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BILIPSCHITZ AND QUASISYMMETRIC EXTENSION PROPERTIES

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1. Introduction

Let X and Y be metric spaces with distance denoted by $|a-b|$. A map $f\colon X\to Y$ is called *L-bilipschitz,* $L\geqq 1$, if

$$|x-y|/L \leqq |f(x)-f(y)| \leqq L|x-y|$$

for all $x, y\in X$. We say that a set $A\subset X$ has the *bilipschitz extension property* (abbreviated BLEP) in (X, Y) if there is $L_0>1$ such that if $1\leqq L\leqq L_0$, then every L-bilipschitz $f\colon A\to Y$ has an L_1-bilipschitz extension $g\colon X\to Y$, where $L_1=L_1(L, A, X, Y)\to 1$ as $L\to 1$.

Similarly, A has the *quasisymmetric extension property* (abbreviated QSEP) in (X, Y) if there is $s_0>0$ such that if $0\leqq s\leqq s_0$, then every s-quasisymmetric $f\colon A\to Y$ has an s_1-quasisymmetric extension $g\colon X\to Y$, where $s_1=s_1(s, A, X, Y)\to 0$ as $s\to 0$. The definition of quasisymmetric maps will be recalled in 2.2.

We also say that A has one of these properties in X if A has this property in (X, X). If A has both the BLEP and the QSEP in (X, Y) or in X, we say that A has the *extension properties* in (X, Y) or in X, respectively.

In this paper we consider the case where X is the euclidean n-space R^n and Y is an inner product space. Without loss of generality, we may assume that Y is a linear subspace of the Hilbert space l_2. The main results are Theorems 5.5 and 6.2. These give sufficient conditions for a set $A\subset R^n$ to have the extension properties, the first one in R^n, the second one in (R^n, Y). Both conditions are somewhat implicit, but we show that the first one applies to all compact DIFF and PL $(n-1)$-manifolds, the second one to all compact convex sets and to all quasisymmetric n-cells.

In a joint paper [TV_4] with Pekka Tukia, we proved that R^p and S^p have the extension properties in R^n for $p\leqq n-1$. In Section 4 we extend these results to the relative case (R^n, Y).

The basic idea of the extension proofs of the present paper is the same as in [TV_4]: We choose a suitable triangulation of $R^n\setminus A$, define the extension g at the vertices, and extend affinely to the simplexes. Thus g will be PL outside A. However,

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to define g at the vertices, we must replace the rather explicit constructions of [TV$_4$] by an auxiliary approximation theorem, which will be given in Section 3.

In Section 7 we give several examples of sets $A\subset R^n$ which do not have the extension properties in R^n or in (R^n, Y). It is not easy to find an example which has only one of these properties. In fact, I conjecture that if A has the QSEP in R^n, it has also the BLEP, and that for $n\neq 4$ the proof can be based on the ideas of [TV$_5$] together with careful estimates on the bilipschitz constants. In 7.5 we give an example of a set $A\subset R^2$ which has the BLEP but not the QSEP in R^2. However, I do not know of any such example where A is connected.

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2. Preliminaries

In this section we give the basic notation and terminology used in this paper, some properties of quasisymmetric maps, and elementary results on affine and PL maps.

2.1. *Notation.* We let l_2 denote the Hilbert space of all square summable sequences of real numbers. Let $(e_1, e_2, \ldots)$ be its natural basis. We identify the euclidean n-space R^n with the linear subspace of l_2 spanned by $e_1, \ldots, e_n$. Then $R^p\subset R^n$ for $p\leqq n$. Open balls in R^n are written as $B^n(x, r)$ and spheres as $S^{n-1}(x, r)$; the superscript may be dropped. We also set

$$B^n(r) = B^n(0, r),\quad B^n = B^n(1),\quad S^{n-1}(r) = S^{n-1}(0, r),\quad S^{n-1} = S^{n-1}(1),$$

$$R^n_+ = \{x\in R^n\colon x_n \geqq 0\},\quad B^n_+ = B^n\cap R^n_+.$$

If $A\subset l_2$, we let $T(A)$ denote the affine subspace spanned by A. In each metric space, $|a-b|$ denotes the distance between a and b. If f and g are maps into l_2, defined on a set X, we set

$$\|f-g\|_X = \sup\{|f(x)-g(x)|\colon x\in X\}.$$

If f is a bounded linear map between normed spaces, we let $|f|$ denote its sup-norm.

2.2. *Quasisymmetric maps.* These maps were introduced in [TV$_1$]. We recall the definition. Let X and Y be metric spaces. An embedding $f\colon X\to Y$ is *quasisymmetric* (abbreviated QS) if there is a homeomorphism $\eta\colon R^1_+\to R^1_+$ such that if $a, b, x\in X$ with $|a-x|\leqq t|b-x|$, then $|f(a)-f(x)|\leqq\eta(t)|f(b)-f(x)|$. We also say that f is $\eta-$QS. If $s>0$, we say that f is $s-$QS if f is QS and satisfies the following condition: If $t\leqq 1/s$ and if $a, b, x\in X$ with $|a-x|\leqq t|b-x|$, then $|f(a)-f(x)|\leqq (t+s)|f(b)-f(x)|$.

In [TV_4] we used a slightly different definition of s-quasisymmetry. We said that f is $s-$QS if it is $\eta-$QS for some η in

$$N(\mathrm{id}, s) = \{\eta\colon\ |\eta(t)-t| \leqq s \quad \text{for} \quad 0 \leqq t \leqq 1/s\}.$$

Clearly this condition implies that f is $s-$QS in the sense given above. Conversely, if f is $s-$QS, then for every $s'>s$ there is $\eta\in N(\mathrm{id}, s')$ such that f is $\eta-$QS.

We say that f is a *similarity* or $0-$QS if there is $L>0$ such that $|f(x)-f(y)|=L|x-y|$ for all $x, y\in X$. In other words, f is $\eta-$QS with $\eta=\mathrm{id}$. Every L-bilipschitz map is $s-$QS with $s=(L^2-1)^{1/2}$. If $|f(x)-f(y)|=|x-y|$ for all $x, y\in X$, f is an *isometry*. An isometry need not be surjective.

If G is open in R^n, $n\geqq 2$, an $\eta-$QS map $f\colon G\to R^n$ is K-quasiconformal (abbreviated $K-$QC) with $K=\eta(1)^{n-1}$. The converse is not in general true but a $K-$QC map $f\colon R^n\to R^n$ is $s-$QS where $s=s(K, n)\to 0$ as $K\to 1$, see [TV_4, 2.6].

It is often a laborious task to prove that a given embedding $f\colon X\to Y$ is QS, since one must consider all triples $a, b, x\in X$. However, it is often possible to exclude triples where the ratio $t=|a-x|/|b-x|$ is small or large. See, for example [TV_1, 2.16, 3.10]. For connected spaces, we prove the following useful result:

2.3. Lemma. *Let X and Y be metric spaces with X connected. Suppose that $0<s\leqq 1/4$ and that $f\colon X\to Y$ is a nonconstant continuous map such that*

$$|f(a)-f(x)| \leqq (t+s)|f(b)-f(x)| \tag{2.4}$$

whenever $|a-x|=t|b-x|$ and $1/2\leqq t\leqq 2$. Then f is $\eta-$QS with a universal η, and also s_1-QS, where $s_1=s_1(s)\to 0$ as $s\to 0$.

Proof. Let a, b, x be distinct points in X with $|b-x|=r$, $|a-x|=tr$. Suppose first that $0<t<1/2$. We show that (2.4) is also valid in this case. Choose an integer $m\geqq 2$ such that $2^{-m}\leqq t<2^{-m+1}$, and set $t_0=t^{1/m}$. Then $1/2\leqq t_0<2^{-1/2}$. Since X is connected, we can choose points $b=x_0, x_1, \dots, x_m=a$ such that $|x_j-x|=t_0^j r$. Since $|x_{j+1}-x|=t_0|x_j-x|$, we have

$$|f(x_{j+1})-f(x)| \leqq (t_0+s)|f(x_j)-f(x)|,$$

and hence

$$|f(a)-f(x)| \leqq (t_0+s)^m|f(b)-f(x)|.$$

Here

$$(t_0+s)^m-t \leqq (2^{-1/2}+s)^m-2^{-1/2} \leqq s,$$

since $2^{-1/2}+s\leqq 2^{-1/2}+1/4<1$. Hence (2.4) is true.

From [TV_1, 2.20] it follows that f is an embedding. It is easy to verify that f satisfies the conditions of [TV_1, 3.10] with $\lambda_1=\lambda_2=3/4$, $h=4/3$, and $H=2$. Hence f is $\eta-$QS with a universal η. Indeed, by [TV_1, 3.11] we can choose $\eta(t)=4\max(t^{5/2}, t^{2/5})$.

To show that f is $s_1(s)-$QS with $s_1(s)\to 0$, let $\varepsilon>0$. Suppose that $t\leqq 1/\varepsilon$. It suffices to show that there is $\delta=\delta(\varepsilon)>0$ such that if $s\leqq\delta$, then

$$t' = \frac{|f(a)-f(x)|}{|f(b)-f(x)|} \leqq t+\varepsilon.$$

If $t\leqq 2$, this is true for $s\leqq\varepsilon$. Suppose that $2<t\leqq 1/\varepsilon$. Choose an integer $n\geqq 2$ such that $2^{n-1}<t\leqq 2^n$. Setting $t_1=t^{1/n}$ we have $1<t_1\leqq 2$. Since X is connected, we can choose points $b=y_0, \ldots, y_n=a$ such that $|y_j-x|=t_1^j r$. Then

$$|f(y_{j+1})-f(x)| \leqq (t_1+s)|f(y_j)-f(x)|,$$

which implies

$$|f(a)-f(x)| \leqq (t_1+s)^n|f(b)-f(x)|.$$

Thus $t'\leqq t+s'$ with

$$s' = (t_1+s)^n-t_1^n \leqq (2+s)^n-2^n = s_1(s, n).$$

Since $2^{n-1}<t\leqq 1/\varepsilon$, n has an upper bound of the form $n\leqq n_1(\varepsilon)$. Hence $s'\leqq s_1(s, n_1(\varepsilon))\to 0$ as $s\to 0$, and thus $s'\leqq\varepsilon$ for small s. □

2.5. Remark. It follows from the proof of 2.3 that for a connected X, f is $s-$QS if it satisfies (2.4) for $t\in[1/2, 1/s]$ and if $s\leqq 1/4$. This is an improvement of [TV$_4$, 2.4].

2.6. *Simplexes and affine maps.* Let $k\geqq 1$ and let $\Delta=a_0\ldots a_k$ be a k-simplex in l_2 with vertices $a_0, \ldots, a_k$. We let b_j denote the distance of a_j from the $(k-1)$-plane spanned by the opposite face, and we set

$$b(\Delta) = \min(b_0, \ldots, b_k).$$

The diameter $d(\Delta)$ of Δ is the largest edge $|a_i-a_j|$. The number

$$\varrho(\Delta) = d(\Delta)/b(\Delta) \geqq 1$$

is called the *flatness* of Δ. We let Δ^0 denote the set of vertices of Δ.

Let $T\subset l_2$ be a finite-dimensional plane (affine subspace), and let $f\colon T\to l_2$ be affine. We let $L_f=L(f)$ and $l_f=l(f)$ denote the smallest and the largest number, respectively, such that

$$l_f|x-y| \leqq |f(x)-f(y)| \leqq L_f|x-y|$$

for all $x, y\in T$. Thus f is a similarity if and only if $l_f=L_f>0$, and an isometry if and only if $l_f=L_f=1$. Moreover, f is injective if and only if $l_f>0$. In this case, the number $H_f=L_f/l_f$ is the *metric dilatation* of f.

Recall that an origin-preserving isometry of an inner product space into an inner product space is linear and preserves the inner product. Such a map is called an *orthogonal* map. A sense-preserving orthogonal map $R^n\to R^n$ is called a *rotation.*

2.7. Lemma. *Suppose that $\Delta\subset l_2$ is an n-simplex, that $f\colon \Delta\to l_2$ is affine and that $h\colon \Delta\to l_2$ is a similarity such that*

$$|h(v)-f(v)| \leqq \alpha L_h b(\Delta)/(n+1)$$

for every vertex v of Δ, *where* $0 \leqq \alpha \leqq 1/2$. *Then*

$$L_f \leqq L_h(1+2\alpha),\quad l_f \geqq L_h/(1+2\alpha),\quad H_f \leqq (1+2\alpha)^2.$$

If $\Delta \subset R^n$ *and* $f, h\colon \Delta \to R^n$, *then h is sense-preserving if and only if f is sense-preserving.*

Proof. Extend h to a bijective similarity $h_1\colon l_2 \to l_2$. Replacing f by $h_1^{-1}f$ we may assume that $h=\mathrm{id}$. The proof of [TV$_4$, 3.3] is then valid also in the present situation. □

2.8. Lemma. *Let* $\Delta = a_0 \dots a_k$ *be a k-simplex in* R^n *with* $a_0=0$. *Suppose that* $g\colon T(\Delta) \to R^n$ *is an orthogonal map such that* $ga_j = a_j$ *for* $0 \leqq j \leqq k-1$ *and* $|ga_k - a_k| \leqq \delta$. *Then there is an orthogonal map* $u\colon R^n \to R^n$ *such that* $ug|\Delta = \mathrm{id}$ *and* $|u-\mathrm{id}| \leqq \delta/b(\Delta)$. *If* $k<n$, *u can be chosen to be a rotation.*

Proof. If $k=n$, either $g=\mathrm{id}$ or g is the reflection in $T(a_0, \dots, a_{k-1})$. In the first case we choose $u=\mathrm{id}$. In the second case we have $\delta \geqq |ga_k - a_k| = 2b_k \geqq 2b(\Delta)$. Since $|g-\mathrm{id}| = 2 \leqq \delta/b(\Delta)$, we can choose $u=g$.

Suppose that $k<n$. Let E be the linear subspace of R^n spanned by $a_1, \dots, a_{k-1}$. Let $q_1\colon R^n \to E$ and $q_2\colon R^n \to E^\perp$ be the orthogonal projections. Let T be a two-dimensional linear subspace of $E^\perp$ containing the vectors $x_1 = q_2 g a_k$ and $x_2 = q_2 a_k$. Since $ga_k = gq_1 a_k + gq_2 a_k$ and since $g|E = \mathrm{id}$, we have $gx_2 = x_1$, and thus $|x_1| = |x_2|$. Consequently, there is a rotation u of T with $ux_1 = x_2$. Extend u to a rotation $u\colon R^n \to R^n$ with $u|T^\perp = \mathrm{id}$. Since $|x_1 - x_2| \leqq \delta$, $|u-\mathrm{id}| \leqq \delta/|x_2|$. Here $|x_2| = d(a_k, E) \geqq b(\Delta)$, and the lemma follows. □

2.9. Lemma. *Let* $\Delta = a_0 \dots a_p$ *be a p-simplex in* R^n *with* $a_0 = 0$. *Suppose that* $h\colon T(\Delta) \to R^n$ *is an orthogonal map such that* $|ha_j - a_j| \leqq \delta$ *for all j. Then there is an orthogonal map* $u\colon R^n \to R^n$ *such that*

$$uh|\Delta = \mathrm{id},\ |h-\mathrm{id}| \leqq |u-\mathrm{id}| \leqq b(\Delta)^{-1} p(1+\varrho(\Delta))^{p-1}\delta.$$

If $p<n$, *u can be chosen to be a rotation.*

Proof. We define inductively orthogonal maps $u_k\colon R^n \to R^n$, $0 \leqq k \leqq p$, as follows: Let $u_0 = \mathrm{id}$. Assume that we have constructed $u_0, \dots, u_{k-1}$ such that setting $g_j = u_{j-1} \dots u_0$, we have $g_j h a_i = a_i$ for $i<j \leqq k$. Apply 2.8 with the substitution

$$k \mapsto k,\ \Delta \mapsto \Delta_k = a_0 \dots a_k,\ g \mapsto g_k h|T(\Delta_k),\ \delta \mapsto \delta_k = \max\{|g_k h a_i - a_i|\colon 1 \leqq i \leqq p\}.$$

We obtain an orthogonal map $u_k\colon R^n \to R^n$ such that $u_k g_k h|\Delta_k = \mathrm{id}$ and $|u_k - \mathrm{id}| \leqq \delta_k/b(\Delta_k)$. Thus

$$g_{k+1}h|\Delta_k = \mathrm{id},\ |u_k - \mathrm{id}| \leqq b(\Delta)^{-1}\delta_k.$$

We show by induction that

$$\delta_k \leqq (1+\varrho)^{k-1}\delta, \tag{2.10}$$

where $\varrho = \varrho(\Delta)$. This is clearly true for $k=1$. Suppose that (2.10) holds for $k<s$.

Since $|a_i| \leqq d(\Delta) \leqq \varrho b(\Delta)$, we obtain

$$\begin{aligned}|g_s h a_i - a_i| &\leqq |u_{s-1} g_{s-1} h a_i - g_{s-1} h a_i| + |g_{s-1} h a_i - a_i| \\ &\leqq b(\Delta)^{-1} \delta_{s-1} |a_i| + \delta_{s-1} \\ &\leqq (\varrho+1)\delta_{s-1} \leqq (\varrho+1)^{s-1}\delta,\end{aligned}$$

which gives (2.10) for $k=s$.

Since

$$|g_{k+1} - \mathrm{id}| \leqq |u_k g_k - g_k| + |g_k - \mathrm{id}| \leqq |u_k - \mathrm{id}| + |g_k - \mathrm{id}|$$

and

$$|u_k - \mathrm{id}| \leqq b(\Delta)^{-1}\delta_k \leqq b(\Delta)^{-1}(1+\varrho)^{p-1}\delta,$$

we obtain

$$|g_{p+1} - \mathrm{id}| \leqq |u_1 - \mathrm{id}| + \ldots + |u_p - \mathrm{id}| \leqq b(\Delta)^{-1} p (1+\varrho)^{p-1}\delta.$$

Since $h = g_{p+1}^{-1} | T(\Delta)$, the lemma is true with $u = g_{p+1}$. □

2.11. Lemma. *Let $\Delta \subset l_2$ be a p-simplex. Suppose that $h, k\colon T(\Delta) \to l_2$ are similarities such that $|h(z) - k(z)| \leqq \delta$ for all $z \in \Delta^0$. Then*

$$|L_h - L_k| \leqq 2\delta / d(\Delta),$$

$$|h(x) - k(x)| \leqq \delta\big(1 + d(\Delta)^{-1} M |x - v|\big)$$

for all $x \in T(\Delta)$ and $v \in \Delta^0$, where

$$M = 4 + 6\varrho(\Delta) p \big(1 + \varrho(\Delta)\big)^{p-1}.$$

Proof. Observing that

$$L_h\, d(\Delta) = d(h\Delta) \leqq d(k\Delta) + 2\delta = L_k\, d(\Delta) + 2\delta$$

and interchanging the roles of h and k, we obtain the first inequality.

To prove the second inequality, we may assume that $\Delta \subset R^p$, that $v=0$ and that $h, k\colon R^p \to R^n$, $n = 2p+1$. Assume first that $h(0) = 0 = k(0)$. Extend h to a similarity $h_1\colon R^n \to R^n$. Then the map $g = (L_h/L_k) h_1^{-1} k\colon R^p \to R^n$ is an orthogonal map. If $z \in \Delta^0$, we have

$$\begin{aligned}|gz - z| &\leqq |(L_h/L_k) h_1^{-1} kz - h_1^{-1} kz| + |h_1^{-1} kz - z| \\ &\leqq |L_h/L_k - 1|\, L_k |z| / L_h + |kz - hz| / L_h \leqq 3\delta / L_h.\end{aligned}$$

Hence 2.9 implies that

$$|g - \mathrm{id}| \leqq b(\Delta)^{-1} L_h^{-1} M_1 \delta$$

with $M_1 = 3p\big(1 + \varrho(\Delta)\big)^{p-1}$. Consequently

$$\begin{aligned}|h - k| &\leqq L_h |h_1^{-1} k - \mathrm{id}| \\ &\leqq L_h |h_1^{-1} k - g| + L_h |g - \mathrm{id}| \\ &\leqq |L_h - L_k| + b(\Delta)^{-1} M_1 \delta \\ &\leqq d(\Delta)^{-1} M \delta / 2.\end{aligned}$$

In the general case, set $h'(x)=h(x)-h(0)$, $k'(x)=k(x)-k(0)$, and apply the inequality above to the linear maps h', k' with δ replaced by 2δ. We obtain

$$|h(x)-k(x)| \leqq |h(0)-k(0)|+|h'(x)-k'(x)| \leqq \delta(1+d(\Delta)^{-1}M|x|). \qquad \square$$

2.12. Lemma. *Let $\Delta\subset R^n$ be an $(n-1)$-simplex. Suppose that h, k: $R^n\to R^n$ are sense-preserving similarities such that $|h(z)-k(z)|\leqq\delta$ for all $z\in\Delta^0$. Then the inequalities of* 2.11 *are true with $p=n-1$ for all $x\in R^n$ and $v\in\Delta^0$.*

Proof. We repeat the proof of 2.11 with a slight modification. When applying 2.9 we first obtain a rotation u: $R^n\to R^n$ satisfying the inequality of 2.9. Since ug is a rotation with $ug|\Delta=\mathrm{id}$, we have $u=g^{-1}$, which implies $|u-\mathrm{id}|=|g-\mathrm{id}|$. The rest of the proof is unchanged. $\square$

2.13. Suppose that K is a simplicial complex. We say that a map f: $|K|\to l_2$ is *simplicial* if f is affine on every simplex of K. We let K^0 denote the set of vertices of K.

The proof of the following lemma is based on an idea of J. Luukkainen.

2.14. Lemma. *Let K be a finite simplicial complex in l_2. Then there is $\alpha_0=\alpha_0(K)>0$ such that if $0\leqq\alpha\leqq\alpha_0$, f: $|K|\to l_2$ is simplicial, h: $K^0\to l_2$ is a similarity and $\|f-h\|_{K^0}\leqq\alpha L_h$, then*

$$L_h|x-y|/A \leqq |f(x)-f(y)| \leqq AL_h|x-y|$$

for all $x, y\in|K|$, where $A=A(\alpha, K)\to 1$ as $\alpha\to 0$.

If u: $|K|\to l_2$ is a similarity, one can choose $\alpha_0(uK)=L_u\alpha_0(K)$ and $A(\alpha, uK)=A(\alpha/L_u, K)$.

Proof. The last statement of the lemma is clear. Replacing f and h by f/L_h and h/L_h, we may assume that h is an isometry.

We say that a pair Δ_1, Δ_2 of simplexes is a proper simplex pair if $\Delta_1\not\subset\Delta_2$ and $\Delta_2\not\subset\Delta_1$. If K has no proper simplex pairs, the lemma follows from 2.7. The lemma is clearly true if $\dim K=0$. Let $0\leqq p\leqq q\geqq 1$ be integers. We make the inductive hypothesis that the lemma holds for all K such that if (Δ_1, Δ_2) is a proper simplex pair of K with $\dim\Delta_1\leqq\dim\Delta_2$, then either $\dim\Delta_2<q$ or $\dim\Delta_2=q$ and $\dim\Delta_1<p$. It suffices to prove the lemma in the case where K has exactly two principal simplexes Δ_1, Δ_2 with $\dim\Delta_1=p$, $\dim\Delta_2=q$. Extending h to a bijective isometry h_1 of l_2 and replacing f by $h_1^{-1}f$, we may assume that $h=\mathrm{id}$. Since $f-\mathrm{id}$ is simplicial, we have $\|f-\mathrm{id}\|_{|K|}\leqq\alpha$.

Set $\Delta=\Delta_1\cap\Delta_2$. Let $x\in\Delta_1\setminus\Delta$, $y\in\Delta_2\setminus\Delta$. We must find an upper and a lower bound for $|f(x)-f(y)|/|x-y|$.

Case 1. $\Delta=\emptyset$. Now $d(\Delta_1, \Delta_2)=\delta>0$, and

$$|f(x)-f(y)| \leqq |x-y|+2\alpha \leqq (1+2\alpha/\delta)|x-y|,$$

$$|f(x)-f(y)| \geqq |x-y|-2\alpha \geqq (1-2\alpha/\delta)|x-y|.$$

Hence we can choose

$$\alpha_0(K) = \min\left(\delta/3, \alpha_0(\Delta_1), \alpha_0(\Delta_2), \Lambda(\alpha, K)\right)$$
$$= \max\left((1-2\alpha/\delta)^{-1}, \Lambda(\alpha, \Delta_1), \Lambda(\alpha, \Delta_2)\right).$$

Case 2. $\Delta \neq \emptyset$. By the inductive hypothesis, the lemma holds for the complexes $K \setminus \{\Delta_1\}$ and $K \setminus \{\Delta_2\}$. Choose $\alpha_0 > 0$ and a function $\Lambda: [0, \alpha_0] \to [1, \infty)$ with the properties given by the lemma for these complexes. Let $0 \leqq \alpha \leqq \alpha_0$. Choose $a \in \Delta$. Then $x \in ab$ and $y \in ac$ for some $b \in \partial \Delta_1 \setminus \Delta$ and $c \in \partial \Delta_2 \setminus \Delta$. We may assume that

$$\frac{|b-a|}{|x-a|} \geqq \frac{|c-a|}{|y-a|} = \lambda.$$

Then there is $z \in xb$ such that $|z-a| = \lambda |x-a|$, and thus $z-c = \lambda(x-y)$. Since f is affine on ab and on ac, we have $f(z)-f(c) = \lambda\left(f(x)-f(y)\right)$, and hence

$$|f(x)-f(y)| = \lambda^{-1}|f(z)-f(c)| \leqq \Lambda\lambda^{-1}|z-c| = \Lambda|x-y|,$$

and similarly

$$|f(x)-f(y)| \geqq \Lambda^{-1}|x-y|.$$

Hence one can choose $\alpha_0(K) = \alpha_0$ and $\Lambda(\alpha, K) = \Lambda(\alpha)$. □

We next give an estimate for the flatness of a simplex. This will be needed in the proof of 5.19.

2.15. Lemma. *Suppose that $\Delta \subset R^p$ is a p-simplex with $\varrho(\Delta) \leqq M$. Suppose that v is a point in R^{p+1} such that $v_{p+1} \geqq \delta d(\Delta)$, $\delta > 0$, and $d(v, \Delta) \leqq c d(\Delta)$. Then $v\Delta$ is a $(p+1)$-simplex with $\varrho(v\Delta) \leqq M_1(M, \delta, c, p)$.*

Proof. Let $a_0, \ldots, a_p$ be the vertices of Δ, and set $\Delta_1 = v\Delta$. We first derive a lower bound for the numbers $b_j = b_j(\Delta_1)$ (see 2.6). Clearly $b_{p+1} = v_{p+1} \geqq \delta d(\Delta)$. For $0 \leqq j \leqq p$ we may assume $j = 0$. Writing $\Delta_0 = a_1 \ldots a_p v$ we have $b_0 = (p+1) m_{p+1}(\Delta_1)/m_p(\Delta_0)$, where m_p is the p-measure. For any p-simplex σ we have

$$p!\, m_p(\sigma) \geqq b(\sigma)^{p-1} d(\sigma),$$

which can easily be proved by induction on p. This implies

$$m_{p+1}(\Delta_1) \geqq \frac{m_p(\Delta)\delta d(\Delta)}{p+1} \geqq \frac{\delta b(\Delta)^{p-1} d(\Delta)^2}{(p+1)p!}.$$

Since

$$m_p(\Delta_0) \leqq d(\Delta_0)^p \leqq (1+c)^p d(\Delta)^p,$$

we obtain

$$b_0 \geqq \frac{\delta d(\Delta)}{p!(1+c)^p M^{p-1}}.$$

Since $d(\Delta_1) \leqq (1+c) d(\Delta)$, the lemma is true with

$$M_1 = p!\,(1+c)^{p+1} M^{p-1}/\delta. \quad \square$$

3. Approximation by similarities and by isometries

Intuitively, an L-bilipschitz map with small L is close to an isometry, and an $s-$QS map with small s is close to a similarity. We shall give a precise meaning for this in this section.

3.1. Theorem. *Let $A\subset R^p$ be compact, let Y be a linear subspace of l_2 with* $\dim Y\geqq p$, *and let $f\colon A\to Y$ be $s-$QS. Then there is a similarity $h\colon R^p\to Y$ such that*

$$\|h-f\|_A \leqq \varkappa(s,p)L_h\,d(A), \tag{3.2}$$

where $s\mapsto\varkappa(s,p)$ is an increasing function and $\varkappa(s,p)\to 0$ as $s\to 0$.

If f is L-bilipschitz and $s=(L^2-1)^{1/2}$, then h can be chosen to be an isometry.

Proof. Suppose that the first part of the theorem is false. Then there exist $\lambda>0$ and a sequence $f_j\colon A_j\to Y_j$ of η_j-QS maps such that each A_j is compact in R^p, $\eta_j\in N(\mathrm{id}, 1/j)$, and

$$\|f_j-h\|_{A_j} \geqq \lambda L_h\,d(A_j) \tag{3.3}$$

for every similarity $h\colon R^p\to Y_j$. Passing to a subsequence we may assume that $\dim T(A_j)=k$ does not depend on j. For each positive integer j we choose points $a_j^0,\dots,a_j^k\in A_j$ as follows: Let $a_j^0\in A_j$ be arbitrary, and let a_j^{i+1} be a point $x\in A_j$ at which the distance $d(x, T(a_j^0,\dots,a_j^i))$ is maximal.

Using auxiliary similarities of R^p and l_2, we may assume that $R^p\subset Y_j$ and that

$$a_j^0=0,\quad a_j^1=e_1,\quad a_j^i\in\operatorname{int}R_+^i\quad\text{for}\quad 2\leqq i\leqq k,$$

$$f(a_j^0)=0,\quad f_j(a_j^1)=e_1,\quad f_j(a_j^i)\in R_+^i\quad\text{for}\quad 2\leqq i\leqq k.$$

Then $A_j\subset\bar B^k$ and $1\leqq d(A_j)\leqq 2$. Applying [TV$_1$, 2.5] with the substitution $A\mapsto\{0,e_1\}$, $B\mapsto A_j$, $f\mapsto f_j$, yields

$$d(f_jA_j)\leqq 2\eta_j(d(A_j))\leqq 2\eta_j(2)\leqq 2(2+1/j)\leqq 5$$

for $j\geqq 2$. Applying (3.3) with $h=\mathrm{id}$ we find $x_j\in A_j$ such that

$$|f_j(x_j)-x_j|\geqq\lambda$$

for all $j\geqq 2$. Passing to a subsequence and performing an auxiliary isometry φ of l_2 with $\varphi|R^k=\mathrm{id}$, we may assume that the following sequences converge as $j\to\infty$:

$$a_j^i\to a^i\in\bar B_+^i,\quad 0\leqq i\leqq k,$$

$$f_j(a_j^i)\to b^i\in\bar B_+^i(5),\quad 0\leqq i\leqq k,$$

$$x_j\to x_0\in\bar B^k,$$

$$f_j(x_j)\to y_0\in\bar B^{k+1}(5).$$

Moreover, $a^0=b^0=0$ and $a^1=b^1=e_1$.

Put $T=T(a^0, \ldots, a^k)$, $s=\dim T$. Then $s\leqq k$ and $a^i\in \operatorname{int} R^i_+$ for $i\leqq s$. Since $\eta_j\in N(\mathrm{id}, 1/j)$, we have

$$|f_j(a_j^2)| = \frac{|f_j(a_j^2)-f_j(0)|}{|f_j(e_1)-f_j(0)|} \to \frac{|a^2-0|}{|e_1-0|} = |a^2|.$$

Hence $|b^2|=|a^2|$. Changing the roles of 0 and e_1, a similar argument shows that $|b^2-e_1|=|a^2-e_1|$. Since $a^2, b^2\in R^2_+$, we obtain $a^2=b^2$. Proceeding inductively, we similarly obtain $a^i=b^i$ for $0\leqq i\leqq s$. Since

$$\lim_{j\to\infty} \max_{x\in A_j} d(x, T) = 0,$$

we have $x_0\in T$. If i and l are distinct integers on $[0, s]$,

$$\frac{|y_0-a^i|}{|a^l-a^i|} = \lim_{j\to\infty} \frac{|f_j(x_j)-f_j(a_j^i)|}{|f_j(a_j^l)-f_j(a_j^i)|} = \lim_{j\to\infty} \frac{|x_j-a_j^i|}{|a_j^l-a_j^i|} = \frac{|x_0-a^i|}{|a^l-a^i|}.$$

Thus $|y_0-a^i|=|x_0-a^i|$ for $0\leqq i\leqq s$. Since $a^0, \ldots, a^s$ are affinely independent in T and $x_0\in T$, this implies $x_0=y_0$. Since $|x_0-y_0|\geqq\lambda$, this is a contradiction.

The bilipschitz case could be proved in a similar manner, but it also follows from the QS case. Assume that $f\colon A\to Y$ is L-bilipschitz. Then f is $s-$QS with $s=(L^2-1)^{1/2}$. Choose a similarity $h\colon R^p\to Y$ satisfying (3.2). We may assume that $0\in A$ and that $h(0)=0$. Then $h_1=h/L_h$ is an isometry. For each $x\in A$ we have

$$\begin{aligned}|f(x)-h_1(x)| &\leqq |f(x)-h(x)|+|h(x)-h_1(x)|\\ &\leqq \varkappa(s, p)L_h\, d(A)+|1-1/L_h|\,|h(x)|\\ &\leqq \varkappa_1(s, p)d(A),\end{aligned}$$

where

(3.4) $$\varkappa_1(s, p) = L_h\varkappa(s, p)+|1-L_h|.$$

On the other hand,

(3.5) $$L_h\, d(A) = d(hA) \leqq d(fA)+2\varkappa(s, p)L_h\, d(A).$$

This implies

$$L_h \leqq \frac{L}{1-2\varkappa(s, p)}$$

as soon as $L-1$ is so small that $2\varkappa(s,p)<1$. Similarly, we obtain a lower bound for L_h, and (3.4) yields

$$\varkappa_1(s, p) = \delta(L, p) \to 0$$

as $L\to 1$. □

3.6. Remarks. 1. Theorem 3.1 is true with (3.2) replaced by the inequality

(3.7) $$\|f-h\|_A \leqq \varkappa(s, p)\, d(fA),$$

replacing $\varkappa$ by another function with the same properties. This follows easily from (3.5).

2. In the QS case of (3.2) and (3.7) one can always choose $\varkappa(s,p)\leqq 2$. By auxiliary similarities we can normalize the situation so that $0\in A$, $f(0)=0$, and $d(A)=d(fA)=1$. Then

$$\|f-\mathrm{id}\|_A \leqq d(fA)+d(A) \leqq 2.$$

This observation is due to J. Luukkainen.

We next prove converse results of 3.1. These are not needed in the rest of the paper.

3.8. Theorem. *Let* $0<\delta<1/2$, *let* $X\subset R^p$, *and let* $f\colon X\to l_2$ *be a map such that for every bounded* $A\subset X$ *there is an isometry* $h\colon R^p\to l_2$ *such that* $\|h-f\|_A\leqq \delta d(A)$. *Then f is L-bilipschitz with* $L=(1-2\delta)^{-1}$.

Proof. Let $a, b\in X$ with $a\neq b$. Set $A=\{a, b\}$, and choose the corresponding isometry h. Now

$$|f(a)-f(b)| \leqq |h(a)-h(b)|+|h(a)-f(a)|+|h(b)-f(b)|$$
$$\leqq (1+2\delta)|a-b| \leqq (1-2\delta)^{-1}|a-b|,$$

and similarly

$$|f(a)-f(b)| \geqq (1-2\delta)|a-b|. \quad \square$$

3.9. Theorem. *Let* $0<\varkappa\leqq 1/25$, *let* $X\subset R^p$ *be connected, and let* $f\colon X\to l_2$ *be a map such that for every bounded* $A\subset X$ *there is a similarity* $h\colon R^p\to l_2$ *such that* $\|h-f\|_A\leqq \varkappa L_h d(A)$. *Then f is* $s-$QS, *where* $s=s(\varkappa)\to 0$ *as* $\varkappa\to 0$.

Proof. We first show that f is injective. Let $a, b\in X$ with $a\neq b$. Set $A=\{a, b\}$ and choose the corresponding similarity h. Then

$$|f(a)-f(b)| \geqq (1-2\varkappa)|h(a)-h(b)| > 0.$$

Now assume that a, b, x are distinct points in X with $|a-x|=t|b-x|$. Set $A=\{a, b, x\}$ and choose the corresponding similarity $h\colon R^p\to l_2$. Since

$$|f(b)-f(x)| \geqq |h(b)-h(x)|-2\varkappa L_h d(A) = L_h|b-x|-2\varkappa L_h d(A),$$

we obtain

$$|f(a)-f(x)| \leqq |h(a)-h(x)|+2\varkappa L_h d(A)$$
$$\leqq t|f(b)-f(x)|+2(1+t)\varkappa L_h d(A).$$

Since

$$d(A) \leqq |a-x|+|b-x| = (1+t)|b-x|,$$

we obtain

$$L_h d(A) \leqq (1+t)|h(b)-h(x)|$$
$$\leqq (1+t)|f(b)-f(x)|+2(1+t)\varkappa L_h d(A).$$

Assume $t\leqq \varkappa^{-1/2}$. Since $\varkappa\leqq 1/25$, we have $2(1+t)\varkappa<1/2$, and thus

$$L_h d(A) \leqq 2(1+t)|f(b)-f(x)|.$$

Consequently, $|f(a)-f(x)|=t'|f(b)-f(x)|$ with

$$t' \leqq t+4\varkappa(1+t)^2. \tag{3.10}$$

Assuming $t \leqq \varkappa^{-1/4}$ this implies

$$t' \leqq t+9\varkappa^{1/2}.$$

Hence, if f is QS, it is s-QS with

$$s = s(\varkappa) = \max(\varkappa^{1/4}, 9\varkappa^{1/2}).$$

To show that f is QS, we verify that f satisfies the conditions (1) and (2) of [TV_1, 3.10] with $\lambda_1=\lambda_2=1/2$, $h=2$, $H=4$. Since X is connected, it is λ-HD. If $t\leqq 2$, then $t\leqq\varkappa^{-1/4}$, and hence $t'\leqq t+9\varkappa^{1/2}<4$. If $t\leqq 1/4$, then (3.10) implies $t'\leqq 1/2$. The quasisymmetry of f follows then from the proof of [TV_1, 3.10] and from [TV_1, 2.21]. □

4. Planes and spheres

In [TV_4] we proved that R^p and S^p have the extension properties in R^n for $p<n$. In this section we show that R^n can be replaced by (R^n, Y) where Y is any linear subspace of l_2 with $\dim Y\geqq n$. The result will be needed in Section 6.

4.1. Theorem. *Let Y be a linear subspace of l_2 with* $\dim Y\geqq n$, *and let* $1\leqq p\leqq n-1$. *Then R^p has the extension properties in (R^n, Y). The numbers in the definition of the extension properties do not depend on Y, thus* $L_0=L_0(n)$, $L_1=L_1(L, n)$, $s_0=s_0(n)$, $s_1=s_1(s, n)$.

Proof. The proof can be carried out by rewriting the proof of [TV_4, 5.3, 5.4] in this more general setting. However, some modifications have to be made. We shall only give these modifications.

The lemmas of [TV_4, Section 3] are easily generalized to the new setting and partly given in Section 2 of the present paper. The results of [TV_4, Section 4] concerning frames are still valid in the general case but in the proofs one cannot make use of the compactness of the space $V_n^0(Y)$ of all orthonormal n-frames of Y. However, the uniform differentiability formulas [TV_4, (4.2), (4.3)] of the Gram—Schmidt map $G\colon V_n(Y)\to V_n^0(Y)$ are still valid in some neighborhood N of $V_n^0(Y)$ in $V_n(Y)$, as easily follows from the definition of G. Hence we obtain the interpolation lemma [TV_4, 4.4] with R^n replaced by Y. The crucial extension lemma [TV_4, 4.9] also remains valid with R^n replaced by Y. Although $V_n^0(Y)$ is not necessarily compact, G is still uniformly continuous in a neighborhood of it. On the other hand, we cannot use the diagonal process to conclude that it suffices to define the map $u\colon \mathscr{I}(p)\to V_n^0(Y)$ only on $\mathscr{I}(p, k)$. Instead, we give a direct construction of u on the whole $\mathscr{I}(p)$.

We again start with the cube $Q_0=J^p$ and define $u_{Q_0}=u_0$. Next we inductively define u_{Q_j} for $Q_j=2^jJ^p$ directly by $u_{Q_{j-1}}$ for *all* positive integers j. For each j we consider the family of the 3^p-1 cubes $R\in\mathscr{I}_j(p)$ with $R\sim Q_j$, $R\neq Q_j$, and define

u_R directly by u_{Q_j}. Next we define u_{P_R} for the principal subcubes P_R of these cubes R directly by u_R. Then we apply the generalized version of [TV_4, 4.4] to define u_Q for every $Q\in\mathscr{I}_{j-1}(p)$ in the convex hull E_{j-1} of the union of these principal subcubes, except for those Q for which u_Q has already been defined directly by $u_{Q_{j-1}}$. Proceeding in this manner, it is easy to see that we obtain a map $u\colon \mathscr{I}(p)\to V_n^0(Y)$ with the desired properties provided that $q\leqq 2^{-p-4}$.

The extension $g\colon R^n\to Y$ of the given L-bilipschitz or s–QS map $f\colon R^p\to Y$ can now be constructed and its continuity proved as in the proofs of Theorems 5.3 and 5.4 of [TV_4]. However, we must give a new proof for the fact that g is L_1-bilipschitz or s_1–QS, because the old one was based on the convexity of gR^n in the bilipschitz case and on the theory of QC maps in the QS case. We shall prove the QS case. The proof for the bilipschitz case is similar but easier; observe that the convexity of R^p implies that g is lipschitz.

To show that g is s_1–QS we use 2.3. Thus assume that a, b, x are distinct points in R^n with $|b-x|=r$, $|a-x|=tr$, $t\leqq 2$. We must find an estimate

$$|g(a)-g(x)| \leqq (t+s_1)|g(b)-g(x)|, \tag{4.2}$$

where $s_1=s_1(q, n)\to 0$ as $q\to 0$.

Using the notation of [TV_4] we again obtain the estimate

$$\|g-h_Q\|_{Z_Q} \leqq Mq\varrho_Q, \quad M = 24n^2 \tag{4.3}$$

[TV_4, (5.9)]. Here Q is an arbitrary cube in $\mathscr{I}(p)$, $h_Q\colon R^p\to Y$ is a similarity, and $\varrho_Q=L(h_Q)\lambda_Q$, where λ_Q is the length of the side of Q. For $Q\in\mathscr{I}(p)$ set

$$Y_Q'' = \cup\{Y_R\colon Y_R\cap Y_Q \neq \emptyset\}.$$

We may assume that $x\in R^n\setminus R^p$. Then there is $Q\in\mathscr{I}(p)$ such that $x\in Y_Q$. We divide the rest of the proof into two cases:

Case 1. $r\leqq\lambda_Q/4$. Now

$$|a-x| \leqq \lambda_Q/2 = d(Y_Q, R^n\setminus Y_Q'').$$

Hence $\{x, a, b\}\subset Y_Q''$. Let W_Q'' be the subcomplex of W with $|W_Q''|=Y_Q''$. Let $\alpha_0=\alpha_0(W_Q'')$ and $\Lambda=\Lambda(\alpha, W_Q'')$ be the numbers given by 2.14. One can choose

$$\alpha_0 = \gamma_0\lambda_Q, \quad \Lambda(\alpha, W_Q'') = \Lambda_0(\alpha/\lambda_Q, n),$$

for some $\gamma_0=\gamma_0(n)>0$ and for some function Λ_0 with $\lim_{\alpha\to 0}\Lambda_0(\alpha, n)=1$. Let R be the unique cube in $\mathscr{I}(p)$ with $\lambda_R=2\lambda_Q$ and $Q\subset R$. Then $Y_Q''\subset Z_R$. Hence (4.3) implies

$$\|g-h_R\|_{Y_Q''} \leqq Mq\varrho_R = \alpha L(h_R)$$

with $\alpha=2Mq\lambda_Q$. We give the new restriction

$$q \leqq \gamma_0/2M.$$

Then $\alpha \leqq \alpha_0 = \gamma_0 \lambda_Q$, and Lemma 2.14 implies (4.2) with

$$s_1 = 2(A_0(2Mq, n)^2 - 1).$$

Case 2. $r > \lambda_Q/4$. Let $Q = R_0 \subset R_1 \subset \dots$ be the unique sequence of cubes of $\mathcal{I}(p)$ such that $k(R_{j+1}) = k(R_j) + 1$. Let m be the smallest integer for which Z_{R_m} contains $\{a, b\}$, and set $R = R_m$. Since $d(Z_{R_j}, R^n \setminus Z_{R_{j+1}}) = \lambda_{R_j}$, we have $r \geqq \lambda_R/8$. From (4.3) we obtain

$$\|g - h_R\|_{Z_R} \leqq MqL(h_R)\lambda_R.$$

Hence

$$|g(a) - g(x)| \leqq L(h_R)(tr + 2Mq\lambda_R),$$

$$|g(b) - g(x)| \geqq L(h_R)(r - 2Mq\lambda_R).$$

Assuming $q < 1/16M$ we obtain

$$\frac{|g(a) - g(x)|}{|g(b) - g(x)|} \leqq \frac{t + 16Mq}{1 - 16Mq},$$

which implies (4.2). □

4.4. Corollary. *Let Y be a linear subspace of l_2, and let $p \leqq n \leqq \dim Y$. Then a set $A \subset R^p$ has the* BLEP *or the* QSEP *in (R^p, Y) if and only if it has the same property in (R^n, Y). In particular, the extension properties in R^n and in (R^p, R^n) are equivalent for $A \subset R^p$.* □

It is natural to ask whether R^n has the extension properties in l_2. I do not know the answer. However, the following result in this direction can be established:

4.5. Theorem. *Every L-bilipschitz $f\colon R^n \to R^n$ can be extended to an L-bilipschitz homeomorphism $g\colon l_2 \to l_2$, and every s-QS $f\colon R^n \to R^n$ can be extended to an s_1-QS homeomorphism $g\colon l_2 \to l_2$ such that $s_1 = s_1(s, n) \to 0$ as $s \to 0$. Moreover, $gY = Y$ for every linear subspace Y of l_2 containing R^n.*

Proof. Let E be the orthogonal complement of R^n in l_2. The bilipschitz case is easy; we define $g(x+y) = f(x) + y$ for $x \in R^n$, $y \in E$.

Suppose that $f\colon R^n \to R^n$ is s-QS. Then f is K-QC with $K = K(s, n)$. By [TV3], f can be extended to a homeomorphism $F\colon R^{n+1}_+ \to R^{n+1}_+$ such that $F|\text{int } R^{n+1}$ is H-bilipschitz in the hyperbolic metric with $H = H(s, n)$. The required homeomorphism g is then the rotation of F around R^n. More precisely, let $e \in E$ be a unit vector. If $x \in R^n$ and $t > 0$, we define $g(x + te) = x' + t'e$, where (x', t') is determined by $x' + t'e_{n+1} = F(x + te_{n+1})$. If a, b, x are points in l_2, there is a linear subspace Y of l_2 with $\dim Y = n+3$ containing these points and R^n. Arguing as in [TV3, 3.13], we see that g defines a K_1-QC map $g_1\colon Y \to Y$ with $K_1 = K_1(s, n)$. Hence g_1 is s_1-QS with $s_1 = s_1(s, n)$. Hence g is s_1-QS.

If s is small, the extension F of f can also be obtained from the fact that R^n has the QSEP in R^{n+1}. Then F is s_2-QS with small s_2. However, we need the fact that the hyperbolic bilipschitz constant H of $F|\text{int } R^{n+1}$ is close to 1. This follows rather easily from the proof of [TV_4, 5.4]. This implies that $s_1(s, n) \to 0$ as $s \to 0$. □

4.6. Theorem. *If Y is a linear supspace of l_2 and if $p < n \leqq \dim Y$, then S^p, R_+^{p+1} and $\bar{B}^{p+1}$ have the extension properties in (R^n, Y).*

Proof. The case $p=0$ needs a separate argument, which is omitted. Assume $p \geqq 1$. The case $A = S^p$ follows from 4.1 by means of auxiliary inversions as in [TV_4, 5.23]. The awkward proof of [TV_4, 5.22] can be essentially simplified by means of quasimöbius maps, see [$Vä_2$, 3.11].

The case $A = R_+^{p+1}$ can be proved by modifying the proof of 4.1. By 4.4, we may assume $n = p+1$. If $f: R_+^n \to Y$ is L-bilipschitz or $s-$QS with small L or s, we define an extension g_0: $R_-^n \to Y$ of $f|R^p$ as in the proof of 4.1. However, when defining the orthogonal frames $u_Q \in V_n^0(Y)$, we do not make use of the results of [TV_4, Section 4]. Instead, we can now define $w_Q^j = f(a_Q + \lambda_Q e_j) - f(a_Q)$ also for $j = n$, and we let u_Q be the Gram—Schmidt orthogonalization of $w_Q = (w_Q^1, \ldots, w_Q^n)$. We obtain an extension g: $R^n \to Y$ of f. We still have to show that g is L_1-bilipschitz or s_1-QS. It follows from the proof of 4.1 that it suffices to show that

$$\|f - h_Q\|_{Z_Q^+} \leqq 24n^2 q\varrho_Q$$

for sufficiently small L or s, where $Z_Q^+ = Z_Q \cap R_+^n$. This follows rather easily from a slightly modified version of [TV_4, 3.10]. We omit the details.

Finally, the case $A = \bar{B}_+^{p+1}$ follows from the preceding case by auxiliary inversions. Alternatively, it is a special case of 6.13.1. □

5. The first condition

In Theorem 5.5 we shall give a sufficient condition for a set $A \subset R^n$ to have the extension properties in R^n. We then show that this condition holds for all compact $(n-1)$-dimensional DIFF and PL manifolds and for certain other sets in R^n.

5.1. *The Whitney triangulation.* Let $G \subset R^n$ be an open set, $\emptyset \neq G \neq R^n$. The *relative size* of a compact set $A \subset G$ is defined as

$$r_G(A) = \frac{d(A)}{d(A, \partial G)}.$$

Let K be the Whitney decomposition of G into closed n-cubes such that

$$\lambda_1 \leqq r_G(Q) \leqq \lambda_2$$

for all $Q \in K$, where λ_1 and λ_2 are positive constants. See e.g. [St, p. 167] or [TV_2, 7.2]. One can choose $\lambda_1 = 1/7$ and $\lambda_2 = \sqrt{n}/2$, but these constants can obviously be chosen to be arbitrarily small.

We define a subdivision of K to a simplicial complex W as follows: Suppose that we have defined a simplicial subdivision W^p of the p-skeleton K^p of K. Let Q be a $(p+1)$-cube of K, and let v_Q be the center of Q. Since ∂Q is the underlying space of a subcomplex L_Q of W^p, the cone construction $v_Q L_Q$ gives a simplicial subdivision of Q, and we obtain W^{p+1}. The complex W is called a *Whitney triangulation of* G.

If σ is an n-simplex of W, we can write

$$\varrho(\sigma) \leqq \varrho_n, \quad a_1 \leqq r_G(\sigma) \leqq a_2, \tag{5.2}$$

where the numbers ϱ_n, a_1, a_2 depend only on n. Indeed, since the simplexes of W belong to a finite number of similarity classes, the first inequality of (5.2) is true. In the second one, we can choose $a_1=\lambda_1/3\sqrt{n}$ and $a_2=\lambda_2/2$.

5.3. *Terminology*. Let $A\subset R^n$. We say that a simplex Δ is a simplex *of* A if $\Delta^0\subset A$. If Δ is a an n-simplex of A and if $f\colon A\to R^n$ is a map, we say that $f|\Delta^0$ is sense-preserving if the unique affine extension $g\colon R^n\to R^n$ of $f|\Delta^0$ is sense-preserving. Two p-simplexes Δ, Δ' of A are said to be *M-related* in A, $M\geqq 1$, if there is a finite sequence $\Delta=\Delta_0, \ldots, \Delta_k=\Delta'$ of p-simplexes of A such that

(1) $\varrho(\Delta_j) \leqq M$ for $0 \leqq j \leqq k$,

(2) $1/M \leqq d(\Delta_j)/d(\Delta_{j-1}) \leqq M$ for $1 \leqq j \leqq k$,

(3) $d(\Delta_{j-1}, \Delta_j) \leqq M \min\left(d(\Delta_{j-1}), d(\Delta_j)\right)$ for $1 \leqq j \leqq k$.

5.4. Lemma. *Let n be a positive integer, let $M\geqq 1$, and let $s=s(M,n)$ be such that $\varkappa(s,n)\leqq 1/10M^3(n+1)$, where $\varkappa$ is the function of* 3.1. *Suppose that $A\subset R^n$, that $f\colon A\to R^n$ is $s-$QS and that the n-simplexes Δ_1, Δ_2 of A are M-related in A. Then $f|\Delta_1^0$ and $f|\Delta_2^0$ are either both sense-preserving or both sense-reversing.*

Proof. We may assume that the sequence $\Delta_0, \ldots, \Delta_k$ of 5.3 is the pair (Δ_1, Δ_2). Suppose that $f|\Delta_1^0$ is sense-preserving. Set

$$F = \{x\in A\colon d(x,\Delta_1) \leqq 2M\, d(\Delta_1)\}.$$

Then $d(F)\leqq 5Md(\Delta_1)$. For every $z\in\Delta_2$ we have

$$d(z, \Delta_1) \leqq d(\Delta_2)+d(\Delta_2, \Delta_1) \leqq 2M\, d(\Delta_1).$$

Hence $\Delta_2^0\subset F$. Applying 3.1 we choose a similarity $h\colon R^n\to R^n$ such that

$$\|h-f\|_F \leqq \varkappa(s, n) L_h\, d(F) \leqq \frac{L_h\, d(\Delta_1)}{2(n+1)M^2}.$$

Since $\varrho(\Delta_1)\leqq M\leqq M^2$, we have

$$|f(z)-h(z)| \leqq \frac{L_h\, d(\Delta_1)}{2(n+1)\,\varrho(\Delta_1)}$$

for every $z \in \Delta_1^0$. By 2.7, h is sense-preserving. Furthermore, since $d(\Delta_1) \leqq Md(\Delta_2)$, we have for every $y \in \Delta_2^0$,

$$|f(y)-h(y)| \leqq \frac{L_h d(\Delta_2)}{2(n+1)\varrho(\Delta_2)}.$$

Again by 2.7, $f|\Delta_2^0$ is sense-preserving. □

5.5. Theorem. *Suppose that* $n \geqq 2$, *that* A *is closed in* R^n, *that* ∂A *is bounded and that* int A *has a finite number of components. For* $x \in R^n \setminus A$ *and* $b>1$ *we set*

$$E(x, b) = A \cap \bar{B}(x, bd(x, A)).$$

Suppose that there exist numbers $b_2 \geqq b_1 > 1$, $M \geqq 1$, *and that for every* $\lambda > 0$ *there is* $r_0 > 0$ *such that if* $x \in R^n \setminus A$ *and* $d(x, A) = r \leqq r_0$, *then one of the following two conditions is satisfied:*

(a) *There is an* $(n-1)$*-simplex* Δ *of* $E(x, b_1)$ *and an* $(n-1)$*-plane* $T \subset R^n$ *such that*

(a_1) $\varrho(\Delta) \leqq M$,

(a_2) $d(\Delta) \geqq r/M$,

(a_3) $E(x, b_1) \subset T + \lambda r \bar{B}^n$.

(b) *There is an n-simplex* Δ *of* $E(x, b_2)$ *such that*

(b_1) $d(\Delta) \geqq r/M$,

(b_2) Δ *is M-related to an n-simplex* Δ' *in* A *with* $d(\Delta') \geqq 1/M$.

Then A *has the extension properties in* R^n.

Proof. Choose an auxiliary parameter $q>0$. To prove the QSEP, it suffices to show that there are $q_0>0$ and for every $q \in (0, q_0]$ a number $s=s(q, A, n)>0$ such that every $s-$QS embedding $f\colon A \to R^n$ has an extension to a $K-$QC map $g\colon R^n \to R^n$, where $K=K(q, A, n) \to 1$ as $q \to 0$. In the bilipschitz case, we find $L=L(q, A, n)$ such that every L-bilipschitz map $f\colon A \to R^n$ has an extension to an L_1-bilipschitz $g\colon R^n \to R^n$ with $L_1 = L_1(q, A, n) \to 1$ as $q \to 0$.

To begin with, we only assume $0<q\leqq 1$. In the course of the proof, we shall give more restrictions on q of the form $q \leqq q_0(A, n)$.

Choose $R \geqq 4$ such that $\partial A \subset B^n(R/2b_2)$, and set $B = \bar{B}^n(R)$. Next choose $r_1>0$ such that every component of int A contains a ball $B(x, r_1) \subset B$. Set $\lambda = q/4$ and choose the corresponding r_0. We may assume $r_0 \leqq 1$. Let $\varkappa$ be the function given by 3.1. Choose $s=s(q, A, n) \in (0, q]$ such that $\varkappa(s, n)$ is smaller than the numbers

$$\frac{1}{10RM^3b_2(n+1)}, \frac{qr_0}{2R}, \frac{q}{4b_2}, \frac{r_1}{5R}. \tag{5.6}$$

We show that s is the required number provided that q is sufficiently small. In the bilipschitz case, we set $L=L(q, A, n)=(s^2+1)^{1/2}$. Then every L-bilipschitz map is $s-$QS.

Suppose that $f\colon A\to R^n$ is $s-$QS. By 3.1, there is a similarity h of R^n such that

$$\|h-f\|_{A\cap B} \leqq \varkappa(s, n)L_h d(A\cap B) \leqq 2RL_h\varkappa(s, n).$$

Replacing f by $h^{-1}f$ we may assume that

$$\|f-\mathrm{id}\|_{A\cap B} \leqq 2R\varkappa(s, n). \tag{5.7}$$

Set

$$G = R^n\setminus A, \quad G(r_0) = \{x\in G\colon d(x, A) < r_0\}.$$

Let G_1 be the set of all points $x\in G(r_0)$ which satisfy the condition (a), and set $G_2=G(r_0)\setminus G_1$. For $x\in G$ we define

$$E_x = E(x, b_1) \quad \text{if} \quad x\in G_1,$$

$$E_x = E(x, b_2) \quad \text{if} \quad x\in G\setminus G_1.$$

We associate to every $x\in G$ a similarity h_x of R^n as follows: If $x\in G\setminus G(r_0)$, we choose $h_x=\mathrm{id}$. Assume $x\in G(r_0)$. Then $d(x, A)=r<r_0$. If $x\in G_1$, we apply 3.1 to find a similarity k_x such that

$$\|k_x-f\|_{E_x} \leqq \varkappa(s, n)L(k_x)\, d(E_x) \leqq 2b_1 rL(k_x)\varkappa(s, n). \tag{5.8}$$

If k_x is sense-preserving, we choose $h_x=k_x$. Otherwise, we set $h_x=k_x\psi$, where ψ is the reflection in the $(n-1)$-plane T given by (a). Finally, if $x\in G_2$, we again apply 3.1 and choose h_x so that

$$\|h_x-f\|_{E_x} \leqq \varkappa(s, n)L(h_x)\, d(E_x) \leqq 2b_2 rL(h_x)\varkappa(s, n). \tag{5.9}$$

Now h_x is defined for all $x\in G$. In the bilipschitz case, h_x is chosen to be an isometry.

We next show that h_x is sense-preserving for every $x\in G$. For $x\in G\setminus G_2$, this follows directly from the construction. Suppose $x\in G_2$. Let Δ and Δ' be the n-simplexes of A given by (b), and let $\Delta=\Delta_0, \dots, \Delta_k=\Delta'$ be the sequence given by the definition 5.3 of M-relatedness, We first show that one can choose Δ' to be a simplex of $A\cap B$. If A is bounded, $A\subset B$, and this is trivial. Assume that A is unbounded. Then A contains $R^n\setminus B^n(R/2)$. If all vertices of Δ' are in $R^n\setminus B^n(R/2)$, we can continuously deform Δ' in $R^n\setminus B^n(R/2)$ to a simplex Δ'' of $B\setminus B^n(R/2)$ with $d(\Delta'')=R/4\geqq 1\geqq 1/M$ without changing its similarity class. Thus Δ is M-related to Δ'' in A. If Δ' has vertices both in $B^n(R/2)$ and $R^n\setminus B$, we choose a translation φ of R^n such that $\varphi\Delta'\cap B^n(R/2)=\emptyset\neq\varphi\Delta'\cap\Delta'$. Then the sequence $\Delta_0, \dots, \Delta_k, \varphi\Delta'$ still satisfies the conditions of 5.3, and the situation reduces to the preceding case.

Since $\varkappa(s, n)\leqq 1/4(n+1)RM^2$ and since $b(\Delta')=d(\Delta')/\varrho(\Delta')\geqq 1/M^2$, (5.7) implies

$$|f(x)-x| \leqq \frac{b(\Delta')}{2(n+1)}$$

for every vertex x of Δ'. By 2.7, $f|(\Delta')^0$ is sense-preserving. Hence, by Lemma 5.4 and by (5.6), $f|\Delta^0$ is sense-preserving. Since $b(\Delta)\geqq r/M^2$, 2.7, (5.6) and (5.9) imply that h_x is sense-preserving.

We next prove the inequality

$$\|h_x-f\|_{E_x} \leqq qd(x, A)L(h_x) \tag{5.10}$$

for every $x\in G$. Set $r=d(x, A)$. We divide the proof into four cases.

Case 1. $r\geqq r_0$. If A is not bounded, $G\subset B^n(R/2b_2)$. Hence $r\leqq R/2b_2$, which implies $E_x\subset A\cap B$. This is clearly also true if A is bounded. Since $h_x=\mathrm{id}$ and since $\varkappa(s, n)\leqq qr_0/2R$, we obtain

$$\|h_x-f\|_{E_x} \leqq \|\mathrm{id}-f\|_{A\cap B} \leqq 2R\varkappa(s, n) \leqq qrL(h_x).$$

Case 2. $r<r_0$, $x\in G_1$, $h_x=k_x$. Now (5.10) follows from (5.8) and from the inequality $\varkappa(s, n)\leqq q/4b_2<q/2b_1$.

Case 3. $r<r_0$, $x\in G_1$, $h_x=\psi k_x$. Now $L(h_x)=L(k_x)$. For every $y\in E_x$, (5.8) yields

$$\begin{aligned}|h_x(y)-f(y)| &\leqq |k_x(\psi(y))-k_x(y)|+|k_x(y)-f(y)| \\ &\leqq 2L(h_x)\lambda r+2b_1 rL(h_x)\varkappa(s, n).\end{aligned}$$

Since $\lambda=q/4$ and since $\varkappa(s, n)\leqq q/4b_2\leqq q/4b_1$, we obtain (5.10).

Case 4. $r<r_0$, $x\in G_2$. Since $\varkappa(s, n)\leqq q/4b_2<q/2b_2$, this case follows from (5.9). Thus (5.10) is proved.

Choose a Whitney triangulation W of G satisfying (5.2). Here we choose

$$a_2 \leqq \min\left(1, (b_1-1)/2\right).$$

The constants a_1 and a_2 depend only on A and n.

For every vertex v of W we set

$$g(v) = h_v(v),$$

and extend g affinely to every simplex of W. Setting $g|A=f$ we obtain a map $g\colon R^n\to R^n$. We claim that g is the desired extension of f.

We first show that g is continuous. This is clearly true in G and in $\mathrm{int}\, A$. Suppose that $x_0\in\partial A=\partial G$, and let $\varepsilon>0$. Since f is continuous, there is $\delta>0$ such that $|f(x)-f(x_0)|\leqq\varepsilon$ whenever $x\in A$ and $|x-x_0|\leqq\delta$. Choose $\delta_1<\delta$ such that $E_v\subset B(x_0, \delta)$ and $d(v, A)\leqq r_0$ whenever v is a vertex of any n-simplex $\sigma\in W$ such that $d(x_0, \sigma)\leqq\delta_1$. Suppose that $x\in G$ with $|x-x_0|\leqq\delta_1$. Choose an n-simplex $\sigma\in W$ containing x. It suffices to find an estimate

$$|g(v)-f(x_0)| \leqq M_1\varepsilon \tag{5.11}$$

for the vertices v of σ with some constant M_1. In what follows, we let $M_2, M_3, \ldots$ denote constants $M_j\geqq 1$ depending only on A and n. Set $r=d(v, A)$, and choose

$y \in A$ with $|y-v|=r$. Since $q \leqq 1$, (5.10) implies

$$\begin{aligned}|g(v)-f(x_0)| &\leqq |h_v(v)-h_v(y)|+|h_v(y)-f(y)|+|f(y)-f(x_0)| \\ &\leqq L(h_v)r+qrL(h_v)+\varepsilon \\ &\leqq 2rL(h_v)+\varepsilon.\end{aligned}$$

Since $r \leqq r_0$, E_v contains points x_1, x_2 with $|x_1-x_2| \geqq r/M$. We give the restriction $q \leqq 1/4M$. Then (5.10) implies

$$\begin{aligned}rL(h_v)/M &\leqq L(h_v)|x_1-x_2| = |h_v(x_1)-h_v(x_2)| \\ &\leqq |f(x_1)-f(x_2)|+2qrL(h_v) \\ &\leqq 2\varepsilon+rL(h_v)/2M,\end{aligned}$$

and hence $rL(h_v) \leqq 4M\varepsilon$. This implies (5.11) with $M_1=8M+1$ and proves the continuity of g.

Let σ be an n-simplex of W, and let v be the vertex of σ which is closest to A. We want to estimate $|h_v-g|$ in σ^0. Set $r=d(v, A)$. If $r \geqq r_0$, $h_v=\mathrm{id}=g$ in σ^0. Assume that $r<r_0$. Set

$$c_1 = \frac{b_1-1}{2(b_2-1)}, \quad r' = c_1 r.$$

Then c_1 depends only on A, and

$$\frac{a_2}{b_2-1} \leqq c_1 \leqq \frac{1}{2}.$$

Choose $y \in A$ with $|y-v|=r$. Let z be the unique point on the segment vy such that $|z-y|=r'$. A direct computation shows

$$|v-z|+b_2 r' = r(1+b_1)/2. \tag{5.12}$$

Moreover, $r'=d(z, A)$. Let $x \in E(z, b_2)$. If $u \in \sigma^0$, then

$$|u-v| \leqq d(\sigma) \leqq a_2\, d(\sigma, A) \leqq (b_1-1)\, r/2.$$

Hence (5.12) gives

$$|x-u| \leqq |x-z|+|z-v|+|v-u| \leqq b_1 r \leqq b_1 d(u, A).$$

Thus

$$E(z, b_2) \subset A \cap \bar{B}(u, b_1 r) \subset E(u, b_1) \subset E_u. \tag{5.13}$$

Since $r'<r_0$, there is an $(n-1)$-simplex Δ of $E(z, b_2)$ such that $\varrho(\Delta) \leqq M$ and $d(\Delta) \geqq r'/M$. If $z \in G_2$, Δ is a suitable face of the n-simplex given by (b). Since $d(u, A) \leqq (1+a_2)r \leqq 2r$, (5.10) and (5.13) yield for every $x \in \Delta^0$:

$$\begin{aligned}|h_v(x)-h_u(x)| &\leqq |h_v(x)-f(x)|+|f(x)-h_u(x)| \\ &\leqq qrL(h_v)+2qrL(h_u).\end{aligned} \tag{5.14}$$

We give the new restriction $q \leqq c_1/8M$. Since $r \leqq Md(\Delta)/c_1$, (5.10) and (5.13) imply

$$\begin{aligned} L(h_u)\,d(\Delta) = d(h_u\Delta) &\leqq d(f\Delta^0)+2\|h_u-f\|_{E_u} \\ &\leqq d(f\Delta^0)+4qrL(h_u) \\ &\leqq d(f\Delta^0)+L(h_u)d(\Delta)/2. \end{aligned}$$

Since $\Delta^0 \subset E_v$, this yields

$$\begin{aligned} L(h_u)\,d(\Delta) \leqq 2d(f\Delta^0) &\leqq 2d(h_v\Delta)+4\|h_v-f\|_{E_v} \\ &\leqq 2L(h_v)d(\Delta)+4qrL(h_v) < 3L(h_v)d(\Delta). \end{aligned}$$

Hence (5.14) gives

$$\|h_v-h_u\|_{\Delta^0} \leqq 7qrL(h_v).$$

By (5.13), $|x-u| \leqq b_1 r$ for every $x \in \Delta^0$. Since $\varrho(\Delta) \leqq M$ and $d(\Delta) \geqq c_1 r/M$, 2.12 yields

$$|h_v(u)-g(u)| \leqq M_2 qrL(h_v).$$

Furthermore,

$$r \leqq d(\sigma)+d(\sigma, A) \leqq (1+a_1^{-1})\varrho_n b(\sigma).$$

We set $M_3=2(1+a_1^{-1})\varrho_n(n+1)M_2$ and give the new restriction $q \leqq 1/M_3$. Then 2.7 implies that $g|\sigma$ is sense-preserving and that

$$(5.15) \qquad \begin{aligned} L(g|\sigma) \leqq L(h_v)(1+M_3 q), \quad l(g|\sigma) \geqq L(h_v)/(1+M_3 q), \\ H(g|\sigma) \leqq (1+M_3 q)^2. \end{aligned}$$

In the bilipschitz case $L(h_v)=1$, and hence $g|\sigma$ is $(1+M_3q)$-bilipschitz.

We use degree theory to show that g is a homeomorphism onto R^n. The topological degree $\mu(y, f, D)$ is an integer defined whenever D is a bounded domain in R^n, $f\colon \bar{D} \to R^n$ is continuous, and $y \in R^n \setminus f\partial D$; see e.g. [Do, IV. 5] or [RR, II. 2]. If $G \subset R^n$ is open and if $f\colon G \to R^n$ is continuous, f is said to be sense-preserving if $\mu(y, f, D) > 0$ whenever $\bar{D}$ is compact in G and $y \in fD \setminus f\partial D$.

We first show that $g|\text{int}\,A = f|\text{int}\,A$ is sense-preserving. Let V be a component of int A. Then there is a ball $B_V = B(x_V, r_1) \subset V \cap B$. By (5.6), $\varkappa(s, n) \leqq r_1/5R$, and therefore

$$\|f-\text{id}\|_{B_V} \leqq 2R\varkappa(s, n) < r_1/2.$$

Consequently, the segmental homotopy $h_t\colon f \simeq \text{id}$ satisfies $h_t(x_V) \notin h_t\partial B_V$, and thus

$$\mu(f(x_V), f, B_V) = \mu(x_V, \text{id}, B_V) = 1.$$

Since $f|V$ is an embedding, $f|\text{int}\,A$ is sense-preserving.

We next show that g is sense-preserving. Let $D \subset R^n$ be a bounded domain, and let $y \in gD \setminus g\partial D$. Set $Y = \partial A \cup |W^{n-1}|$, where W^{n-1} is the $(n-1)$-skeleton of W. Then int $Y = \emptyset$. Since $g|\partial A$ is an embedding and since $g|G$ is PL, we have int $gY = \emptyset$. Let U be the y-component of $R^n \setminus g\partial D$. Then $D_0 = D \cap g^{-1}U$ is open and nonempty, and so is $D_0 \setminus Y$. Since $g|R^n \setminus Y$ is an immersion, $g[D_0 \setminus Y]$ is open. Hence we can choose a point $z \in gD_0 \setminus gY$. Since $z \in U$, $\mu(y, g, D) = \mu(z, g, D)$. On the other hand, $D\cap g^{-1}(z)$ is a finite nonempty subset of $D\setminus Y$, and $g|D\setminus Y$ is a sense-preserving immersion. Hence

$$\mu(z, g, D) = \text{card}\,(D\cap g^{-1}(z)) > 0,$$

which implies that g is sense-preserving.

Clearly each fiber $g^{-1}(y)$ is countable. Consequently, g is light and sense-preserving, hence discrete and open [TY, Corollary, p. 333]. Furthermore, $g|R^n\setminus B$ is a homeomorphism onto a neighborhood of ∞. Indeed, if A is bounded, $g|R^n\setminus B=\text{id}$. If A is unbounded, $g|R^n\setminus B=f|R^n\setminus B$ is a QS embedding, and $g(x)\to\infty$ as $x\to\infty$. Hence there is a ball $B_1=B^n(R_1)$ containing B such that $gB\cap g[R^n\setminus B_1]=\emptyset$. Let V be the bounded component of $R^n\setminus g\partial B_1$. Then $\mu(y, g, B_1)=k$ is independent of $y\in V$. Choosing $y\in V\setminus gB$ we see $k=1$. Hence we obtain for every $y\in V$

$$1 = \mu(y, g, B_1) = \Sigma\{i(x, g)\colon x\in B_1\cap g^{-1}(y)\} \geqq \text{card}\, g^{-1}(y),$$

where $i(x, g)$ is the local degree of g at x. Thus g is a homeomorphism onto R^n.

In the bilipschitz case, it follows from (5.15) that g is L_1-bilipschitz with $L_1=\max(L, 1+M_3q)$. In the QS case, it follows from (5.15) and from a standard removability theorem [Vä$_1$, 35.1] that $g|G$ is $(1+M_3q)^{2n-2}-$QC. If ∂A is of σ-finite $(n-1)$-measure, [Vä$_1$, 35.1] implies that g is $K-$QC with

$$K = \max\left((1+s)^{n-1}, (1+M_3q)^{2n-2}\right),$$

and thus A has the QSEP. Since this case is sufficient in the applications 5.17 and 5.19, and since a detailed proof of the general case would take several pages, we only give a sketch of it.

To show that g is QC, it suffices to find a uniform upper bound for the metric dilatation $H(x, g)$, see [Vä$_1$, 34.1]. Once this has been done, the desired estimate for the dilatation of g is easily obtained by considering the derivative of g at points of density of ∂A.

Let $z\in\partial A$ and $x\in G$ with $|x-z|=r$, where r is small. Choose a suitable $c_2>1$ and apply 3.1 to find a similarity h such that $|h(y)-f(y)|\leqq M_4qL_hr$ for $y\in A\cap B(z, c_2r)$. It suffices to find M_5 such that

$$L_hr/M_5 \leqq |g(x)-f(z)| \leqq M_5L_hr. \tag{5.16}$$

The second inequality is fairly easy. With a small loss of generality, assume $x\in W^0$. Let $r_1=d(x, A)$ and choose $y\in A$ with $|y-x|=r_1$. We may assume that $E_x\subset A\cap B(z, c_2r)$. Then

$$\begin{aligned}|g(x)-f(z)| &\leqq |h_x(x)-h_x(y)|+|h_x(y)-f(y)|+|f(y)-h(y)|\\ &\quad+|h(y)-h(z)|+|h(z)-f(z)|\\ &\leqq L(h_x)r_1+L(h_x)qr_1+2M_4qL_hr+c_2L_hr.\end{aligned}$$

Thus it suffices to show that $L(h_x)r_1\leqq M_6L_hr$. For this, observe that E_x contains a point a with $r_1 \leqq 2M|a-y|$. Then

$$L(h_x)|a-y| = |h_x(a)-h_x(y)| \leqq |f(a)-f(y)|+4L(h_x)qM|a-y|.$$

Since we may assume that $q \leqq 1/8M$, this gives

$$L(h_x)|a-y| \leqq 2|f(a)-f(y)| \leqq 2L_h|a-y|+4M_4 qL_h r,$$

and hence

$$L(h_x)r_1 \leqq (2+M_4)L_h r.$$

The first inequality of (5.16) is harder. We first replace b_1 by $b_1' = \max(b_1, 2(2+a_1)(1+a_1))$ and show that this is no loss of generality. Choose σ with $x \in \sigma \in W$. Consider a vertex v of σ, set $\alpha r = d(v, A)$, and consider separately three cases: (1) $\alpha \leqq \alpha_0$ for a suitable small α_0, (2) $\alpha_0 < \alpha \leqq 1/b_1'$, (3) $\alpha > 1/b_1'$. □

5.17. Theorem. *Let $A \subset R^n$ be a compact $(n-1)$-dimensional C^1-manifold, with or without boundary. Then A has the extension properties in R^n.*

Proof. If $n=1$, then A is a finite set, and the result is obvious. Suppose $n \geqq 2$. For every $y \in A$, let $T(y)$ be the tangent $(n-1)$-plane of A at y, and let P_y: $R^n \to T(y)$ be the orthogonal projection. For $t>0$, set

$$D(y, t) = T(y) \cap B^n(y, t), \quad Z(y, t) = P_y^{-1} D(y, t).$$

Let $A(y, t)$ be the y-component of $A \cap Z(y, t)$. There it $t_0 > 0$ such that if $t \leqq t_0$, then $P_y|A(y, t)$ is injective and $A \cap B^n(y, t) \subset A(y, t)$. By compactness, we can choose t_0 to be independent of y. If $\partial A = \emptyset$, we have $P_y A(y, t) = D(y, t)$, but in any case, we can choose t_0 so that for $t \leqq t_0$, $P_y A(y, t) = C(y, t)$ contains a regular $(n-1)$-simplex Δ with $d(\Delta) = t/2$.

Let φ_y: $C(y, t) \to A(y, t)$ be the local inverse of P_y, satisfying $P_y \varphi_y = \mathrm{id}$. By differentiability, we can write

$$|\varphi_y(y+h)-(y+h)| \leqq |h|\varepsilon(|h|), \tag{5.18}$$

where ε: $[0, t_0] \to R^1$ is an increasing function and $\varepsilon(t) \to 0$ as $t \to 0$. By compactness, ε can be chosen to be independent of y.

We show that A satisfies the condition (a) of 5.5 with $b_1 = 3$. Let $0 \leqq \lambda \leqq 1$. Choose t_1, $0 < t_1 \leqq t_0$, such that $\varepsilon(t_1) \leqq \lambda/5$, and set $r_0 = t_1/5$. Assume that $x \in R^n \setminus A$ with $d(x, A) = r \leqq r_0$. Choose $y \in A$ with $|x-y| = r$. Then, with the notation of 5.5 we have

$$E(x, 3) \subset A \cap B(y, 5r) \subset A(y, 5r) \subset T(y) + \lambda r \bar{B}^n.$$

Let Δ_1 be a regular $(n-1)$-simplex in $C(y, r)$ with $d(\Delta_1) = r/2$, and let Δ be the simplex with $\Delta^0 = \varphi_y \Delta_1^0$. Since $\varepsilon(r) \leqq 1/5$, (5.18) implies that $d(\Delta) \leqq 2d(\Delta_1)$. Since $b(\Delta) \geqq b(\Delta_1)$, we have

$$\varrho(\Delta) \leqq 2\varrho(\Delta_1) = \varrho_0,$$

where ϱ_0 depends only on n. Furthermore, for every $z \in A(y, r)$ we have

$$|z-x| \leqq |z-P_y z|+|P_y z-y|+|y-x| \leqq r\varepsilon(r)+r+r < 3r.$$

Hence $A(y, r)\subset E(x, 3)$, which implies that Δ is a simplex of $E(x, 3)$. The theorem follows now from 5.5. □

5.19. Theorem. *Let $n\geqq 2$, and let $A\subset R^n$ be a finite union of simplexes of dimensions n and $n-1$. Then A has the extension properties in R^n.*

Proof. Suppose that A is a finite union of n-simplexes σ_j and $(n-1)$-simplexes Δ_k. We show that the conditions of 5.5 are satisfied with $b_1=2$, $b_2=3$.

Let M_0 be the maximum of all numbers $\varrho(\sigma_j)$ and $\varrho(\Delta_k)$, and let c_1 and c_2 be the minimum and the maximum, respectively, of the diameters of the simplexes. Choose $\alpha>0$ such that if H is a component of $R^n\setminus T(\Delta_j)$ and if Δ_k meets H, then Δ_k has a vertex v in H with $d(v, T(\Delta_j))\geqq\alpha$. Set $r_0=\min(c_1/3, \alpha/4)$.

Let $x\in R^n\setminus A$ with $d(x, A)=r\leqq r_0$. We divide the proof into three cases:

Case 1. $E(x, 2)$ is contained in some $(n-1)$-plane T. Now the condition (a_3) of 5.5 is trivially true for all λ. Choose $y\in A$ with $|y-x|=r$. Then $y\in\Delta_j$ for some j. There is an $(n-1)$-simplex Δ which is similar to Δ_j and satisfies the conditions

$$y\in\Delta\subset\Delta_j\cap\bar{B}(x, 2r),\quad d(\Delta)\geqq r.$$

Hence (a) is true with $M=M_0$.

Case 2. $E(x, 2)$ meets σ_j for some j. Now there is an n-simplex σ which is similar to σ_j and satisfies the conditions

$$\sigma\subset\sigma_j\cap\bar{B}(x, 3r),\quad d(\sigma)\geqq r.$$

Obviously σ is M_0-related to σ_j in σ_j and hence in A. Thus (b) is true with $M=\max(M_0, 1/c_1)$.

Case 3. The cases 1 and 2 do not occur. Now $E(x, 2)$ meets two $(n-1)$-simplexes Δ_j and Δ_k which are not contained in an $(n-1)$-plane. Choose $y\in A$ with $|x-y|=r$. We may assume that $y\in\Delta_j$. To simplify notation, we assume that $y=0$ and that $T(\Delta_j)=R^{n-1}$. Choose $z_0\in\Delta_k\cap\bar{B}(x, 2r)$. We may assume that $z_0\in R^n_+$ and that $\Delta_k\cap\operatorname{int} R^n_+\neq\emptyset$. Let $P\colon R^n\to R^1$ be the projection $P(x)=x_n$, and choose a point $v\in\Delta_k$ at which P attains its maximum. Then $P(v)\geqq\alpha$. Since $|v-z_0|\geqq\alpha-3r\geqq r$, there is a point z on the segment vz_0 with $|z-z_0|=r$. Choose $\mu\in(0, 1]$ such that the $(n-1)$-simplex $\Delta_0=\mu\Delta_j$ is contained in $\bar{B}(r)$ and meets $S(r)$. Then $r\leqq d(\Delta_0)\leqq 2r$ and thus $r/c_2\leqq\mu\leqq 2r/c_1$. Furthermore, $\sigma=z\Delta_0$ is an n-simplex of $E(x, 3)$. It suffices to show that σ is M-related to $v\Delta_j$ in $\Delta_j\cup\Delta_k$ for some M depending only on A. This is done by deforming $z\Delta_0$ to $v\Delta_j$ by a parameter $t\in[0, 1]$ so that an intermediate simplex is $\sigma_t=z_t\Delta_t$ where

$$z_t=(1-t)z+tv,\quad \Delta_t=\mu_t\Delta_0,\quad \mu_t=1-t+t/\mu.$$

By 2.15, it suffices to find an upper bound for the numbers

$$\beta_1=\frac{d(z_t, \Delta_t)}{d(\Delta_t)},\quad \beta_2=\frac{d(\Delta_t)}{P(z_t)}.$$

We have

$$\beta_1 \leqq \frac{|z_t|}{\mu_t r} \leqq \frac{(1-t)|z|+t|v|}{(1-t+t/\mu)r} = \gamma_1(t).$$

Since γ_1 is monotone and since

$$\gamma_1(0) = \frac{|z|}{r} \leqq \frac{|z-z_0|+|z_0|}{r} \leqq 4,$$

$$\gamma_1(1) = \frac{\mu|v|}{r} \leqq 2(|v-z_0|+|z_0|)/c_1 \leqq 4c_2/c_1,$$

we obtain $\beta_1 \leqq 4c_2/c_1$. To estimate β_2 observe first that

$$\frac{|z-z_0|}{|v-z_0|} = \frac{P(z)-P(z_0)}{P(v)-P(z_0)} \leqq \frac{P(z)}{P(v)} \leqq \frac{P(z)}{\alpha},$$

which implies $P(z) \geqq \alpha r/c_2$. Since

$$\beta_2 = \frac{\mu_t d(\Delta_0)}{P(z_t)} \leqq \frac{2r(1-t+t/\mu)}{(1-t)P(z)+tP(v)},$$

we can argue as above and obtain $\beta_2 \leqq 2c_2/\alpha$. □

5.20. Corollary. *Let* $n \geqq 2$, *and let* $A \subset R^n$ *be a compact* PL *manifold of dimension n or* $n-1$, *with or without boundary. Then A has the extension properties in* R^n. □

5.21. *Remarks.* It is not possible to extend 5.17 and 5.20 to LIP manifolds. For example, a LIP circle in R^2 need not have the extension properties in R^2, see 7.10.

One can also consider bilipschitz and QS extension without the condition that the bilipschitz constants or the dilatations are small. For example, Gehring [Ge_1, Corollary 2, p. 218] proved that if A is a QS circle in R^2, every L-bilipcshitz $f\colon A \to R^2$ can be extended to an L_1-bilipschitz $g\colon R^2 \to R^2$, $L_1 = L_1(L, A)$. In higher dimensions $n \neq 4$, similar extension is possible if, for example, A and fA are QC spheres, see [TV_5, 2.19].

5.22. *Open problems.* 1. Are 5.17 and 5.20 true for p-dimensional manifolds, $p \leqq n-2$?

2. Does A in 5.17 and in 5.20 have the extension properties in (R^n, Y) for $\dim Y > n$?

3. Does every compact polyhedron in R^n have the extension properties in R^n?

5.23. Example. Let $A \subset R^2$ be the well-known snow-flake curve see e.g. [Ma, p. 42]. There is a family of equilateral triangles associated with A in a natural way. It is easy to see that these are mutually M-related in A with some M. It follows from 5.5 that A has the extension properties in R^2. A stronger result will be given in 6.13.2.

6. Thick sets

6.1. In this section we give a sufficient condition for a set $A\subset R^p$ to have the extension properties in (R^n, Y) for $p\leqq n\leqq \dim Y$. The condition is somewhat similar to the condition (b) of Theorem 5.5, but it does not involve the notion of M-relatedness. On the other hand, it must be valid at all boundary points and there is no choice between two conditions as in 5.5. We show that the condition applies, for example, to all compact convex sets and to QS p-cells.

We say that a set $A\subset R^p$ is *thick* in R^p if there are $r_0>0$ and $\beta>0$ such that if $y\in\partial A$ and if $0<r\leqq r_0$, then there is a p-simplex Δ such that $\Delta^0\subset A\cap\bar{B}(y, r)$ and $m_p(\Delta)\geqq\beta r^p$. This implies that $\varrho(\Delta)\leqq M$ and $d(\Delta)\geqq r/M$ for some $M=M(\beta, p)$. Conversely, these inequalities imply that $m_p(\Delta)\geqq\beta r^p$ with $\beta=\beta(M, p)>0$. Examples of thick sets are given in 6.13.

6.2. Theorem. *Suppose that A is closed and thick in R^p and that either A or $R^p\setminus A$ is bounded. Then A has the extension properties in* (R^n, Y) *whenever Y is a linear subspace of l_2 and* $p\leqq n\leqq\dim Y$.

Proof. By 4.4 it suffices to show that A has the extension properties in (R^p, Y). We again choose an auxiliary parameter $q\in(0, 1]$. To prove the QSEP it suffices to find $q_0\in(0, 1]$ and for every $q\in(0, q_0]$ a number $s=s(q, A)>0$ such that every $s-$QS map $f\colon A\to Y$ has an s_1-QS extension $g\colon R^p\to Y$ where $s_1=s_1(q, A)\to 0$ as $q\to 0$. In the bilipschitz case, we find $L=L(q, A)>1$ such that every L-bilipschitz $f\colon A\to Y$ has an L_1-bilipschitz extension $g\colon R^p\to Y$ where $L_1=L_1(q, A)\to 1$ as $q\to 0$.

The basic idea of the proof is the same as in 5.5. However, the number b corresponding to the constant b_2 of 5.5 will depend on q. In fact, $b\to\infty$ as $q\to 0$. No use will be made of sense-preservation.

Let $r_0>0$ and $\beta>0$ be the numbers given in the definition of thickness, and let $M=M(\beta, p)\geqq 1$ be as in 6.1. Set

$$c = q^{-1/3}, \quad b = 2+3c,$$

and choose $R>0$ such that $\partial A\subset\bar{B}^p(R/b)$. Choose $s=s(q, A)\in(0, q]$ such that

$$\varkappa(s, p) \leqq \min(q^2 r_0/2R, q/2b), \tag{6.3}$$

where $\varkappa$ is given by 3.1. We show that s is the required number provided that q is sufficiently small. In the bilipschitz case we set $L=(s^2+1)^{1/2}$.

We may assume that $R^p\subset Y$. Suppose that $f\colon A\to Y$ is $s-$QS By 3.1, there is a similarity $h\colon R^p\to Y$ such that setting $B=\bar{B}^p(R)$ we have

$$\|h-f\|_{A\cap B} \leqq \varkappa(s, p) L_h\, d(A\cap B) \leqq q^2 r_0 L_h.$$

Extending h to a bijective similarity h_1 of Y and replacing f by $h_1^{-1}f$ we may assume that

$$\|f-\mathrm{id}\|_{A\cap B} \leqq q^2 r_0. \tag{6.4}$$

Set $G=R^p\setminus A$. For every $x\in R^p$ we set $r_x=d(x, A)$ and choose $a_x\in A$ with $|a_x-x|=r_x$. We also set $E_x=A\cap\bar{B}(x, br_x)$. Then clearly $E_x\subset B$ for all $x\in G$.

We first show that

$$r_y \leqq (1+c)r_x, \quad A\cap\bar{B}(a_y, r_y)\subset E_x\cap E_y \tag{6.5}$$

whenever $x, y\in G$ with $|y-x|\leqq cr_x$. The first inequality is obvious. If $z\in A\cap \bar{B}(a_y, r_y)$, then

$$|z-x| \leqq |z-a_y|+|a_y-y|+|y-x| \leqq r_y+r_y+cr_x \leqq br_x.$$

Hence $z\in E_x$. Furthermore, since $b\geqq 5$, we have

$$|z-y| \leqq |z-a_y|+|a_y-y| \leqq br_y,$$

which implies $z\in E_y$ and proves (6.5).

We associate to every $x\in G$ a similarity $h_x\colon R^p\to Y$ as follows: If $r_x\geqq qr_0$, we set $h_x=\mathrm{id}$. If $r_x<qr_0$, we apply 3.1 and choose h_x such that

$$\|h_x-f\|_{E_x} \leqq \varkappa(s, p)L(h_x)\, d(E_x) \leqq 2br_xL(h_x)\varkappa(s, p).$$

By (6.3) this yields

$$\|h_x-f\|_{E_x} \leqq qr_xL(h_x). \tag{6.6}$$

By (6.4), this is valid for all $x\in G$. In the bilipschitz case h_x is chosen to be an isometry.

In what follows, we let $M_1, M_2, \ldots$ denote numbers $M_j\geqq 1$ depending only on A. We next show that

$$L(h_y)r_y \leqq 5MbL(h_x)r_x, \quad |h_x(y)-h_y(y)| \leqq M_1q^{2/3}r_xL(h_x), \tag{6.7}$$

whenever $x, y\in G$, $|y-x|\leqq cr_x$ and $r_x\leqq r_0/(1+c)$.

By (6.5), we have $r_y\leqq (1+c)r_x\leqq r_0$. Hence there is a p-simplex Δ of $A\cap\bar{B}(a_y, r_y)$ such that $d(\Delta)\geqq r_y/M$ and $\varrho(\Delta)\leqq M$. By (6.5) we have $\Delta^0\subset E_x\cap E_y$. We give the restriction

$$q \leqq 1/4M.$$

Now (6.6) implies

$$\begin{aligned} L(h_y)r_y &\leqq ML(h_y)\, d(\Delta) = Md(h_y\Delta) \\ &\leqq M\big(d(f\Delta^0)+2qr_yL(h_y)\big) \\ &\leqq Md(f\Delta^0)+r_yL(h_y)/2. \end{aligned}$$

Hence

$$L(h_y)r_y \leqq 2M\, d(f\Delta^0) \leqq 2M\big(L(h_x)\, d(\Delta)+2qr_xL(h_x)\big).$$

Since $d(\Delta)\leqq d(E_x)\leqq 2br_x$ and since $q\leqq 1\leqq b/4$, we obtain the first inequality of (6.7).

To prove the second inequality, we first obtain

$$\begin{aligned}\|h_x-h_y\|_{\Delta^0} &\leqq \|h_x-f\|_{E_x}+\|f-h_y\|_{E_y}\\ &\leqq qr_xL(h_x)+qr_yL(h_y)\\ &\leqq (1+5Mb)qr_xL(h_x).\end{aligned}$$

Since

$$1+5Mb \leqq 6Mb \leqq 30Mc = 30Mq^{-1/3},$$

and since $\varrho(\Delta)\leqq M$, 2.11 yields

$$|h_x(y)-h_y(y)| \leqq 30Mq^{2/3}r_xL(h_x)(1+d(\Delta)^{-1}M_2|y-z_1|),$$

where z_1 is an arbitrary vertex of Δ. Since $d(\Delta)\geqq r_y/M$ and $|y-z_1|\leqq|y-a_y|+|a_y-z_1|\leqq 2r_y$, we obtain the second inequality of (6.7).

Choose a Whitney triangulation W of G as in 5.1. Thus the p-simplexes σ of W satisfy the conditions (5.2):

$$\varrho(\sigma) \leqq \varrho_p, \quad a_1 \leqq r_G(\sigma) \leqq a_2,$$

where ϱ_p, a_1, a_2 depend only on p. We may assume that $a_2\leqq 1$. As in the proof of 5.5, we define $g(v)=h_v(v)$ for every vertex v of W, extend affinely to all simplexes of W, and set $g|A=f$. We shall show that g is the desired extension of f provided that q is sufficiently small.

Since $a_2\leqq 1$, we see that $g(x)=x$ whenever $r_x\leqq 2qr_0$.

We omit the proof for the continuity of g, since it is similar to the corresponding proof in 5.5.

We first show that

$$|h_x(x)-g(x)| \leqq M_1q^{2/3}r_xL(h_x) \tag{6.8}$$

whenever $x\in G$ and $r_x\leqq r_0/(1+c)$. Choose a p-simplex $\sigma\in W$ containing x. For every vertex v of σ we have

$$|v-x| \leqq d(\sigma) \leqq a_2r_x \leqq cr_x.$$

Hence (6.7) implies

$$|h_x(v)-g(v)| \leqq M_1q^{2/3}r_xL(h_x).$$

Since h_x-g is affine in σ, this yields (6.8).

We next show that

$$|h_x(y)-g(y)| \leqq M_3q^{1/3}r_xL(h_x) \tag{6.9}$$

whenever $x\in G$, $r_x\leqq r_0/(1+c)^2$ and $|y-x|\leqq cr_x$. If $y\in A$, then $y\in E_x$, and (6.9) follows from (6.6) with $M_3=1$. Suppose that $y\in G$. Since $r_y\leqq(1+c)r_x\leqq r_0/(1+c)$, (6.7) and (6.8) imply

$$\begin{aligned}|h_x(y)-g(y)| &\leqq |h_x(y)-h_y(y)|+|h_y(y)-g(y)|\\ &\leqq M_1q^{2/3}r_xL(h_x)+M_1q^{2/3}r_yL(h_y).\end{aligned}$$

Since $b\leqq 5c=5q^{-1/3}$, (6.9) follows from the first inequality of (6.7) with $M_3=26MM_1$.

We must show that g is s_1-QS, where $s_1=s_1(q, A)\to 0$ as $q\to 0$. In the bilipschitz case, we must show that g is L_1-bilipschitz, where $L_1=L_1(q, A)\to 1$ as $q\to 0$. We omit the proof of the QS case, since it would take several pages of elementary and dull reasoning, where one would consider several cases and subcases according to the situation of a triple (a, b, x) of points in R^p. We give in detail the proof for the bilipschitz case, which is simpler. Assume that $f\colon A\to Y$ is L-bilipschitz satisfying (6.3) with $s=(L^2-1)^{1/2}$. Now each h_x is an isometry, and thus $L(h_x)=1$.

For every p-simplex σ of W, we let K_σ denote the subcomplex of W generated by all p-simplexes meeting σ. The underlying polyhedron $N(\sigma)=|K_\sigma|$ is a neighborhood of σ in R^p. From the construction of W it follows that there are positive numbers a_3 and a_4 depending only on p such that

$$d(\sigma, R^p\setminus N(\sigma)) \geqq a_3 r_x, \quad |y-x| \leqq a_4 r_x,$$

whenever $\sigma\in W$ is a p-simplex, $x\in\sigma$, and $y\in N(\sigma)$. Moreover, the complexes K_σ belong to a finite number of similarity classes. By 2.14, there exist a number $\alpha_0=\alpha_0(p)>0$ and for every $\alpha\in(0, \alpha_0]$ a number $L_2=L_2(\alpha, p)$ such that $L_2(\alpha, p)\to 1$ as $\alpha\to 0$ and such that if $\varphi\colon N(\sigma)\to l_2$ is affine on each simplex of $K(\sigma)$ and if $|\varphi(v)-h(v)|\leqq\alpha d(\sigma)$ for some isometry $h\colon R^p\to l_2$ and for every vertex v of $K(\sigma)$, then φ is L_2-bilipschitz.

We give the following new restrictions on q:

$$(6.10)\qquad q \leqq a_4^{-3}, \quad q \leqq \alpha_0^3 a_1^3 M_3^{-3}, \quad 2q \leqq (1+c)^{-2},$$

which are of the form $q\leqq q_0(A)$. We show that for every p-simplex $\sigma\in W$, $g|N(\sigma)$ is L_3-bilipschitz with $L_3=L_3(q, A)=L_2(M_3q^{1/3}/a_1, p)$. Choose $x\in\sigma$ with $r_x=d(\sigma, A)$. If $r_x\geqq r_0/(1+c)^2$, the last inequality of (6.10) implies that $r_y\geqq qr_0$ for all $y\in N(\sigma)$, and hence $g|N(\sigma)=\mathrm{id}$. If $r_x\leqq r_0/(1+c)^2$, then (6.10) implies that $N(\sigma)\subset B(x, cr_x)$. Hence (6.9) yields

$$|h_x(y)-g(y)| \leqq \alpha d(\sigma)$$

for $y\in N(\sigma)$ with $\alpha=M_3q^{1/3}/a_1\leqq\alpha_0$. Thus $g|N(\sigma)$ is L_3-bilipschitz.

It follows that g is L_4-lipschitz with $L_4=L_4(q, A)=\max(L, L_3)$. We assume that q is so small that $L_4\leqq 2$. It remains to find $L_5=L_5(q, A)$ such that

$$(6.11)\qquad |g(x)-g(y)| \geqq |x-y|/L_5$$

for all $x, y\in R^p$ and such that $L_5(q, A)\to 1$ as $q\to 0$. We may assume that $r_y\leqq r_x$. We consider three cases.

Case 1. $r_x=0$. Now $x, y\in A$, and (6.11) holds with $L_5=L$.

Case 2. $0<r_x\leqq r_0/(1+c)^2$. Choose a p-simplex $\sigma\in W$ containing x. If $y\in N(\sigma)$, (6.11) holds with $L_5=L_3$. If $y\in B(x, cr_x)\setminus N(\sigma)$, then $|x-y|\geqq a_3r_x$, and (6.9)

yields

$$\begin{aligned}|g(x)-g(y)| &\geqq |h_x(x)-h_x(y)|-|h_x(x)-g(x)|-|h_x(y)-g(y)|\\ &\geqq |x-y|-2M_3q^{1/3}r_x\\ &\geqq |x-y|(1-2M_3q^{1/3}/a_3),\end{aligned}$$

which implies (6.11) for small q. Finally assume that $|x-y|\geqq cr_x$. Since g is 2-lipschitz, we obtain

$$\begin{aligned}|g(x)-g(y)| &\geqq |g(a_x)-g(a_y)|-|g(x)-g(a_x)|-|g(y)-g(a_y)|\\ &\geqq |a_x-a_y|/L-2r_x-2r_y\\ &\geqq (|x-y|-2r_x)/L-4r_x\\ &\geqq |x-y|(1-2q^{1/3})/L-4q^{1/3}|x-y|,\end{aligned}$$

which gives (6.11) for small q.

Case 3. $r_x\geqq r_0/(1+c)^2$. Now $g(x)=x$. If $r_y\geqq 2qr_0$, then $g(y)=y$, and (6.11) is trivial. Assume that $r_y\leqq 2qr_0$. If $y\in G$ or if $|y|\leqq R$, (6.4) gives

$$\begin{aligned}|g(x)-g(y)| &\geqq |x-y|-|y-a_y|-|a_y-g(a_y)|-|g(a_y)-g(y)|\\ &\geqq |x-y|-2qr_0-q^2r_0-2|a_y-y|\\ &\geqq |x-y|-7qr_0.\end{aligned}$$

Since $|x-y|\geqq r_0/(1+c)^2-2qr_0$, we again obtain (6.11) for small q. Finally, assume that $y\in A$ and $|y|\geqq R$. Now $r_x\leqq R/b$ and $|x-y|\geqq R-R/b$. Since g is 2-lipschitz and since $b-1=1+3c>q^{1/3}$, we obtain

$$\begin{aligned}|g(x)-g(y)| &\geqq |g(a_x)-g(y)|-|g(x)-g(a_x)|\\ &\geqq |a_x-y|/L-2r_x\\ &\geqq |x-y|(1-q^{1/3})/L-2q^{1/3}|x-y|,\end{aligned}$$

which again implies (6.11) for small q. □

6.12. Remarks. Inspection of the proof of 6.2 gives the following information on the constants L_0 and L_1 of the BLEP: L_0 depends only on $r_0/d(\partial A)$, β and n, and L_1 depends, in addition, only on L. In particular, these numbers do not depend on Y. In the case $p=n$, we can choose g to be an isometry outside a given neighborhood U of A; then L_0 depends also on U.

A similar statement is true for the QSEP. Then one can choose $g|R^p\setminus U$ to be a similarity.

6.13. Examples. 1. Suppose that a domain $D\subset R^p$ is a John domain, see e.g. [MS]. It is then easy to see that $\bar{D}$ is thick in R^p, and has therefore the extension properties in (R^n, Y). In particular, this is true if D is a bounded uniform domain [GM, 2.18]; in particular, if D is a QS ball [Vä$_2$, 5.6]; in particular if D is bounded and convex. It follows that every convex compact set in R^n has the extension properties in (R^n, Y).

2. Let A be the snow-flake curve in R^2. It is easy to see that A is thick in R^2, and has thus the extension properties in (R^n, Y) for $2 \leqq n \leqq \dim Y$. This strengthens the result of 5.23. Since A is a QS circle, we see that thickness is not a QS invariant property.

3. The Cantor middle-third set is thick in R^1.

4. If A_1 is thick in R^p and A_2 is thick in R^q, then $A_1 \times A_2$ is thick in R^{p+q}.

5. If A is any closed set in R^p and if $r_0 > 0$, then $A + r_0 \bar{B}^p$ is thick in R^p with constants r_0 and $\beta = \beta(p)$. This observation will be used in Section 8.

7. Examples

7.1. In this section we give several examples of sets $A \subset R^n$ which have neither of the extension properties in R^n or in (R^n, Y) for some Y. To show this, it suffices to construct a sequence of L_k-bilipschitz maps $f_k\colon A \to Y$ such that $L_k \to 1$ and such that there are no s_k–QS extensions $g_k\colon R^n \to Y$ of f_k such that $s_k \to 0$. In 7.5, we give an example of a set which has the BLEP but not the QSEP in R^2.

7.2. Lemma. *Let $n \geqq 2$, let $1 \leqq L < b$, and let x_0, y_0 be points in $R^n \setminus \{0\}$ such that $|x_0|/L \leqq |y_0| \leqq L|x_0|$. Then there is an L_1-bilipschitz map $h\colon R^n \to R^n$ such that*

(1) $h(x_0) = y_0$,

(2) $h(x) = x$ *if* $|x| \leqq |x_0|/b$ *or* $|x| \geqq b|x_0|$,

(3) $L_1 = L_1(L, b) \to 1$ *as* $L \to 1$ *and* $b \to \infty$.

If, in addition, $|y_0 - x_0| \leqq \delta |x_0|$, *one can replace* (3) *by*

(3′) $L_1 = L_1(\delta, b) \to 1$ *as* $\delta \to 0$.

Proof. We may assume $n=2$. The map h can be constructed as the map f on p. 205 of [Ge$_1$], combined with a simple radial map. The last statement is clear. □

7.3. Let $x_1, x_2, \ldots$ be a strictly decreasing sequence of positive numbers such that $x_{k+1}/x_k \to 0$ and thus $x_k \to 0$. Then $A = \{0\} \cup \{x_k\colon k \in N\}$ has neither of the extension properties in R^1. To see this, define $f_k\colon A \to R^1$ by $f_k(x_k) = -x_k$ and by $f_k(x) = x$ for $x \neq x_k$. Then f_k is L_k-bilipschitz with $L_k \to 1$, but f_k has no extension to a homeomorphism $g\colon R^1 \to R^1$.

7.4. Let A be as in 7.3. We show that A has the BLEP in R^n for $n \geqq 2$. Suppose that $f\colon A \to R^n$ is L-bilipschitz with L close to one. We may assume that $f(0) = 0$ and that $\|f - \mathrm{id}\|_A$ is small (Theorem 3.1). Choose disjoint annuli $A_j = \{x \in R^n\colon x_j/b_j < |x| < b_j x_j\}$ where $b_j \to \infty$ as $j \to \infty$. Then Lemma 7.2 gives easily an L_1-bilipschitz extension $g\colon R^n \to R^n$ of f such that L_1 is close to one.

7.5. Let A be as in 7.3 and in 7.4 with $x_n = e^{-n!}$. We show that A does not have the QSEP in any connected set. In particular, A has the BLEP but not the QSEP in R^2. Fix a positive integer k, and define a map $f_k\colon A \to R^1$ as follows: Set $\varphi(x) = -1/\log x$. Then $f_k(0)=0, f_k(x)=\varphi(x)$ for $x \leqq x_k$, and $f_k(x)=\varphi(x)+\varphi'(x_k)(x-x_k)$ for $x \geqq x_k$. An elementary but tedious proof shows that f_k is s_k–QS where $s_k \to 0$ as $k \to \infty$. However, f_k has no QS extension to any connected set, since by [TV$_1$, 3.14], this extension would be Hölder continuous at the origin.

7.6. Let $A \subset R^2$ be the union of R^1 and the line segments $J_k = 2^k \times [0, 1]$, $k \in N$. Define $f_k\colon A \to R^2$ by $f_k(x, y) = (x, -y)$ for $(x, y) \in J_k$ and by $f_k|(A \setminus J_k) = \mathrm{id}$. Then f_k is L_k-bilipschitz where $L_k \to 1$ as $k \to \infty$. Since f_k has no extension to a homeomorphism of R^2, A has neither of the extension properties in R^2.

7.7. The preceding example can easily be modified to a compact set $A \subset R^2$ with the same property. This set consists of the horizontal segment $I = [0, 1]$ and of the vertical segments $\{1/k\} \times [0, 2^{-k}]$.

7.8. We modify the preceding example so that A will be an arc. Set

$$E = \{(x, y) \in R^2\colon |x| \leqq 1, y = 1 - |x|^{1/2}\}.$$

The intervals $\Delta_k = [1/k - 2^{-k}, 1/k + 2^{-k}]$ are disjoint for $k \geqq 7$. Let A be the arc obtained from I by replacing each Δ_k, $k \geqq 7$, by $E_k = 2^{-k}A + 1/k$. Define again $f_k\colon A \to R^2$ by $f_k(x, y) = (x, -y)$ for $(x, y) \in E_k$ and by $f_k|(A \setminus E_k) = \mathrm{id}$. Then f_k is L_k-bilipschitz with $L_k \to 1$. Now f_k has an extension to a homeomorphism $g\colon R^2 \to R^2$, but g cannot be QC and hence not bilipschitz.

A related example has recently been given by Gehring [Ge$_2$].

7.9. We replace the arc E of 7.8 by the PL arc E' with consecutive vertices $-e_1$, $-e_1+e_2$, e_1+e_2, e_1. We obtain an arc $A' \subset R^2$. Define $f_k\colon A' \to R^2$ as before. Again f_k is L_k-bilipschitz with $L_k \to 1$. Now f_k has an extension to a bilipschitz homeomorphism $g_k\colon R^2 \to R^2$, but g_k cannot be chosen to be s_k–QS with $s_k \to 0$, since g_k maps an angle $\pi/2$ onto an angle $3\pi/2$.

Observe that A' is a LIP arc, that is, a bilipschitz image of I. By 5.17 and 5.10, all DIFF and PL arcs in R^2 have the extension properties.

7.10. It is easy to enlarge the arc A' of the preceding example to a LIP circle A'' which has neither of the extension properties in R^2. On the other hand, if D is the bounded component of $R^2 \setminus A''$, then D is a bilipschitz disc, and hence $\bar{D}$ has the extension properties in R^2 by 6.13.1.

7.11. Similar examples can be given in higher dimensions. For example, a LIP arc in R^3 without the extension properties can be obtained from the preceding example by replacing the arc E' by the PL arc with vertices $-e_1$, $-e_1+e_2$, $-e_1+e_2+e_3$, $e_1+e_2+e_3$, e_1+e_2, e_1.

7.12. We next give an example of a set $A \subset R^3$ without the extension properties such that A is the closure of a domain. I do not know whether such an example exists in R^2. Set

$$D_1 = R^2 \times (1, \infty),$$
$$D_2 = R^2 \times (-\infty, 0),$$
$$Z_k = B^2(ke_1, 1/k) \times I,$$
$$G = D_1 \cup D_2 \cup Z_2 \cup Z_3 \cup \dots .$$

Now G is a domain in R^3. We shall prove that $A = \bar{G}$ has neither of the extension properties in R^3.

For $k \geqq 2$ define a homeomorphism $f_k \colon A \to A$ as follows: Outside Z_k, f_k is the identity map. In Z_k, f_k is the twist

$$f_k(k + re^{i\varphi}, t) = (k + re^{i(\varphi + 2\pi t)}, t).$$

Since the cylinders Z_k become very thin for large k, it is easy to see that f_k is L_k-bilipschitz with $L_k \to 1$.

We show that f_k has no extension to a homeomorphism $g \colon R^3 \to R^3$. Suppose that g is such an extension, and assume $k \geqq 3$. Define a path $\alpha \colon I \to R^3$ by $\alpha(s) = 2e_1 + se_3$. Next choose a natural path homotopy $H_t \colon I \to R^3$ of α such that H_t is a PL path with vertices $2e_1, x_t, x_t + e_3, 2e_1 + e_3$, where

$$x_t = \big(2(1-t) + t(k - 1/k)\big)e_1 .$$

Let $P \colon R^3 \to R^2$ be the orthogonal projection. Now $PgH_t \colon I \to R^2$ is a path homotopy in $R^2 \setminus \{ke_1\}$. Hence PgH_1 is null-homotopic in $R^2 \setminus \{ke_1\}$, which is clearly a contradiction.

7.13. We can easily modify the preceding example to a compact set A which consists of a closed 3-ball B together with a sequence of handles Z_k which are very thin for large k. We can choose these handles so that $d(Z_k) \to 0$. Now remove a thin slice E_k from each handle Z_k. In the situation of 7.12 E_k could correspond to the set $B^2(ke_1, 1/k) \times (0, 2^{-k})$. We obtain a set Q, which is a locally flat TOP 3-cell. If $f_k \colon Q \to Q$ is defined as in 7.12, f_k can be extended to a homeomorphism $g \colon R^3 \to R^3$. However, one can show that g cannot be QC. Hence Q has neither of the extension properties in R^3. Remember that by 6.13.1, every QS n-cell has the extension properties in R^n.

7.14. We give an example of a set $A \subset R^2$ which has the extension properties in R^2 but not in (R^2, R^3), or equivalently, in R^3. Let Q_0 be the square $I \times I$, and set inductively $Q_k = [a_k, a_k + 1] \times I$ where $a_0 = 0$, $a_k = a_{k-1} + 1 + 1/k$. Removing the squares $Q_0, Q_1, \dots$ from R^2 we obtain a domain G. We show that $A = \bar{G}$ has the BLEP in R^2; the QSEP can be proved in a similar manner. Suppose that $f \colon A \to R^2$ is L-bilipschitz with L close to one. By 5.19, ∂Q_j has the BLEP in R^2. Hence $f | \partial Q_j$ can be extended to an L_1-bilipschitz $g_j \colon Q_j \to R^2$ where $L_1 = L_1(L) \to 1$ as $L \to 1$.

Since each Q_j is a square, L_1 does not depend on j. These maps give an extension $g: R^2 \to R^2$ of f, which is L_2-bilipschitz with $L_2 = \max(L, L_1)$.

To show that A does not have the extension properties in (R^2, R^3) define $f_k: A \to R^3$ as follows. Let R_k be the rectangle $[a_k+1, a_{k+1}] \times I$. Let f_k be the identity outside R_k, and let $f_k|R_k$ be a twist, see 7.12. Then f_k is L_k-bilipschitz with $L_k \to 1$. As in 7.12, one can show that f_k has no extension to a homeomorphism of R^3.

7.15. Suppose that X and Y are linear subspaces of l_2 with $\infty > \dim X \leqq \dim Y$, and let $A \subset X$. It is natural to ask how the extension properties of A in (X, Y) depend on X and Y. By 4.4, they are independent of X. However, the examples 7.3, 7.4 and 7.14 show that they depend essentially on Y. More precisely, if $Y_1 \subset Y_2$ with $\dim Y_1 < \dim Y_2$, the extension properties of A in (X, Y_1) do not imply and are not implied by the extension properties of A in (X, Y_2).

7.16. Suppose that A is an infinite-dimensional linear subspace of l_2 with $\bar{A} \neq l_2$. Then there is an isometry $f: A \to l_2$ such that fA is dense in l_2. Hence A has neither of the extension properties in l_2. It seems to the author that the notions BLEP and QSEP are only useful for finite-dimensional sets A.

8. Supplementary results

In this section we give some general remarks on the extension properties. We first show that if A is compact, the extensions can be chosen to be very elementary outside a given neighborhood of A.

8.1. Theorem. *Suppose that $A \subset R^n$ is compact and has the* BLEP *in (R^n, Y), where Y is a linear subspace of l_2. Let U be a neighborhood of A. Then there is $L_0 > 1$ such that if $1 \leqq L \leqq L_0$, then every L-bilipschitz $f: A \to Y$ has an L_1-bilipschitz extension $g: R^n \to Y$ such that $L_1 = L_1(L, A, U, n, Y) \to 1$ as $L \to 1$ and such that $g|R^n \setminus U$ is an isometry.*

A similar statement is true for the QSEP; *then $g|R^n \setminus U$ is a similarity.*

Proof. We prove the first part of the theorem; the proof for the QS case is similar. Let $L_0' > 1$ and $L_1'(L, A, n, Y)$ be the numbers given by the BLEP of A in (R^n, Y). Set $r_0 = d(A, \partial U)/2$ and $E = A + r_0 \bar{B}^n$. By 6.2 and 6.13.5, E has the BLEP in (R^n, Y). More precisely, it follows from 6.12 that there is $L_0'' = L_0''(r_0/d(E), n) > 1$ such that if $1 \leqq L \leqq L_0''$, then every L-bilipschitz $f: E \to Y$ has an L_1''-bilipschitz extension $g: R^n \to Y$ such that $L_1'' = L_1''(L, r_0/d(E), n) \to 1$ as $L \to 1$ and such that $g|R^n \setminus U$ is an isometry. Choose $L_0 > 1$ such that $L_0 \leqq L_0'$ and such that $L_1'(L, A, n, y) \leqq L_0''$ for $1 \leqq L \leqq L_0$. Suppose that $1 \leqq L \leqq L_0$ and that $f: A \to Y$ is L-bilipschitz. Then f has an $L_1'(L, A, n, Y)$-bilipschitz extension $h: R^n \to Y$. Now there is an extension $g: R^n \to Y$ of $h|E$ such that $g|R^n \setminus U$ is an isometry and g is L_1-bilipschitz with

$$L_1 = L_1''(L_1'(L, A, n, Y), r_0/d(E), n). \quad \square$$

Observe that the proof of 8.1 made use only of the fact that A has the BLEP in (U, Y). Hence we obtain:

8.2. Theorem. *Suppose that* $A\subset R^n$ *is compact, that* U *is a neighborhood of* A, *that* Y *is a linear subspace of* l_2, *and that* A *has the* BLEP *or the* QSEP *in* (U, Y). *Then* A *has the same property in* (R^n, Y). □

8.3. Remark. On the other hand, there seems to be genuine problems if we consider *local* extension properties. Suppose, for example, that A is compact in R^n and that Y is a linear subspace of l_2. Suppose also that each point in A has a neighborhood U such that $A\cap\bar{U}$ has the BLEP in (R^n, Y). I do not know whether this implies that A has the BLEP in (R^n, Y).

8.4. *Addendum*. In a recent paper [Tr], D. A. Trotsenko announces results related to our results on the QSEP. He uses the notion of h-similarity, which for small h is close to s-quasisymmetry with small s, cf. 3.9. However, the examples of Section 7 (e.g. 7.6) seem to contradict Theorem 1 of [Tr], unless [Tr] tacitly assumes that all similarities are sense-preserving.

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