann. of Math. 72, 1960, 385–404.

[6] Bers, L.: Spaces of Riemann surfaces. - Proceedings of the International Congress of Mathematicians (Edinburgh, 1958). Cambridge University Press, Cambridge, 1960, 349–361.

[7] Bers, L.: Automorphic forms and general Teichmüller spaces. - Proceedings of the Conference on Complex Analysis (Minneapolis, 1964). Springer-Verlag, Berlin, 1965, 109–113.

[8] Bers, L.: A non-standard integral equation with applications to quasiconformal mappings. - Acta Math. 96, 1966, 113–134.

[9] Bers, L., and H.L. Royden: Holomorphic families of injections. - Acta Math. 157, 1986, 259–286.

[10] BEURLING, A., and L.V. AHLFORS: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125–142.

[11] DOUADY, A., and C.J. EARLE: Conformally natural extension of homeomorphisms of the circle. - Acta Math. 157, 1986, 23–48.

[12] EARLE, C.J.: Some explicit real analytic trivializations of the Teichmüller curves. - Proc. Amer. Math. Soc. (to appear).

[13] EARLE, C.J., and C. MCMULLEN: Quasiconformal isotopies. - Holomorphic functions and moduli I. Mathematical Sciences Research Institute Publications 10. Springer-Verlag, New York–Berlin–Heidelberg, 1988, 143–154.

[14] EARLE, C.J., and S. NAG: Conformally natural reflections in Jordan curves with applications to Teichmüller spaces. - Holomorphic functions and moduli II. Mathematical Sciences Research Institute Publications 11. Springer-Verlag, New York–Berlin–Heidelberg, 1988, 179–194.

[15] GARDINER, F.P.: Teichmüller theory and quadratic differentials. - Wiley-Interscience, John Wiley and Sons, New York, 1987.

[16] LEHTO, O.: Univalent functions and Teichmüller spaces. - Graduate Texts in Mathematics 109. Springer-Verlag, New York–Berlin–Heidelberg, 1987.

[17] MAÑÉ, R., P. SAD, and D. SULLIVAN: On the dynamics of rational maps. - Ann. Sci. École Norm. Sup. 16, 1983, 193–217.

[18] NAG, S.: The complex analytic theory of Teichmüller spaces. - Wiley-Interscience, John Wiley and Sons, New York, 1988.

[19] ROYDEN, H.L.: Automorphisms and isometries of Teichmüller space. - Advances in the theory of Riemann surfaces. Ann. Math. Studies 66. Princeton University Press, Princeton, N. J. 1971, 369–383.

[20] SULLIVAN, D.P., and W.P. THURSTON: Extending holomorphic motions. - Acta Math. 157, 1986, 243–257.

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Received 15 January 1988