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ON THE VARIATIONAL PRINCIPLE OF GERSTENHABER AND RAUCH

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1. Introduction

Let $w=f(z)$ be a sense preserving homeomorphism of a Riemann surface S_1 onto a Riemann surface S_2, z, w denoting local parameters, and let $\varrho(w)$ be a weight function, $\varrho(w)\geqq 0$, $\iint_{S_2}\varrho(w)\ du\,dv=1$ $(w=u+iv)$. Let us assume that $f\in\mathscr{F}$, and $\varrho\in\mathscr{P}_0$, where the members of the classes $\mathscr{F}$ and $\mathscr{P}_0$ are sufficiently nice so that the Douglas–Dirichlet functional

$$\mathscr{D}_\varrho[f]=\iint_{S_1}(|f_z|^2+|f_{\bar z}|^2)\,\varrho(f(z))\,dx\,dy \quad (z=x+iy),$$

makes sense. If $\mathscr{F}$ is a class of qc (quasiconformal) mappings the maximal dilatation $K[f]$ is finite for each $f\in\mathscr{F}$. In the theory of extremal qc mappings one is concerned with the infinum,

$$K^*_{\mathscr{F}}=\inf_{f\in\mathscr{F}}K[f], \tag{1.1}$$

typically when all $f\in\mathscr{F}$ belong to a given homotopy class. Motivated by the fact that the ess sup norm by which $K[f]$ is defined has technical disadvantages one may ask whether the number K^* can be determined by extremal problems involving *finite* order means; in particular, we may ask whether the functional $\mathscr{D}_\varrho$ can be used to determine K^*. This idea is not new. In fact, partly on the basis of heuristic considerations, Gerstenhaber and Rauch [4] were led to formulate the following *principle:*

$$\sup_{\varrho\in\mathscr{P}_0}\inf_{f\in\mathscr{F}}\mathscr{D}_\varrho[f]=\frac{1}{2}\left(K^*_{\mathscr{F}}+\frac{1}{K^*_{\mathscr{F}}}\right).$$

In [4], Gerstenhaber and Rauch had mainly compact Riemann surfaces in mind for the domain and range of the mappings. At approximately the same time the fundamental paper of Ahlfors [1] provided a rigorous foundation to the theory of qc mappings; including the extremal problem (1.1), from an approach quite different from that of [4]. Some work in the directions suggested by [4] was however taken up by a number of writers, e.g. [8, 12, 13]. The survey article [3] contains an extensive compilation of relevant literature. See also the recent paper [6].

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Our contributions shall be for the case when S_1 and S_2 are the unit disk U, and when $\mathscr{F}$ is determined as follows. Let H denote a homeomorphism of ∂U onto ∂U that is realizable as the restriction $f_1|_{\partial U}$ of some qc mapping, $w=f_1(z)$, of U onto U. Then $\mathscr{F}$ shall be the family $Q(H)$ of qc mappings of U onto U such that $f|_{\partial U}=H$. The following notation is standard:

$$\mu_f(z)=\frac{f_{\bar z}}{f_z},\quad k_f(z)=|\mu_f(z)|,\quad D_f(z)=\frac{1+k_f(z)}{1-k_f(z)}=\frac{|f_z|+|f_{\bar z}|}{|f_z|-|f_{\bar z}|},$$

$$k[f]=\operatorname*{ess\,sup}_{z\in U} k_f(z),\quad K[f]=\operatorname*{ess\,sup}_{z\in U} D_f(z).$$

We shall denote $K^*_{Q(H)}$ by K^*, for short. As the class $\mathscr{P}_0$, we shall take the class $\mathscr{P}$ of measurable functions $\varrho(w)$, $\varrho(w)\geqq 0$, $\iint_U \varrho(w)\,du\,dv=1$.

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2. A class of variational problems

Since

$$K[f]=\sup_{\varrho\in\mathscr{P}}\iint_U D_f(z)\varrho(z)\,dx\,dy,$$

it is clear that

$$\inf_{f\in Q(H)}\sup_{\varrho\in\mathscr{P}}\iint_U D_f(z)\varrho(z)\,dx\,dy=K^*. \tag{2.1}$$

We will see (Theorem 2.2) that the operations, inf, sup, in (2.1) can be reversed. This fact is non-trivial, and, in a somewhat generalized form, is the import of the following result.

Theorem 2.1. *Let $G(t)$, $t\geqq 1$, be an increasing convex function of t. Then*

$$\sup_{\varrho\in\mathscr{P}}\inf_{f\in Q(H)}\iint_U G(D_f(z))\varrho(z)\,dx\,dy=G(K^*). \tag{2.2}$$

Proof. Let $z_{jn}=e^{2\pi ij/n}$, $j=1,2,\dots,n$, $n=4,5,6,\dots$, and let $w_{jn}=H\circ z_{jn}$. It is known [15] that there exists a qc mapping $f_n(z)$ of U onto U with the following properties:

$$f_n(z_{jn})=w_{jn},\quad \mu_{f_n}(z)=k_n\frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}.$$

Here k_n is a non-negative constant, and $\varphi_n(z)$ is holomorphic in U,

$$\iint_U |\varphi_n(z)|\,dx\,dy = 1. \tag{2.3}$$

Moreover, [9, Section 3.3],

$$\lim_{n\to\infty} K_n = K^* \quad \left(K_n = \frac{1+k_n}{1-k_n}\right). \tag{2.4}$$

Furthermore, according to [9, Theorem 7],

$$2\iint_U \operatorname{Re}\frac{\mu_f(z)\varphi_n(z)}{1-|\mu_f(z)|^2}\,dx\,dy+\iint_U \frac{1+|\mu_f(z)|^2}{1-|\mu_f(z)|^2}|\varphi_n(z)|\,dx\,du \geqq K_n, \tag{2.5}$$

for every $f\in Q(H)$. From (2.5) we deduce that

$$\iint_U D_f(z)|\varphi_n(z)|\,dx\,dy = \iint_U \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}|\varphi_n(z)|\,dx\,dy \geqq K_n.$$

In view of (2.3), Jensen's inequality implies that

$$\iint_U G(D_f(z))|\varphi_n(z)|\,dx\,dy \geqq G\left[\iint_U D_f(z)|\varphi_n(z)|\,dx\,dy\right] \geqq G(K_n),$$

for all $f\in Q(H)$, and $n=1, 2, 3, \ldots$. Since $|\varphi_n|\in\mathcal{P}$,

$$\sup_{\varrho\in\mathcal{P}}\inf_{f\in Q(H)}\iint_U G(D_f(z))\varrho(z)\,dx\,dy \geqq G(K_n), \quad n=1, 2, \ldots.$$

Due to the continuity of G, and condition (2.4), it therefore follows that

$$\sup_{\varrho\in\mathcal{P}}\inf_{f\in Q(H)}\iint_U G(D_f(z))\varrho(z)\,dx\,dy \geqq G(K^*). \tag{2.6}$$

To obtain an inequality in the other direction, choose $f=f^*$, so that $K[f^*]=K^*$, $f^*\in Q(H)$. Since $D_{f^*}(z)\leqq K^*$ a.e., and since G is increasing,

$$\iint_U G(D_{f^*}(z))\varrho(z)\,dx\,dy \leqq \iint_U G(K^*)\varrho(z)\,dx\,dy = G(K^*).$$

Hence,

$$\inf_{f\in Q(H)}\iint_U G(D_f(z))\varrho(z)\,dx\,dy \leqq G(K^*), \quad \text{for all} \quad \varrho\in\mathcal{P}.$$

Theorem 2.2. *Let $\mathcal{D}_p[f]$ denote the Douglas–Dirichlet functional,*

$$\mathcal{D}_\varrho[f] = \iint_U (|f_z|^2+|f_{\bar z}|^2)\,\varrho(f(z))\,dx\,dy \quad (f\in Q(H),\ \varrho\in\mathcal{P}). \tag{2.7}$$

Then, the principle of Gerstenhaber–Rauch holds; that is,

$$\sup_{\varrho\in\mathcal{P}}\inf_{f\in Q(H)}\mathcal{D}_\varrho[f] = \frac{1}{2}\left(K^*+\frac{1}{K^*}\right). \tag{2.8}$$

Proof. Let $z=g(w)$ be the mapping inverse to $w=f(z)$. Let

$$G(t) = (1/2)(t+1/t),$$

so that

$$G(D_g(w)) = \frac{|g_w|^2+|g_{\bar w}|^2}{|g_w|^2-|g_{\bar w}|^2} = \frac{|f_z|^2+|f_{\bar z}|^2}{|f_z|^2-|f_{\bar z}|^2},$$

and

$$\iint_U G(D_g(w))\varrho(w)\,du\,dv$$

$$= \iint_U \frac{|f_z|^2+|f_{\bar z}|^2}{|f_z|^2-|f_{\bar z}|^2}\,\varrho(f(z))(|f_z|^2-|f_{\bar z}|^2)\,dx\,dy = \mathscr{D}_\varrho[f].$$

We apply Theorem 2.1 to $Q(H^{-1})$. Since K^* is the same for H^{-1} as for H, the result follows.

3. Harmonic quasiconformal mappings

If I is the identity mapping with domain ∂U, and if $F\in Q(I)$, $f\in Q(H)$, then $\tilde f = f\circ F^{-1}\in Q(H)$. As a computation shows, the corresponding variation of $\mathscr{D}[f]$ is [4]

$$\delta\mathscr{D}_\varrho[f] = \mathscr{D}_\varrho[\tilde f]-\mathscr{D}_\varrho[f] = 2\iint_U \frac{|f_z|^2+|f_{\bar z}|^2}{|F_z|^2-|F_{\bar z}|^2}|F_{\bar z}|^2\varrho(f(z))\,dy\,dy$$

$$-4\,\mathrm{Re}\iint_U \frac{f_z\bar f_{\bar z}\bar F_z F_{\bar z}}{|F_z|^2-|F_{\bar z}|^2}\,\varrho(f(z))\,dx\,dy.$$

If

$$F(z) = z+\varepsilon\lambda(z)+o(\varepsilon),\quad \lambda(z)|_{\partial U} = 0,$$

then, formally,

$$\delta\mathscr{D}_\varrho[f] = -4\,\mathrm{Re}\left[\varepsilon\iint_U f_z\bar f_{\bar z}\varrho(f(z))\lambda_{\bar z}\,dx\,dy\right]+o(\varepsilon).$$

This leads to the expectation that if there are mappings $f\in Q(H)$ for which $\inf_{f\in Q(H)} D_\varrho[f]$ is attained, then such mappings[1]) necessarily satisfy

(3.1) $$f_z\bar f_{\bar z}\varrho(f(z)) = \varphi(z)\quad \text{for a.a.}\quad z\in U,$$

where $\varphi(z)$ is holomorphic in U.

One refers to a homeomorphism f satisfying (3.1) as a *harmonic* mapping relative to the weight function ϱ. We shall assume that φ is not identically 0.

[1]) In [11] the assertion is made that if $K>K^*$, then $\inf\{\mathscr{D}_\varrho[f]: f\in Q_k(H)\}$ is attained for a mapping f satisfying (3.1). However, the reasoning involves the Hahn–Banach theorem in a manner for which justification is missing, thereby leaving [11, Theorem 6] open to doubt.

Theorem 3.1. *Suppose* $\varrho \in \mathscr{P}$, *and* $f(z)$, $z \in U$, *is a qc mapping. If* f *is a harmonic mapping relative to* ϱ, *and the holomorphic function* φ *is defined by* (3.1), *then*

$$\mu_f(z) = k_f(z) \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad k_f(z) > 0 \text{ a.e.}, \tag{3.2}$$

and

$$\iint_U \frac{|\varphi(z)|}{k_f(z)}\, dx\, dy < \infty. \tag{3.3}$$

Conversely, if $\mu_f(z)$ *has the form* (3.2), *where* $\varphi(z)$ *is holomorphic in* U *and satisfies* (3.3), *then* f *is a harmonic mapping relative to a normalized weight function* ϱ *determined by* (3.1).

Proof. To deduce necessity let $\varphi(z)$ be defined by (3.1). Then,

$$\overline{\varphi(z)} = \varrho(f(z)) f_z \overline{f_{\bar z}}, \quad \text{and} \quad |\varphi(z)| = \varrho(f(z)) |f_z f_{\bar z}|.$$

Therefore, (3.2) holds, and $k_f(z) > 0$ a.e. Also,

$$\varrho(f(z)) J(f(z)) = \frac{|\varphi|}{|f_z f_{\bar z}|}(|f_z|^2 - |f_{\bar z}|^2) = \frac{1-k_f(z)^2}{k_f(z)} |\varphi(z)|.$$

Hence,

$$1 = \iint_U \varrho(w)\, du\, dv = \iint_V \frac{1-k_f(z)^2}{k_f(z)} |\varphi(z)|\, dx\, dy, \tag{3.4}$$

so that (3.3) follows. Conversely, if (3.2) and (3.3) hold, one defines ϱ by

$$\varrho(f(z)) = \frac{\varphi(z)}{f_z \overline{f_{\bar z}}}.$$

Since (3.3) holds, and $0 < k_f(z) \leqq k[f] < 1$, we can normalize $\varphi(z)$ so that (3.4) is satisfied. □

For a harmonic qc mapping f belonging to $Q(H)$, with $k_f(z)$ constant, it is easily verified that $\mathscr{D}_\varrho[f] = (1/2)(K^* + 1/K^*))$ for all $\varrho \in \mathscr{P}$. Such a mapping is of course just a Teichmüller map corresponding to a quadratic differential with finite norm, so that [14] $K[f] = K^*$ in this case. It is natural to ask what the most general harmonic qc f_0 with $K[f_0] = K^*$ can be. The next theorem implies that, under a uniqueness hypothesis, there are in fact no possibilities other than the classical Teichmüller case.

Theorem 3.2. *Suppose* f *is a qc mapping of* U *with complex dilatation of the form* (3.2), *where* $\varphi(z)$ *is holomorphic*[2]) *in* U. *If* f *is the unique mapping in* $Q(H)$ *such that* $K[f] = K^*$, *then*

$$k_f(z) = k^* = \frac{K^*-1}{K^*+1} \quad \text{a.e.}$$

[2]) Note that the assumption $\iint_U |\varphi(z)|\, dx\, dy < \infty$ is not required.

Before proceeding with the proof we require a preliminary fact:

Lemma[3]). *Suppose* $f \in Q(H)$, *where* $H = f|_{\partial U}$, *and*

$$\mu_f(z) = k_f(z), \quad z \in U.$$

If $K[f] = K^*$ *then* $k_f(z) \equiv k^* = (K^* - 1)/(K^* + 1)$.

Proof. Assume that $k_f(z) < k^*$ on a set of positive measure. Since $K[f] = K^*$, there exists [5, 9] a sequence $\{\varphi_n(z)\}$ of functions holomorphic in U, with $\iint_U |\varphi_n(z)|\, dx\, dy = 1$, such that

$$\lim \iint_U \mu_f(z)\, \varphi_n(z)\, dx\, dy = k^*. \tag{3.5}$$

Since $|\mu_f(z)| \leqq k^*$, this is possible (cf. e.g. [10], Corollary of Lemma 0.3) only if

$$\lim_{n\to\infty} \varphi_n(z) = 0 \quad \text{locally uniformly in } U. \tag{3.6}$$

Now,

$$\iint_U \mu_f \varphi_n\, dx\, dy = \iint_U \left(\mu_f - \frac{k^*}{2}\right) \varphi_n\, dx\, dy + \frac{k^*}{2} \iint_U \varphi_n\, dx\, dy$$

$$= \iint_U \left(k_f(z) - \frac{k^*}{2}\right) \varphi_n(z)\, dx\, dy + \frac{\pi k^*}{2}\, \varphi_n(0).$$

But,

$$\iint_U \left|k_f(z) - \frac{k^*}{2}\right| |\varphi_n(z)|\, dx\, dy \leqq \frac{k^*}{2} \iint_U |\varphi_n(z)|\, dx\, dy = \frac{k^*}{2},$$

and, by (3.6), $\lim \varphi_n(0) = 0$. This produces a contradiction with (3.5).

Proof of Theorem 3.2. We have

$$\operatorname*{ess\,sup}_{z \in D} k_f(z) = k^* \quad \text{for every disk } D, D \subset U;$$

otherwise, the uniqueness of f could be contradicted by constructing a variation of f within D. Let $Z = \{z \in U : \varphi(z) = 0\}$. Suppose $z_0 \notin Z$. Let us consider a neighbourhood V of z_0 in which $\zeta = \Phi(z) = \int \sqrt{\varphi(z)}\, dz$ is schlicht, and such that $\Phi(V)$ is a disk $\tilde{V}$. In $\tilde{V}$, $f \circ \Phi^{-1}$ has complex dilatation

$$\varkappa(\zeta) = k_f(\Phi^{-1}(\zeta)).$$

According to the Lemma, then, unless $k_f(z) \equiv k^*$ in V, there exists a K-qc mapping g of $\tilde{V}$, with the boundary values of $f \circ \Phi^{-1}$, and $K < K^*$. This would imply that $g \circ \Phi$ is a K-qc mapping of V with the boundary values of f, and hence the uniqueness hypothesis for f would again be contradicted. We conclude that every point $z_0 \in U \setminus Z$ possesses a neighbourhood in which $k_f(z) = k^*$ a.e. Since Z has measure zero, this is readily seen to imply that $k_f(z) \equiv k^*$ a.e. in U. □

[3]) This is a special case of [2, Theorem 2]. For convenience of the reader we have included a proof here.

We note that there do exist examples of harmonic qc mappings f of U for which $K[f]=K^*$, but where $k_f(z)<k^*$ on a set of positive measure. Such examples can be constructed with e.g. the help of the known example of [14] of a case where $Q(H)$ contains distinct mappings f for which $K[f]=K^*$.

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