

A REMARK ON DOMAINS QUASICONFORMALLY EQUIVALENT TO A BALL

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1. Introduction. An important problem in the theory of higher dimensional quasiconformal mappings is to characterize geometrically those domains D in \bar{R}^n , the one point compactification of euclidean n -space R^n , which can be mapped quasiconformally onto the unit ball B^n in R^n . The following rather implicit criterion appears to be the best result known in this direction at present [6].

Theorem 1. *Suppose that D is a domain in \bar{R}^n and suppose that for some neighborhood U of ∂D there exists a quasiconformal mapping f of $D \cap U$ into B^n such that $|f(x)| \rightarrow 1$ as $x \rightarrow \partial D$ in $D \cap U$. Then D is quasiconformally equivalent to B^n .*

Theorem 1 can be localized as follows for the case when D is a Jordan domain, that is, when ∂D is homeomorphic to ∂B^n , [6] and [13].

Theorem 2. *Suppose that D is a Jordan domain in \bar{R}^3 and suppose that for each $x \in \partial D$ there exists a neighborhood U of x and a quasiconformal mapping f of $D \cap U$ into B^3 such that $|f(y)| \rightarrow 1$ as $y \rightarrow \partial D$ in $D \cap U$. Then D is quasiconformally equivalent to B^3 .*

It seems likely that Theorem 2 is still true with 3 replaced by n throughout. However the proof given in [6] depends crucially on Ahlfors' affirmative solution, when $n = 2$, of the lifting conjecture for quasiconformal mappings [1].

Conjecture. *Each quasiconformal mapping $f: R^n \rightarrow R^n$ can be extended to a quasiconformal mapping $g: R^{n+1} \rightarrow R^{n+1}$.*

Carleson has recently established this conjecture for $n = 3$ [4], and his proof suggests a method to settle the conjecture for all $n \geq 4$.

If we apply Theorem 2 and Theorem 17.12 of [15], we obtain a simple geometric condition sufficient to guarantee that a domain D in R^3 is quasiconformally equivalent to B^3 .

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COROLLARY 1. *Suppose that D is a bounded Jordan domain in R^3 , that D has a tangent plane T at each point of ∂D , and that T is continuous in ∂D . Then D is quasiconformally equivalent to B^3 .*

Unfortunately the example given below shows that this sufficient condition is very far from being necessary. (See also [7].)

EXAMPLE 1. *For each $n \geq 3$ there exists a domain D in R^n such that D is quasiconformally equivalent to B^n and such that D does not have a tangent plane at any point of ∂D .*

Proof. Let

$$H = \{x = (x_1, \dots, x_n) : x_n < 0\},$$

let ∂H denote the usual imbedding of \bar{R}^{n-1} in \bar{R}^n , and choose a sequence of open $(n-1)$ -balls B_j in R^{n-1} so that the \bar{B}_j are pairwise disjoint and so that $\cup_j B_j$ is dense in R^{n-1} . Next set

$$D = H \cup \left(\bigcup_j P^{-1}(B_j) \right),$$

where P denotes orthogonal projection of R^n into R^{n-1} . Then the argument in [8] or on page 462 of [3] shows that D is quasiconformally equivalent to B^n , while the fact that D is dense in R^n implies that D does not have a tangent plane at any point of its boundary.

On the other hand the following example shows that the geometric hypotheses in Corollary 1 cannot be weakened significantly. (See also [11].)

EXAMPLE 2. *For each $n \geq 3$ there exists a bounded Jordan domain D in R^n such that D has a tangent plane T at each point of ∂D , such that T is continuous at all but one point of ∂D , and such that D is not quasiconformally equivalent to B^n .*

Examples 1 and 2 show that there is no satisfactory way to characterize the domains D in R^n which are quasiconformally equivalent to B^n in terms of their tangential properties.

The purpose of this note is to establish the existence of the domain mentioned in Example 2. The proof depends on an inequality, given in Section 3, between the $(n-1)$ -modulus of a family of curves in the boundary and the n -modulus of a corresponding family in the interior of a ball or half space.

2. Moduli of curve families. For $q \in [0, \infty)$ the *normalized q -dimensional Hausdorff outer measure* of a set E in R^n is defined by

$$\mathcal{H}^q(E) = \lim_{t \rightarrow 0} \left(\inf \sum_j \alpha(q) 2^{-q} \text{dia}(E_j)^q \right),$$

where the infimum is taken over all countable coverings of E by sets E_j with $\text{dia}(E_j) < t$ and where

$$(1) \quad \alpha(q) = \frac{\Gamma\left(\frac{1}{2}\right)^q}{\Gamma\left(\frac{q}{2} + 1\right)}.$$

If q is an integer in $[1, n]$ and T a q -dimensional hyperplane in R^n , then $\mathcal{H}^q \upharpoonright T$ coincides with the Lebesgue q -dimensional outer measure m_q in T .

For $p, q \in [1, \infty)$ the q -dimensional p -modulus of a curve family Γ in \bar{R}^n is defined as

$$M_p^q(\Gamma) = \inf_{R^n} \int h^p d\mathcal{H}^q,$$

where the infimum is taken over all h in $\text{adm}(\Gamma)$, the collection of all Borel functions $h: R^n \rightarrow [0, \infty]$ such that $\int_\gamma h ds \geq 1$ for all locally rectifiable curves γ in Γ . One often writes $M_p(\Gamma)$ for $M_p^n(\Gamma)$ and $M(\Gamma)$ for $M_n(\Gamma)$.

We will need a symmetry principle for the moduli of curve families, namely the n -dimensional analogue of Lemma 3.3 in [8]. We take the opportunity to establish here the following more general result by means of a somewhat different argument.

L e m m a 1. *Suppose that D is an open half space in R^n , that E and F are disjoint compact sets in \bar{D} , and that E^* and F^* are the symmetric images of E and F in ∂D . If Γ_1 and Γ_2 are the families of curves joining E to F in D and $E \cup E^*$ to $F \cup F^*$ in \bar{R}^n , respectively, then*

$$(2) \quad M_p^n(\Gamma_2) = 2 M_p^n(\Gamma_1).$$

The proof depends upon the following result.

L e m m a 2. *Suppose that D, E, F, Γ_1 are as in Lemma 1 and that Γ is the family of curves joining E to F in \bar{D} . Then*

$$(3) \quad M_p^n(\Gamma) = M_p^n(\Gamma_1).$$

Proof. Since $\Gamma_1 \subset \Gamma$, it is sufficient to prove that

$$(4) \quad M_p^n(\Gamma) \leq M_p^n(\Gamma_1).$$

Let e denote the inner unit normal for ∂D , choose $t \in (0, 1)$, and set

$$f(x) = x + t \text{dist}(x, E \cup F) e$$

for $x \in R^n$ and $f(\infty) = \infty$. Then f is a homeomorphism of \bar{R}^n onto itself and

$$(5) \quad (1-t) |x-y| \leq |f(x) - f(y)| \leq (1+t) |x-y|$$

for all $x, y \in R^n$.

Choose $g \in \text{adm}(\Gamma_1)$ and set $h = (1+t)g \circ f$ in R^n . Then each locally rectifiable curve γ in Γ contains a subcurve γ' which f maps into Γ_1 and

$$\int_{\gamma} h \, ds \geq \int_{\gamma'} (1+t)g \circ f \, ds \geq \int_{f(\gamma')} g \, ds \geq 1$$

by (5). Thus $h \in \text{adm}(\Gamma)$ and

$$M_p^n(\Gamma) \leq \int_{R^n} h^p \, dm_n \leq \frac{(1+t)^p}{(1-t)^n} \int_{R^n} g^p \, dm_n.$$

Taking the infimum over all such g gives

$$M_p^n(\Gamma) \leq \frac{(1+t)^p}{(1-t)^n} M_p^n(\Gamma_1),$$

and we obtain (4) by letting $t \rightarrow 0$.

Proof for Lemma 1. Let $r: \bar{R}^n \rightarrow \bar{R}^n$ denote reflection in ∂D . Then Γ_1 and $r(\Gamma_1)$ are separate subfamilies of Γ_2 and

$$(6) \quad M_p^n(\Gamma_2) \geq M_p^n(\Gamma_1) + M_p^n(r(\Gamma_1)) = 2 M_p^n(\Gamma_1).$$

Next let Γ be as in Lemma 2, choose $g \in \text{adm}(\Gamma)$, and set $h = g \circ f$ in R^n , where

$$f(x) = \begin{cases} x & \text{if } x \in \bar{D}, \\ r(x) & \text{if } x \in r(\bar{D}). \end{cases}$$

If γ is a locally rectifiable curve in Γ_2 , then $f(\gamma) \in \Gamma$ and

$$\int_{\gamma} h \, ds \geq \int_{f(\gamma)} g \, ds \geq 1.$$

Hence $h \in \text{adm}(\Gamma_2)$,

$$M_p^n(\Gamma_2) \leq \int_{R^n} h^p \, dm_n = \int_D g^p \, dm_n + \int_{r(D)} (g \circ r)^p \, dm_n \leq 2 \int_{R^n} g^p \, dm_n,$$

and taking the infimum over all such g yields

$$(7) \quad M_p^n(\Gamma_2) \leq 2 M_p^n(\Gamma).$$

The desired conclusion then follows from (3), (6) and (7).

3. An inequality. We establish next an inequality between the moduli of two families of curves joining the same pair of disjoint continua.

Lemma 3. *Suppose that $2 \leq m < n < \infty$, that R^m is the usual imbedding of euclidean m -space in R^n , and that E and F are nondegenerate disjoint continua in \bar{R}^m . If Γ_1 and Γ_2 are the families of curves which join E to F in \bar{R}^m and \bar{R}^n , respectively, then*

$$(8) \quad -2n < \left(\frac{M_m^m(\Gamma_1)}{\beta(m)} \right)^{1/(1-m)} - \left(\frac{M_n^n(\Gamma_2)}{\beta(n)} \right)^{1/(1-n)} < 2m,$$

where $\beta(q) = q \alpha(q)$ and $\alpha(q)$ is as in (1).

Proof. By performing a preliminary Möbius transformation, we may assume that $0 \in E$ and that $\infty \in F$. Arguing as in Lemma 3.5 of [8], we can find continua C_0 and C_1 in \bar{R}^m such that $\partial C_0 \subset E \subset C_0$, $\partial C_1 \subset F \subset C_1$ and such that $R = \bar{R}^m \sim (C_0 \cup C_1)$ is an m -dimensional ring domain. Then $S = \bar{R}^n \sim (E \cup F)$ is an n -dimensional ring domain,

$$\text{cap}_m(R) = M_m^m(\Gamma_1), \quad \text{cap}_n(S) = M_n^n(\Gamma_2),$$

and (8) is equivalent to the inequality

$$(9) \quad -2n < \text{mod}_m(R) - \text{mod}_n(S) < 2m.$$

To establish (9) set $a = \sup \{ |x| : x \in E \}$, $b = \inf \{ |x| : x \in F \}$. Then by a well known estimate for the moduli of rings,

$$\text{mod}_m(R) \leq \log \lambda_m^2 \left(\frac{b}{a} + 1 \right),$$

where λ_m is a constant which depends only on m . (See [3], [5], [12] and [14].) If $a < b$, then E and F are separated in \bar{R}^n by the spherical ring

$$T = \{ x \in R^n : a < |x| < b \}.$$

In this case,

$$\log \frac{b}{a} = \text{mod}_n(T) \leq \text{mod}_n(S),$$

and we obtain

$$(10) \quad \text{mod}_m(R) - \text{mod}_n(S) < \log 2 \lambda_m^2.$$

If $a \geq b$, then (10) follows trivially since $\text{mod}_n(S) > 0$. From estimates in [2] it follows that $\log 2 \lambda_m^2 < 2m$, and we obtain the second half of (9) from (10). The first half follows from interchanging the roles of m and n in the above argument.

Remark. The hypothesis that E and F be connected is essential in Lemma 3.

Proof. Let $h : (0, 1) \rightarrow (0, \infty)$ denote the Hausdorff measure function $h(t) = (\log(1/t))^{1-n}$. Then

$$\lim_{t \rightarrow 0} h(t) \left(\log \frac{1}{t} \right)^{m-1} = 0.$$

By Theorem 4.4 in [16], there exists a compact linear set E in the unit ball $B^m \subset R^m$ such that E is of positive conformal capacity as a subset of R^m and such that E is of Hausdorff h -measure zero. By Theorem 4.3 in [16], this last fact implies that E is of zero conformal capacity as a subset of R^n .

Let $F = \bar{R}^m \sim B^m$, let Γ_1 and Γ_2 be as in Lemma 3, and let Γ'_2 denote the family of curves joining E to $F' = \bar{R}^n \sim B^n$ in \bar{R}^n . Then each γ in Γ_2 contains a subcurve γ' in Γ'_2 , and

$$M_m^n(\Gamma_1) > 0, \quad M_n^n(\Gamma_2) \leq M_n^n(\Gamma'_2) = 0$$

by known results relating extremal length and conformal capacity. (See, for example, [10].) Hence inequality (8) cannot hold in this case.

Corollary 2. Suppose that $n \geq 3$, that D is an open ball or half space in R^n , and that for $j = 1, 2, \dots$, E_j and F_j are pairs of nondegenerate disjoint continua in ∂D . If $\Gamma_{1,j}$ and $\Gamma_{2,j}$ are the families of curves joining E_j to F_j in ∂D and in D , respectively, and if either

$$(11) \quad \lim_{j \rightarrow \infty} M_{n-1}^{n-1}(\Gamma_{1,j}) = 0 \quad \text{or} \quad \lim_{j \rightarrow \infty} M_n^n(\Gamma_{2,j}) = 0,$$

then

$$(12) \quad \lim_{j \rightarrow \infty} \frac{M_{n-1}^{n-1}(\Gamma_{1,j})^{(n-1)/(n-2)}}{M_n^n(\Gamma_{2,j})} = c,$$

where c is a positive constant which depends only on n .

Proof. By performing a preliminary Möbius transformation, we may assume that

$$D = \{ x = (x_1, \dots, x_n) : x_n > 0 \}$$

and that ∂D is the usual imbedding of \bar{R}^{n-1} in \bar{R}^n . If $\Gamma_{3,j}$ denotes the family of curves joining E_j to F_j in \bar{R}^n , then

$$M_n^n(\Gamma_{3,j}) = 2 M_n^n(\Gamma_{2,j})$$

by Lemma 1. Hence

$$\left| \left(\frac{M_{n-1}^{n-1}(\Gamma_{1,j})}{\beta(n-1)} \right)^{1/(2-n)} - \left(\frac{2 M_n^n(\Gamma_{2,j})}{\beta(n)} \right)^{1/(1-n)} \right| < 2n$$

by Lemma 3, and (11) implies (12) with

$$c = 2 \beta(n-1)^{(n-1)/(n-2)} \beta(n)^{-1}.$$

4. Proof for Example 2. For $j = 1, 2, \dots$, set $a_j = 4^{-j}$ and define $f: [0, \infty) \rightarrow (-1, 0]$ by

$$f(t) = \begin{cases} a_j^2(\cos(\pi a_j^{-3} t) - 1) & \text{if } t \in (a_j, 2a_j) \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is continuously differentiable in $(0, \infty)$, $|f(t)| \leq 2t^2$ in $[0, \infty)$, and

$$(13) \quad 1 \leq \int_{a_j}^{2a_j} |f'(t)| dt \leq \pi$$

for all j . Let

$$(14) \quad D = \{x = (x_1, \dots, x_n) : x_n > f(r)\}, \quad r = (x_1^2 + \dots + x_{n-1}^2)^{1/2},$$

and let $g: \bar{R}^n \rightarrow \bar{R}^n$ be any Möbius transformation for which $g^{-1}(\infty) \notin \bar{D}$. Then $g(D)$ is a bounded Jordan domain, $g(D)$ has a tangent plane T at each point of its boundary, and T is continuous except at the single point $g(0)$.

To complete the proof it suffices to show that D , and hence $g(D)$, is not quasiconformally equivalent to B^n . For each j let

$$S_j = \{x \in \partial D : r \in (a_j, 2a_j)\}, \quad E_j = \{x \in \partial D : r \in [2a_j, 4a_j]\},$$

where r is as in (14). Next let $\Gamma_{1,j}$ and $\Gamma_{2,j}$ denote the families of curves which join E_j to E_{j+1} in ∂D and in D , respectively.

To obtain an upper bound for $M_{n-1}^{n-1}(\Gamma_{1,j})$ set h equal to the characteristic function of S_j . If γ is a locally rectifiable curve in $\Gamma_{1,j}$, then γ contains a subcurve γ' which lies, except for its endpoints, in S_j . Hence with (13) we have

$$\int_{\gamma'} h ds \geq \int_{\gamma'} ds \geq \int_{a_j}^{2a_j} |f'(t)| dt \geq 1,$$

$h \in \text{adm}(\Gamma_{1,j})$, and

$$(15) \quad M_{n-1}^{n-1}(\Gamma_{1,j}) \leq \int_{S_j} d\mathcal{H}^{n-1} \\ \leq \beta(n-1) \int_{a_j}^{2a_j} r^{n-2} (|f'(r)| + 1) dr \leq c_1 a_j^{n-2},$$

where c_1 is a finite constant which depends only on n .

To obtain a lower bound for $M_n^n(\Gamma_{2,j})$ let

$$H = \{x \in (x_1, \dots, x_n) : x_n > 0\}.$$

Then $E_j \subset \partial H$ and $H \subset D$. Let $\Gamma_{3,j}$ denote the family of curves which join E_j to E_{j+1} in H . Then $\Gamma_{2,j} \supset \Gamma_{3,j}$, and since the Möbius transformation $k(x) = 4^{j-1}x$ maps $\Gamma_{3,j}$ onto $\Gamma_{3,1}$, we obtain

$$(16) \quad M_n^n(\Gamma_{2,j}) \geq M_n^n(\Gamma_{3,j}) = M_n^n(\Gamma_{3,1}) = c_2,$$

where c_2 is a positive constant which depends only on n .

Now suppose there exists a K -quasiconformal mapping h which maps D onto a ball D' . Then h has a homeomorphic extension h^* which maps ∂D onto $\partial D'$. Let $\Gamma_{4,j}$ denote the family of curves which join E_j to E_{j+1} in S_j , and let $\Gamma'_{i,j}$ denote the image of $\Gamma_{i,j}$ under h^* . Since h is K -quasiconformal in D ,

$$(17) \quad M_n^n(\Gamma_{2,j}) \leq K M_n^n(\Gamma'_{2,j}).$$

Next since D has a continuous tangent plane at each point of S_j , S_j is an admissible boundary surface for D and

$$(18) \quad M_{n-1}^{n-1}(\Gamma'_{4,j}) \leq K M_{n-1}^{n-1}(\Gamma_{4,j})$$

by Theorem 2.23 in [9]. (See also [3] where (18) is derived with the factor K replaced by K^{n-2} .) Since $\Gamma_{4,j} \subset \Gamma_{1,j}$ and since each curve in $\Gamma_{1,j}$ contains a subcurve in $\Gamma_{4,j}$, (18) implies that

$$(19) \quad M_{n-1}^{n-1}(\Gamma'_{1,j}) \leq K M_{n-1}^{n-1}(\Gamma_{1,j}).$$

But (15), (16), (17) and (19) imply that

$$\lim_{j \rightarrow \infty} M_{n-1}^{n-1}(\Gamma'_{1,j}) = 0, \quad \lim_{j \rightarrow \infty} \frac{M_{n-1}^{n-1}(\Gamma'_{1,j})^{(n-1)/(n-2)}}{M_n^n(\Gamma'_{2,j})} = 0,$$

contradicting the conclusion of Corollary 2. Thus D is not quasiconformally equivalent to a ball, and the proof is complete.

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