ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

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DUAL C^*-ALGEBRAS, WEAKLY SEMI-COMPLETELY CONTINUOUS ELEMENTS, AND THE EXTREME RAYS OF THE POSITIVE CONE

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HELSINKI 1975
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1975.599

ISSN 0066-1953
ISBN 951-41-0216-9

Communicated 19 September 1974

KESKUSKIRJAPAINO
HELSINKI 1975

1. Introduction

B. J. Tomiuk and P. K. Wong have defined in [14, p. 654] an element u of a Banach algebra A to be *weakly semi-completely continuous* (w.s.c.c.) if the map $x \mapsto uxu$ is a weakly compact operator [4, p. 482] on A. One of their results [14, Theorem 5.2] states that a C^*-algebra A is *dual* (cf. e.g. [3, p. 99]) if and only if every element of A is w.s.c.c. In this paper we extend this theorem in two directions. In section 2 we show that A is already dual if for every positive $u \in A$ the image of the unit ball of A is relatively weakly (i.e. $\sigma(A, A^*)$) compact under some mapping $x \mapsto F_u(x)$ where $F_u(x)$ is a finite product of positive powers of u and x (its form depending on u). On the other hand, it is known that a C^*-algebra is dual if and only if its every element is *compact* [1, p. 17]. (An element u of a Banach algebra A is called compact, if the map $x \mapsto uxu$ is a compact operator on A.) In section 3 we prove that even for a single element of a C^*-algebra compactness and weak semi-complete continuity are equivalent. Combined with Theorem 3.1 in [18] this result shows that, for a C^*-algebra A and $u \in A$, the weak compactness of the two-sided multiplication operator $x \mapsto uxu$ already implies the weak compactness of the operators $x \mapsto ux$ and $x \mapsto xu$. This is in striking contrast with the corresponding situation for compact instead of weakly compact multiplication operators. Indeed, it follows from Theorem 3 in [15] that if E is any Banach space and $L(E)$ the Banach algebra of all bounded linear operators on E, then the map $X \mapsto TXT$ is a compact operator on $L(E)$ for every compact operator $T \in L(E)$, but neither $X \mapsto TX$ nor $X \mapsto XT$ is a compact operator on $L(E)$ for any nonzero $T \in L(E)$, unless E is finite-dimensional.

In section 4 (Corollary 3) we give one further characterization of dual C^*-algebras. A C^*-algebra turns out to be dual if and only if its positive cone is the closed convex hull of the union of its extreme rays. This criterion is obtained as a consequence of a characterization of the extreme rays of the positive cone of a C^*-algebra (Theorem 4.1).

2. Monomial actions and dual C^*-algebras

Definition. Let A be an algebra and $u \in A$. The mapping $F_u : A \to A$ is called a *monomial action* of u on A, if there is a finite sequence $d_1, \ldots, d_p$

where each d_j is 1 or -1 (both 1 and -1 occurring at least once), such that

$$F_u(x) = \prod_{j=1}^{p} 2^{-1}[(u+x) + d_j(u-x)]\,, \quad x \in A\,.$$

We now give a generalization and alternative proof of Theorem 5.2 in [14].

Theorem 2.1. *The C^*-algebra A is dual if and only if for every positive $u \in A$ there is some monomial action of u on A which maps the unit ball of A into a weakly compact set.*

Proof. If A is dual, then the operator $x \mapsto ux$ is weakly compact for all $u \in A$ (see [9, p. 21], [3, p. 99]). Let us suppose, conversely, that the monomial action condition is satisfied. Let B be an arbitrary maximal commutative $*$-subalgebra of A. Let $u \in B$ be positive and $x \mapsto F_u(x)$ a monomial action of u on B such that F_u maps the unit ball S of B into a $\sigma(B, B^*)$-compact subset of B. (Observe that B is $\sigma(A\,, A^*)$-closed and the relative $\sigma(A\,, A^*)$-topology on B is the same as $\sigma(B\,, B^*)$.) By commutativity $F_u(x) = u^m x^n$, $x \in B$, for some positive integers m and n. Denote $S_+ = \{y \in B | y \geq 0\,, \|y\| \leq 1\}$. By spectral theory $S_+ = \{y^n | y \in S_+\}$, and so $u^m S_+$ is relatively $\sigma(B\,, B^*)$-compact. Since the operator $x \mapsto u^m x$ maps S into $u^m(S_+ - S_+ + i(S_+ - S_+))$, u^m is a weakly completely continuous (w.c.c.) element of B in the sense of [8] and [18], and so is $u = (u^m)^{1/m}$, which is a norm limit of polynomials in u^m having no constant term. (The w.c.c. elements of B form a closed ideal.) Thus every element of B is w.c.c. since its every positive element is w.c.c. and $B = B_+ - B_+ + i(B_+ - B_+)$, where $B_+ = \{x \in B | x \geq 0\}$. By Lemma 5 in [9, p. 20] the maximal ideal space of every maximal commutative $*$-subalgebra of A is discrete, and so A is dual by [10, Theorem 1].

Corollary. *Let $x \circ y = 2^{-1}(xy + yx)$ denote the Jordan product in the C^*-algebra A. A is dual if and only if the map $x \mapsto u \circ x$ is a weakly compact operator on A for every $u \in A$.*

Proof. If A is dual, the operator $x \mapsto u \circ x$ is weakly compact, because the operators $x \mapsto ux$ and $x \mapsto xu$ are so (see [9, p. 21], [3, p. 99]). The converse follows from the above theorem (or [14, Theorem 5.2]) and the well-known identity

$$uxu = 2(u \circ (u \circ x)) - u^2 \circ x\,.$$

3. A characterization of weakly semi-completely continuous elements

If A is a Banach algebra with identity and $x \in A$, we let $Sp_A x$ denote the spectrum of x. For an arbitrary C^*-algebra A and $x \in A$,

$Sp_A'x$ will stand for $Sp_{A_1}x$, where A_1 is the C^*-algebra obtained by adjoining an identity to A.

Lemma 3.1. *If u is a self-adjoint w.s.c.c. element of the C^*-algebra A, then $Sp_A'u$ consists of a countable number of points, which can accumulate only in the origin.*

Proof. Let A_u be the sub-C^*-algebra of A generated by u. Since A_u is commutative, the map $x \mapsto u^2x$ is a weakly compact operator on A_u. (Observe that A_u is $\sigma(A, A^*)$-closed and the relative $\sigma(A, A^*)$-topology on A_u agrees with $\sigma(A_u, A_u^*)$.) By theorem 3.1 in [18] u^2 is a compact element of A_u, so that by Theorems 1.4 and 1.6 in [16] $Sp_{A_u}'u^2$ is countable and can accumulate only in the origin. Theorem 1.6.10 in [11, p. 32] shows that $Sp_{A_u}'u$ has the same properties, and so has $Sp_A'u$, because $Sp_A'u = Sp_{A_u}'u$ [3, Proposition 1.3.10].

The next lemma is due to Ogasawara [8, p. 361]. The proof of Proposition 2 in [12, p. 661] contains a different method. We sketch still another proof based on a characterization of semi-simple finite-dimensional Banach algebras due to I. Kaplansky.

Lemms 3.2. *Every reflexive C^*-algebra is finite-dimensional.*

Proof. Let A be a reflexive C^*-algebra. Then A has an identity 1 by the theorems of Alaoglu-Bourbaki and Krein-Milman, and Proposition 1.6.1 in [13]. Alternatively, apply [3, Cor. 12.1.3, p. 236]. For any $u \in A$, let A_u denote the closed subalgebra of A generated by u and 1. Choose an arbitrary $x \in A$ and write $x = a + ib$ with a and b self-adjoint. Then A_a is isometrically isomorphic to $C(X)$ for some compact Hausdorff space X. Since $C(X)$ is reflexive [4, p. 67], the characteristic function of every singleton is continuous, and so X is finite. Similarly, A_b is finite-dimensional. It follows that A_x is the linear span of a finite set $\{1, a, b\} \cup \{a^jb^k | 1 \leq j \leq m, 1 \leq k \leq n\}$ so that $Sp_Ax \subset Sp_{A_x}x$ is finite. An application of Lemma 7 in [7, p. 376] completes the proof.

We are now in a position to prove the main result of this section.

Theorem 3.1. *Let A be a C^*-algebra and $u \in A$. The operator $x \mapsto uxu$ on A is compact if and only if it is weakly compact.*

Proof. The weakly compact operators on A form a two-sided ideal of $L(A)$ [4, p. 484]. It follows easily that y^*, xy and yx are w.s.c.c. if $y \in A$ is w.s.c.c. and $x \in A$. Assume that $u \in A$ is w.s.c.c. Then u^*u is a self-adjoint w.s.c.c. element of A. If A is realized as a sub-C^*-algebra of $L(H)$ for a Hilbert space H, then $Sp_A'u^*u = Sp_{L(H)}'u^*u$ [3, p. 8], so that the spectrum of u^*u as an operator on H is by Lemma 3.1 countable and can accumulate only in the origin, the nonzero part in that spectrum being the same as in $Sp_{L(H)}'u^*u$. Therefore the method used in the proof of Theorem 3.8 in [16] yields a representation $u^*u = \sum \lambda_n e_n$, where each

e_n is a w.s.c.c. projection, $\lambda_n > 0$, and the series converges in norm. Since the closed unit ball of e_nAe_n is $\sigma(A, A^*)$-compact and thus $\sigma(e_nAe_n, (e_nAe_n)^*)$-compact, the C^*-algebra e_nAe_n is reflexive [4, p. 425], and therefore finite-dimensional by Lemma 3.2. It follows from Theorem 3.10 in [16] that $|u| = (u^*u)^{1/2} = \sum \lambda_n^{1/2} e_n$ is a compact element of A, and so is $|u^*| = (uu^*)^{1/2}$ by a similar argument. Since the map $x \mapsto |u| \, x \, |u^*|$ is a compact operator on A by Theorem 3.9 in [16], so is the map $x \mapsto uxu = s|u| \, x \, |u^*| t^*$, where $u = s|u|$ and $u^* = t|u^*|$ are the polar decompositions of u and u^*. The »only if» part being obvious, the theorem is proved.

Combined with the theory of the compact elements of C^*-algebras the above theorem yields many corollaries. We state explicitly the analogue of Corollary 2 in [18], section 3.

Corollary. *Let H be a Hilbert space and A an irreducible sub-C^*-algebra of $L(H)$. Then $T \in A$ is a compact operator on H if and only if the map $X \mapsto TXT$ is a weakly compact operator on A.*

Proof. Since the compactness of the operator $X \mapsto TXT$ on A is equivalent to the compactness of T (see [15, Theorem 3] and [16, Corollary 2, p. 15]), the corollary is a consequence of Theorem 3.1.

4. The extreme rays of the positive cone

There are several equivalent definitions of an extreme ray of a convex cone in a real vector space (see e.g. [2, pp. 98—99]). For example, a geometric characterization requires the relative complement of the ray with respect to the cone to be convex. In dealing with the extreme rays of the positive cone of a C^*-algebra the description we give below, valid for any convex proper (pointed) cones (see [2, pp. 100—101]), turns out to be particularly convenient. Let A be a C^*-algebra and $a \in A$, $0 \leq a \neq 0$. Denote $\mathbf{R}_+ = \{\lambda \in \mathbf{R} | \lambda \geq 0\}$. The ray $\delta_a = \mathbf{R}_+ a$ generated by a is called an *extreme ray* of the positive cone $P = \{u \in A | u \geq 0\}$ if $a - b \in P$ for $b \in P$ implies $b \in \delta_a$. The definition is clearly independent of the choice of the generator a of the ray. Note that we have adopted the convention of including the endpoint 0 in the rays of P. To avoid trivialities, the C^*-algebras we consider are assumed to contain nonzero elements.

Theorem 4.1. *Let A be a C^*-algebra, $P = \{u \in A | u \geq 0\}$, and $0 \neq a \in P$. Then $\delta_a = \mathbf{R}_+ a$ is an extreme ray of P if and only if aAa is one-dimensional, in which case $a = \alpha e$ for some projection $e \in A$ and $\alpha \in \mathbf{R}_+$.*

Proof. Suppose δ_a is an extreme ray of P. Denote $e = \|a\|^{-1}a$. Then $e - e^2 \in P$, since $\|e\| = 1$, so that by assumption $e^2 = \lambda e$ for some

$\lambda \in \mathbf{R}_+$. As $\|e\| = \|e^2\| = 1$, we have $\lambda = 1$, and so e is a projection. We consider A realized as a sub-C^*-algebra of $L(H)$ for a Hilbert space H. Take $b \in P$. For every $\xi \in H$ we have $(ebe\xi, \xi) = (be\xi, e\xi) \leq \|b\|\|e\xi\|^2 = \|b\|(e\xi, \xi)$, and so by assumption $ebe \in \delta_a$. Since A is the linear span of P, we have thus $aAa = eAe = \mathbf{C}a$. Suppose, conversely, that $\dim(aAa) = 1$. The technique used in the proof of Theorem 3.8 in [16] shows that $a = \mu e$ for some nonzero projection $e \in A$ and $\mu \in \mathbf{R}_+$. As $e \in eAe$, we have thus $aAa = eAe = \mathbf{C}e$. We show that $\delta_e (= \delta_a)$ is an extreme ray of P. If $0 \leq b \leq e$, then $eb = be = b$ (see e.g. [5, p. 89]). Thus $b = ebe \in eAe = \mathbf{C}e$, and so $b = \nu e$ for some $\nu \in \mathbf{R}_+$.

Corollary 1. *The extreme rays of the positive cone of a C^*-algebra A are precisely the rays generated by the projections $e \in A$ with* $\dim(eAe) = 1$. *In particular, the extreme rays of the cone of all positive operators on a Hilbert space are those generated by the (orthogonal) projections onto one-dimensional subspaces.*

In the next two corollaries A is a C^*-algebra, whose socle in the sense of [11, p. 46], if it exists, is denoted by S. As before, $P = \{u \in A \mid u \geq 0\}$. We let U denote the union of the extreme rays of P.

Corollary 2. *The socle S of A exists if and only if P has at least one extreme ray. If this is the case, $P \cap S$ is the convex hull of U, and S is the linear span of U.*

Proof. All statements follow from the above theorem in conjunction with Theorems 5.1, 3.3, 3.10, 3.8 and 4.2 in [16].

Corollary 3. *Let C denote the set of the compact (equivalently, weakly completely continuous, or weakly semicompletely continuous, see* [18] *and section* 3) *elements of A. P has no extreme rays if and only if $C = \{0\}$. If $C \neq \{0\}$, then C is the closed linear span of U and $C \cap P$ is the closed convex hull of U. The following three conditions are equivalent:*

(i) *A is dual,*

(ii) *P is the closed convex hull of U,*

(iii) *A is the closed linear span of U.*

Proof. The first two assertions follow from Corollary 2 and the fact that $C \neq \{0\}$ if and only if S exists, in which case C is the norm closure of S (see Theorems 5.1 and 3.10 in [16]). Since every positive $u \in C$ can be approximated in norm with elements from $S \cap P$ by the same theorems and Theorem 3.8 in [16], Corollary 2 shows that $C \cap P$ is the closed convex hull of U, if $C \neq \{0\}$. A is known to be dual if and only if $A = C$ [1, Corollary 8.3], and on the other hand $A = C$ if and only if $A \cap P = C \cap P$, since A is the linear span of P and C is a linear subspace of A [16, Theorem 3.10]. Therefore the last claim follows from the first part of the corollary.

Remark. All previously known characterizations of a dual C^*-algebra involve directly the multiplicative structure of the algebra. The existence of a characterization only in terms of the structure of an ordered Banach space, as above, is not surprising, however. For if A and B are C^*-algebras and $T : A \to B$ a linear, isometric order isomorphism, then T preserves self-adjointness and the Jordan algebra structure of the spaces of self-adjoint elements in A and B (see [6, p. 502] or [17, p. 33]), and it follows easily that $T(2^{-1}xy + 2^{-1}yx) = T(x \circ y) = Tx \circ Ty$ for all $x, y \in A$, so that by the corollary in section 2, A is dual if and only if B is dual. Our present approach yields a slightly sharper result in this direction:

Corollary 4. *Let A and B be C^*-algebras and $T : A \to B$ a vector space isomorphism such that $Ta \geq 0$ if and only if $a \geq 0$. Then T maps the socle of A (if it exists) onto that of B, and A is dual if and only if B is dual.*

Proof. As T preserves the property of being an extreme ray of the positive cone, the first assertion is a consequence of Corollary 2. Since T is continuous (see the proof of Theorem 3.1 in [17]), the second assertion follows from the first and the fact that a C^*-algebra is dual if and only if it has a dense socle (see [1, Corollary 8.4] or [3, p. 99]).

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Printed in January 1975