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ON ROTATION AUTOMORPHIC FUNCTIONS WITH DISCRETE ROTATION GROUPS

RAUNO AULASKARI

In the paper [4] we defined a rotation automorphic function f with respect to some Fuchsian group Γ. In [1]—[4] we supposed the rotation automorphic function f to satisfy in a fundamental domain F of Γ the condition

$$\iint_F f^*(z)^2\, d\sigma_z < \infty, \tag{1}$$

where $f^*(z)$ is the spherical derivative of f and $d\sigma_z$ is the euclidean area element. Further, in [1], [2] and [4], we showed that, by suitable restrictions related to Γ, f is a normal function in D, that is, $\sup_{z\in D}(1-|z|^2)f^*(z)<\infty$ (cf. [6]). In the meanwhile, in [3], we constructed a non-normal rotation automorphic function f satisfying the condition (1).

1. In this paper we shall take another point of view, that is, we let Γ be arbitrary but restrict the rotation group $\Sigma=\{S_T|\ T\in\Gamma\}$ acting on the Riemann sphere $\hat{\mathcal{C}}$. Because of the compactness of $\hat{\mathcal{C}}$ we shall see that the condition "Σ is discrete" alone or the equivalent assumption "Σ is finite" will imply the normality of f. I want to thank prof. T. Erkama for our discussions on this subject.

Let D and ∂D be the unit disk and the unit circle, respectively. We shall denote the hyperbolic distance by $d(z_1, z_2)$ $(z_1, z_2\in D)$ and the hyperbolic disk $\{z|\ d(z, z_0)<r\}$ by $U(z_0, r)$. Suppose that Γ is a Fuchsian group acting on D and let f be a meromorphic function in D. Then f is called rotation automorphic with respect to Γ if

$$f(T(z)) = S_T(f(z)),\ z\in D,\ T\in\Gamma,$$

where S_T is a rotation of $\hat{\mathcal{C}}$.

The points $z, z'\in\bar{D}=D\cup\partial D$ are called Γ-equivalent if there exists a mapping $T\in\Gamma$ such that $z'=T(z)$. A domain $F\subset D$ is called a fundamental domain of Γ if it does not contain two Γ-equivalent points and if every point in D is Γ-equivalent to some point in the closure $\bar{F}$ of F. We fix the fundamental domain F of Γ to be a normal polygon in D.

If we suppose the rotation group Σ to have a representation by matrices, then Σ is said to be discrete provided the identity is an isolated element.

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We shall need the following lemma (cf. [4, Lemma]) in the proof of our theorem:

Lemma. *Let* $(z_n)\subset F$ *be a sequence of points such that* $|z_n|\to 1$ *as* $n\to\infty$. *If* $r>0, 0<R<1$ *and* $D_R=\{z \mid |z|<R\}$, *then* $T(U(z_n, r))\cap D_R\neq\Phi$ *for finitely many* $T\in\Gamma$ *and* $n\in N$ *only.*

Theorem. *Let* f *be a rotation automorphic function with respect to* Γ *for which*

$$\iint_F f^*(z)^2\, d\sigma_z <\infty \tag{1.1}$$

holds. If the rotation group Σ *corresponding to* Γ *is discrete, then* f *is a normal function in* D.

Proof. Suppose, on the contrary, that f is not a normal function in D. Then there is a sequence of points $(z_n)\subset F$ such that

$$(1-|z_n|^2)f^*(z_n) \to\infty \tag{1.2}$$

as $n\to\infty$. We choose the hyperbolic disks $U(z_n, r)$, $r>0$, for which

$$U(z_n, r) = \bigcup_{m=0}^{k_n} U(z_n, r)\cap T_m(\bar{F}), \tag{1.3}$$

where $T_m\in\Gamma$. By (1.1) we have

$$\iint_{U(z_n,r)\cap F} f^*(z)^2\, d\sigma_z \to 0$$

as $n\to\infty$. By [5, 5.1 Theorem] the group Σ is finite. Suppose that Σ contains i_0 rotations. We may choose $R>0$ such that

$$\iint_{F\cap D\setminus D_R} f^*(z)^2\, d\sigma_z < \pi/i_0. \tag{1.4}$$

Let

$$f_n(\zeta) = f\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right).$$

By Lemma we may assume that in (1.3), for all $n\geqq n_0$, $T_m^{-1}(U(z_n, r))\cap\bar{F}\subset\bar{F}\cap D\setminus D_R$ for each T_m^{-1}, $m=0, \dots, k_n$. Further,

$$\bigcup_{n=n_0}^{\infty} f_n(U(0,r)) = \bigcup_{n=n_0}^{\infty} f(U(z_n, r))\subset\bigcup_{i=1}^{\infty} f(T_i(\bar{F}\cap D\setminus D_R)) \tag{1.5}$$

$$=\bigcup_{i=1}^{i_0} S_{T_i}(f(\bar{F}\cap D\setminus D_R)),$$

where T_i, $i=1, 2, \dots$, runs through all transformations of Γ. Since $\iint_{U(z_n,r)} f^*(z)^2\, d\sigma_z=\iint_{U(0,r)} f_n^*(\zeta)^2 d\sigma_\zeta=$ the spherical area of $f_n(U(0, r))$, we have by (1.4) and (1.5) that $\{f_n\}_{n=n_0}^{\infty}$ omits at least three values in $U(0, r)$. Thus $\{f_n\}_{n=n_0}^{\infty}$

forms a normal family in $U(0, r)$ and Marty's criterion implies

$$(1-|z_n|^2)f^*(z_n) = f_n^*(0) \leqq M < \infty$$

for each $n \geqq n_0$. This contradicts (1.2) and thus the theorem is proved.

Remark 1. If we reject the finiteness condition of Σ, we shall find a Fuchsian group Γ, a rotation group Σ and a rotation automorphic function f corresponding to Γ and Σ such that Σ is generated by infinitely many rotations with one rotation axis only (0∞-axis) and f satisfies (1.1) but is not a normal function in D (cf. [3]).

Remark 2. The assertion of the above theorem can be proved also if f is considered to be an automorphic function with respect to a certain subgroup of Γ and after that a theorem of Pommerenke is used (cf. [7, Corollary 1]).

References

[1] Aulaskari, R.: On rotation-automorphic functions with respect to a finitely generated Fuchsian group. - J. London Math. Soc. To appear.

[2] Aulaskari, R.: Rotation-automorphic functions near the boundary. - Math. Scand. To appear.

[3] Aulaskari, R., and P. Lappan: On additive automorphic and rotation automorphic functions. - Arkiv för Matematik To appear.

[4] Aulaskari, R., and T. Sorvali: On rotation-automorphic functions. - Math. Scand. 41, 1981, 222—228.

[5] Beardon, A. F.: The geometry of discrete groups. - Discrete groups and automorphic functions (Proceedings of an Instructional Conference organized by the London Mathematical Society and the University of Cambridge, Cambridge, 1975) edited by W. J. Harvey, Academic Press, London—New York—San Francisco, 1977, 47—72.

[6] Lehto, O., and K. I. Virtanen: Boundary behaviour and normal meromorphic functions. - Acta Math. 97, 1957, 47—65.

[7] Pommerenke, Ch.: On normal and automorphic functions. - Michigan Math. J. 21, 1974, 193—202.

University of Joensuu
Department of Mathematics and Physics
SF-80100 Joensuu 10
Finland

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