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ON IRREDUCIBLE MODULES OF A LIE ALGEBRA WHICH ARE COMPOSED OF FINITE-DIMENSIONAL MODULES OF A SUBALGEBRA

BY

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1. Introduction

Let G be a Lie algebra and K a subalgebra of G. If K is semi-simple (or at least reductive) then the finite-dimensional K-modules are well-known. We can then pose the following question: What are the irreducible G-modules which, when regarded as a K-module, are direct sums of irreducible finite-dimensional K-modules? We call such modules K-finite.

This problem has been extensively studied in the following special case (see e.g. [1]—[3], [5], [7]): Let $\mathcal{G}$ be a non-compact semisimple Lie group and let $\mathcal{K}$ be the maximal compact subgroup of $\mathcal{G}$. Let G (resp. K) be the Lie algebra of $\mathcal{G}$ (resp. $\mathcal{K}$). As was shown by Harish-Chandra, study of unitary irreducible representations of $\mathcal{G}$ in a Hilbert space leads in a natural way to a study of irreducible K-finite G-modules.

In this paper G is an arbitrary (finite-dimensional) complex Lie algebra and K is a semi-simple (or reductive) subalgebra of G. The work is divided into two parts. In section 3 we study irreducible G-modules admitting a vector of maximal weight λ with respect to a Cartan subalgebra H_T of G such that $H = K \cap H_T$ is a Cartan subalgebra of K. We prove that for a »special» subalgebra K (Definition 3.7) and for any weight λ such that the restriction $\lambda|_H$ is a dominant integral weight of K there exists a unique equivalence class of K-finite G-modules which have the maximal weight λ.

In section 4 we study irreducible G-modules V with the help of the minimal component $V_{\min}$ of V; if α is a dominant integral weight of K we denote by V_α the sum of all irreducible finite-dimensional K-modules in V which have α as their maximal weight; by definition $V_{\min} = V_\alpha$ if $V_\alpha \neq 0$ and $V_\beta = 0$ for all $\beta < \alpha$. Let rank $G =$ rank K. We prove that if α is »large enough» (see Definition 4.1) then there exists a unique equivalence class $[V]$ of irreducible K-finite G-modules V such that $V_{\min} = V_\alpha$. Such G-modules are called discrete because they are completely characterized by the weight α i.e. by a sequence of integers consisting of the components of α.

We take profit at crucial steps (Theorem 3.9 and Lemma 4.7) of the results of J. Lepowsky and G. W. McCollum, [6]: If V is a G-module such that $V_\alpha \neq 0$ for some dominant integral weight α and V is generated by V_α ($V = \mathcal{E}(C)V_\alpha$ where $\mathcal{E}(G)$ is the enveloping algebra of G) then V is K-finite. In addition V is completely determined by the action of $\mathcal{E}(K)\,C$ on V_α where C is the centralizer of K in $\mathcal{E}(G)$. These results have earlier been obtained by Harish-Chandra, [1] and [2], in case G is semi-simple.

See also for related recent results by van den Hombergh in »A note on Mickelsson's step algebra» and »On some Harish-Chandra modules» (to appear in Indagationes Mathematicæ).

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2. Notation

In this paper all Lie algebras are finite-dimensional. All algebras and vector spaces are over **C**, the field of complex numbers. If A is an algebra, V an A-module, S a subset of A and X a subset of V we denote by SX the subset $\{sx \mid s \in S\,,\ x \in X\}$ of V.

We denote by V^* the dual space of a vector space V.

Let K be a semi-simple Lie algebra, H a Cartan subalgebra of K and $\langle\ ,\ \rangle$ the Killing form on K. Because $\langle\ ,\ \rangle$ is non-degenerate, there exists (for each $\alpha \in H^*$) an element $h_\alpha \in H$ such that $\langle h_\alpha\,, h\rangle = \alpha(h)$ for all $h \in H$. Let $(\ ,\ )$ be the symmetric non-degenerate bilinear form on H^* defined by $(\alpha\,,\beta) = \langle h_\alpha\,, h_\beta\rangle$.

For any Lie algebra G, we denote by $x \mapsto \operatorname{ad} x$ the adjoint representation of G, $\operatorname{ad}x(y) = [x\,, y]$.

If L is any Lie algebra then $\mathcal{E}(L)$ is the universal enveloping algebra of L. If L' is a subalgebra of L then $\mathcal{E}(L')$ can be identified in a natural way with a subalgebra of $\mathcal{E}(L)$.

If $\{x_1\,, x_2\,, \ldots, x_n\}$ is a basis of L then the monomials $x_{i(1)}x_{i(2)}\ldots x_{i(k)}$ ($i(1) \leqq i(2) \leqq \ldots \leqq i(k)$) along with **1** give a basis of $\mathcal{E}(L)$ (Poincaré-Birkhoff-Witt theorem).

For any subset S of $\mathcal{E}(L)$ we denote by $\mathcal{J}(S)$ the left ideal generated by S, $\mathcal{J}(S) = \mathcal{E}(L)S$.

Let $\mathcal{B}$ be an associative algebra, $\mathcal{A}_1$ an ideal of $\mathcal{B}$ and V a $\mathcal{B}/\mathcal{A}_1$-module. If $\mathcal{A}_2$ is any ideal of $\mathcal{B}$ such that $\mathcal{A}_2 \subset \mathcal{A}_1$ then by the extension of V into a $\mathcal{B}/\mathcal{A}_2$-module we mean the $\mathcal{B}/\mathcal{A}_2$-module V, where the action of $\mathcal{B}/\mathcal{A}_2$ is defined by

$$(b + \mathcal{A}_2)v = i(b)v\,;\ b \in \mathcal{B}\,,\ v \in V\,,$$

where $i : \mathcal{B} \to \mathcal{B}/\mathcal{A}_1$ is the canonical projection.

3. Modules with maximal weight

If L is any Lie algebra and H_L a Cartan subalgebra of L then we can write

$$L = H_L \oplus \bigoplus_{\alpha \neq 0} L_\alpha$$

where L_α is the root subspace corresponding to the non-zero root α. Because we assume L to be finite-dimensional, the sum is finite. By definition L_α consists of all elements $x \in L$ such that

$$(\operatorname{ad} h - \alpha(h))^n x = 0 \text{ for some positive integer } n .$$

Let H_L^* be the dual of H_L and $\{h_1, h_2, \ldots, h_l\}$ a fixed basis of H_L. Let $\lambda, \mu \in H_L^*$ with $\lambda \neq \mu$. We say that λ is bigger than μ $(\lambda > \mu)$ if the first non-zero number in the sequence

$$\lambda(h_1) - \mu(h_1), \ldots, \lambda(h_l) - \mu(h_l)$$

is of the form $x + iy$ with $x > 0$ or $x = 0$ and $y > 0$. We denote by $L_+(L_-)$ the solvable subalgebra of L generated by the subspaces L_α with $\alpha > 0$ $(\alpha < 0)$.

Definition 3.1. Let V be an L-module. We denote by V^+ the subspace of V consisting of all vectors v with the property $xv = 0$ for all $x \in L_+$. We say that V is bounded above if $1 \leqq \dim V^+ < \infty$.

Lemma 3.2. *If V is an irreducible L-module which is bounded above then $\dim V^+ = 1$ and there exists $\lambda \in H_L^*$ such that $hv = \lambda(h)v$ for each $v \in V^+$ and $h \in H_L$. (We say that λ is the maximal weight of V and v is a maximal vector.)*

Proof. Because $1 \leqq \dim V^+ < \infty$ and H_L is nilpotent there is a common eigenvector $v \in V^+$ of all $h \in H_L$ (V^+ is clearly H_L-invariant). Because of the irreducibility of V and of the Poincaré-Birkhoff-Witt theorem we have $V = \mathcal{E}(L_-)v$. If $v' = uv$, $u \in \mathcal{E}(L_-)L_-$, is another vector in V^+ then

$$V' = \mathcal{E}(L)v' = \mathcal{E}(L_-)\mathcal{E}(H_L)v'$$

is a non-trivial $(v \notin V')$ invariant subspace if $v' \neq 0$. Thus $v' = 0$ and $\{v\}$ is a basis of V^+.

Definition 3.3. Let V be an L-module. For each $\alpha \in H_L^*$ the weight subspace V_α consists of all vectors $v \in V$ for which

$$(h - \alpha(h))^n v = 0 \text{ for all } h \in H_L \text{ and for some positive integer } n .$$

Theorem 3.4. *Let L be a Lie algebra and let H_L be a Cartan subalgebra of L. For each $\lambda \in H_L^*$ such that $\lambda|_{[H_L, H_L]} = 0$ there exists a unique equivalence class of irreducible L-modules which are bounded above and have maximal weight λ. Any such an L-module is a direct sum of weight subspaces* *of finite dimension.*

Proof. We define $W^\lambda = \mathcal{E}(L)/\mathcal{J}_\lambda$ where $\mathcal{J}_\lambda$ is an ideal,

$$\mathcal{J}_\lambda = \mathcal{J}(L_+) + \mathcal{J}(\{h - \lambda(h) \cdot \mathbf{1} \mid h \in H_L\}) .$$

W^λ is an L_- (and $\mathcal{E}(L) -$) module in a natural way. Let $v_0 = \mathbf{1} + \mathcal{J}_\lambda$. Then

$$L_+ v_0 = 0 , \ h v_0 = \lambda(h) v_0 \ (h \in H_L) , \ W^\lambda = \mathcal{E}(L_-) v_0 .$$

It follows that a basis of W^λ is given by vectors of the type

$$(*) \qquad e_{\beta_1} e_{\beta_2} \ldots e_{\beta_k} v_0 \ (k = 0 , 1 , 2 , \ldots)$$

where e_{β_i} $(i = 1 , 2 , \ldots , k)$ is any element of some fixed basis of L_{β_i} and $0 > \beta_1 \geqq \beta_2 \geqq \ldots \geqq \beta_k$ are negative roots of L. Let W^λ_α be the subspace of W^λ spanned by the vectors (*) for which

$$\alpha = \beta_1 + \ldots + \beta_k + \lambda .$$

We show by induction on k than W^λ_α is a weight subspace with weight α. Assume that the vector v has weight β,

$$(h - \beta(h))^n v = 0 \ \text{ for all } \ h \in H_L .$$

Let γ be a root and $e_\gamma \in L_\gamma$,

$$(\text{ad } h - \gamma(h))^m \, e_\gamma = 0 \ \text{ for all } h \in H_L .$$

Then

$$(h - (\beta + \gamma)(h))^{n+m} e_\gamma v = \sum_{k=0}^{n+m} (\text{ad } h - \gamma(h))^k e_\gamma \cdot \binom{n+m}{k} \cdot$$
$$\cdot \, (h - \beta(h))^{n+m-k} v = 0 .$$

Thus the we'ght of $e_\gamma v$ is $\beta + \gamma$. It follows that each $v \in W^\lambda_\alpha$ is of weight $\alpha \leqq \lambda$. It is clear that dim $W^\lambda_\alpha < \infty$ and each vector of weight α belongs to W^λ_α. Note that W^λ_λ is spanned by the vector $v_0 = \mathbf{1} + \mathcal{J}_\lambda$. Let N^λ be the sum of all invariant subspaces in W^λ which do not contain v_0. Then $v_0 \notin N^\lambda$ and we define

$$V^\lambda = W^\lambda / N^\lambda .$$

The L-module V^λ is irreducible, has $v_0 + N^\lambda$ as the maximal vector and λ is the maximal weight. The uniqueness part of the proof goes as in the case of a semisimple Lie algebra (see [4, p. 109]).

Let G be a Lie algebra, K a semi-simple subalgebra of G and H a Cartan subalgebra of K. Because K is semi-simple, there exists a subspace T in G such that $G = K \oplus T$ and $[K , T] \subset T$. We denote by T_0 the null component of H in T ,

$$T_0 = \{x \in T \mid [h, x] = 0 \quad \forall\, h \in H\}.$$

Lemma 3.5. *Let H_T be a Cartan subalgebra of the Lie algebra $H + T_0 \subset G$. Then $H \subset H_T$ and H_T is even a Cartan subalgebra of G.*

Proof. There exists $x \in H + T_0$ such that

$$H_T = \{y \in H + T_0 \mid (\mathrm{ad}\, x)^n y = 0 \ \text{ for some } n \in N\},$$

[4, pp.79—80]. Now $[x, H] = 0$ for all $x \in H + T_0$, thus $H \subset H_T$. Next let S be the normalizer of H_T in G. From $[S, H_T] \subset H_T$ it follows that $[S, H] = 0$ and therefore $S \subset H + T_0$. Because H_T is a Cartan subalgebra of $H + T_0$ it follows that $S = H_T$ and we can conclude that H_T is a Cartan subalgebra of G.

Let Φ be the set of roots of K relative to H, $\Delta \subset \Phi$ is a set of simple roots and Φ^+ (resp. Φ^-) is the set of positive (resp. negative) roots with respect to Δ. Next we divide T into weight subspaces,

$$T_\lambda = \{x \in T \mid [h, x] = \lambda(h)\, x,\ \forall\, h \in H\}.$$

We denote by Ψ the set of weights of K in T, Ψ^+ (resp. Ψ^-) is the set of positive (resp. negative weights in Ψ relative to an ordered basis $\{h_1, \ldots, h_l\}$ of H which is dual to the basis $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\}$,

$$\alpha_i(h_j) = \langle h_{\alpha_i}, h_j\rangle = \delta_{ij}.$$

Here $\alpha_1, \ldots, \alpha_l$ are the distinct simple roots of K.

Definition 3.6. The semi-simple subalgebra K of G is a special subalgebra if

$$\mathbf{N}(\{\alpha\}) \cap \mathbf{N}(\Psi^+) = \{0\}$$

for all $\alpha \in \Delta$. If $\Omega \subset H^*$ is any subset, we denote by $\mathbf{N}(\Omega)$ the linear span of Ω with non-negative integral coefficients.

Example 3.7. Let $G = \mathrm{gl}\,(n, \mathbf{C})$, the Lie algebra with basis $\{e_{ij}\}_{i,j=1}^n$ and commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}.$$

Let K be the subalgebra spanned by the vectors $(2 \leqq p \leqq n - 2)$

$$e_{ij},\ i \neq j,\ 1 \leqq i, j \leqq p;\ e_{ii} - e_{i+1\, i+1},\ i = 1, 2, \ldots, p-1;$$

and

$$e_{ij},\ i \neq j,\ p+1 \leqq i, j \leqq n;\ e_{ii} - e_{i+1\, i+1},\ i = p+1, p+2, \ldots, n-1.$$

Note that K is isomorphic to $A_{p-1} \oplus A_{n-p-1}$. As H we can take the subalgebra of K spanned by the vectors $e_{ii} - e_{i+1\, i+1}$, $1 \leqq i \leqq n-1$, $i \neq p$. It is easily seen that in this case

$$T_0 = \{a \sum_{i=1}^{p} e_{ii} + b \sum_{i=p+1}^{n} e_{ii} \mid a , b \in \mathbf{C}\}$$

and $H_T = H + T_0$. Using the properties of the roots of the classical simple Lie algebras A_l it is not difficult to verify that K is special.

We return to the general case. Let K be a special subalgebra of G and $l = \operatorname{rank} K$, $p = \operatorname{rank} G$. We fix an ordered basis $\{h_1, h_2, \ldots, h_p\}$ of H_T such that $\{h_1, h_2, \ldots, h_l\}$ is the basis of H described above. We define the following subalgebras of G:

$$G_+ = K_+ + \sum_{\lambda>0} T_\lambda + S_+ ,$$

$$G_- = K_- + \sum_{\lambda<0} T_\lambda + S_-$$

where K_+ (resp. K_-) is the subalgebra of K spanned by the vectors belonging to positive (resp. negative) roots of K. We define

$$H + T_0 = S_+ + S_- + H_T$$

to be the corresponding decomposition for $H + T_0$. Because of our choice of basis of H_T (see also Lemma 3.5) it is clear that $G = G_+ + G_- + H_T$ is a similar decomposition for G relative to the Cartan subalgebra H_T.

Definition 3.8. A G-module V is K-finite if it is a sum of finite-dimensional K-modules when considered as a K-module by restriction to K.

Let Λ be the set of dominant integral elements in H^*:

$$\Lambda = \{\lambda \in H^* \mid (\lambda , \alpha) \text{ is a non-negative integer for all } \alpha \in \Delta\} .$$

Theorem 3.9. *Let K be a special subalgebra of G. Then for each $\lambda \in H_T^*$ such that $\lambda|_H \in \Lambda$ and $\lambda|_{[H_T, H_T]} = 0$ there exists a unique equivalence class of K-finite irreducible G-modules which are bounded above and have λ as the maximal weight.*

Proof. The uniqueness follows from Theorem 3.4. We have to prove the existence. We define an ideal

$$\mathfrak{I}_\lambda = \mathfrak{I}(G_+) + \mathfrak{I}(\{h - \lambda(h) \cdot \mathbf{1} \mid h \in H_T\})$$

and $W^\lambda = \mathscr{E}(G)/\mathfrak{I}_\lambda$. Consider the subset S_λ of W^λ,

$$S_\lambda = \{e_{-\alpha}^{n_\alpha+1} + \mathfrak{I}_\lambda \mid \alpha \in \Delta\}$$

where $e_{-\alpha}$ belongs to the root $-\alpha$ and

$$n_\alpha = 2 \cdot \frac{(\lambda|_H , \alpha)}{(\alpha , \alpha)} , \quad \alpha \in \Delta .$$

Let $U^\lambda = \mathscr{E}(G)\, S_\lambda$ be the submodule of W^λ generated by S_λ. We claim that U^λ does not contain the vector $\mathbf{1} + \mathfrak{I}_\lambda$. It is well-known that S_λ

is annihilated by K_+ (see [4, p. 115]). Now W^λ is a direct sum of weight subspaces, W^λ_λ is spanned by the vector $\mathbf{1} + \mathcal{I}_\lambda$ and λ is the highest weight in W^λ (compare the proof of Theorem 3.4). Suppose that $\mathbf{1} + \mathcal{I}_\lambda \in U^\lambda$; using the Poincaré-Birkhoff-Witt theorem it is easily seen that then there exists $\beta_1, \beta_2, \ldots, \beta_k \in \Psi^+$ such that

$$\beta_1 + \beta_2 + \ldots + \beta_k - (n_\alpha + 1) \cdot \alpha = 0$$

for at least one weight $\alpha \in \Delta$. But this is impossible because K is a special subalgebra of G.

Let again N^λ be the sum of all invariant subspaces of W^λ not containing the vector $\mathbf{1} + \mathcal{I}_\lambda$. It is clear that $U^\lambda \subset N^\lambda$. We define

$$V^\lambda = W^\lambda / N^\lambda .$$

The G-module V^λ is irreducible and has a maximal vector $v = \mathbf{1} + \mathcal{I}_\lambda + N^\lambda$ of weight λ. Furthermore, V^λ contains a finite-dimensional K-module, namely $\mathcal{E}(K)v$ ([4, p. 115]). It follows from proposition 4.2, [6], that V^λ is K-finite. (See also [1, Theorem 1.])

4. Discrete G-modules

If not otherwise stated, the notation of the previous sections is in force also in this section.

Let C be the centralizer of K in $\mathcal{E}(G)$. The algebra C is a finitely generated subalgebra of $\mathcal{E}(G)$ (see [8, p. 162, Theorem 2.3.1.4]).

Let an ordered basis $\{t_1, t_2, \ldots, t_r\}$ be given for the subspace T of G, such that

$$[h, t_i] = \lambda_i(h) t_i , \quad h \in H , \quad i = 1, 2, \ldots, r ,$$

where $\lambda_i \in \Psi$ $(i = 1, 2, \ldots, \nu)$ and $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r$. We complete this to an ordered basis of G,

$$\{t_i , e_{\alpha_i} , h_i , e_{\beta_i}\}$$

where the order is defined through the ordering of roots,

$$\alpha_1 < \alpha_2 < \ldots < \alpha_q < 0 < \beta_1 < \beta_2 < \ldots < \beta_q ,$$

and through a labelling of the basis elements h_i of H. According to the Poincaré-Birkhoff-Witt theorem this ordering induces a basis for $\mathcal{E}(G)$ by ordered monomials in the basis elements of G. If $u \in \mathcal{E}(G)$ is such a basis vector we denote by deg (u) the number of vectors t_i contained in u. If $v \in \mathcal{E}(G)$ is an arbitrary (finite) linear combination of ordered monomials,

$$v = \sum_{k=1}^{n} a_k u_k \quad (a_k \in \mathbf{C}) ,$$

we define $\deg (v) = \max_{k=1,\ldots,n} \deg (u_k)$.

Definition 4.1. Let $c_1, c_2, \ldots, c_\zeta$ be a generating sequence of C. We define

$$n_c = \max_{k=1,2,\ldots,\zeta} \deg (c_k) .$$

We call an element λ of Λ, the set of dominant integral weights of K, large if

$$\lambda + \omega_1 + \omega_2 + \ldots + \omega_k \in \Lambda \text{ for all } \omega_i \in \Psi , \; k = 1, 2, \ldots, n_c .$$

Let V be any G-module. Consider V as a K-module by restriction. For any $\lambda \in \Lambda$ we denote by V_λ the sum of all irreducible finite-dimensional K-submodules of V with maximal weight λ. We define

$$V_\lambda^+ = \{x \in V_\lambda \mid e_\beta x = 0 \quad \forall \beta \in \Phi^+\} ,$$

the subspace of vectors with maximal weight in V_λ; in other words,

$$V_\lambda^+ = \{x \in V_\lambda \mid hx = \lambda(h)x \quad \forall h \in H\} .$$

We denote by $\mathcal{I}_\lambda$ the annihilator in $\mathcal{E}(K)$ of the maximal vector in an irreducible finite-dimensional K-module with maximal weight λ; according to [4, p. 115],

$$\mathcal{I}_\lambda = \mathcal{J}(K_+) + \mathcal{J}(\{h - \lambda(h) \cdot \mathbf{1} \mid h \in H\}) + \mathcal{J}(\{e_{-\alpha}^{n_\alpha + 1} \mid \alpha \in \Delta\})$$

where K_+ and the numbers n_α are defined as in section 3.

For all β and α in Λ we define $A_{\beta,\alpha}$ to be the subset of $\mathcal{E}(G)$ for which

$$A_{\beta,\alpha} V_\alpha^+ \subset V_\beta^+$$

for any G-module V.

Lemma 4.2. $A_{\beta,\alpha} = \{u \in \mathcal{E}(G) \mid \mathcal{I}_\beta u \subset \mathcal{E}(G) \mathcal{I}_\alpha\}$.

Proof. Let V be a G-module such that $V_\alpha \neq 0$. Any such G-module is a factor module of the left-module $\mathcal{E}(G)/\mathcal{E}(G)\mathcal{I}_\alpha$. It follows that

$$A_{\beta,\alpha} = \{u \in \mathcal{E}(G) \mid u \, V_\alpha^+ \subset V_\beta^+\}$$

where $V = \mathcal{E}(G)/\mathcal{E}(G)\mathcal{I}_\alpha$. Let now $u \in A_{\beta,\alpha}$. If $x = \mathbf{1} + \mathcal{E}(G)\mathcal{I}_\alpha$ then $x \in V_\alpha^+$ and

$$ux = u + \mathcal{E}(G)\mathcal{I}_\alpha \in V_\beta^+$$

and therefore $\mathcal{I}_\beta u \subset \mathcal{E}(G)\mathcal{I}_\alpha$. To prove the converse, assume that $\mathcal{I}_\beta u \subset \mathcal{E}(G)\mathcal{I}_\alpha$. Let $x \in V_\alpha^+$. Then

$$\mathcal{I}_\beta ux \subset \mathcal{E}(G)\mathcal{I}x = 0 .$$

It follows that $\mathcal{E}(K)ux$ is a finite-dimensional K-module with ux as the vector of maximal weight (which is β) and thus $ux \in V_\beta^+$.

Lemma 4.3. *Let* ω *be an element of* Λ *such that* $\omega + \lambda \in \Lambda$ *for any* $\lambda \in \Psi$. *Then for each* $t_i \in T$ *there exists* $u \in A_{\omega+\lambda_i,\omega}$ *of the form*

$$u = t_i + \sum_{j,\lambda_j > \lambda_i} t_j v_j$$

where $v_j \in \mathcal{E}(K_-)$.

Proof. We can write

$$(*) \qquad T = \bigoplus_\nu T^{(\nu)}$$

where $T^{(\nu)}$ is the irreducible component of T under the adjoint action of K, with maximal weight ν. We can assume that the basis $\{t_j\}_{j=1}^r$ of T is chosen in such a way that it is compatible with the decomposition (*); thus we may assume that $t_i \in T^{(\nu)}$ for some weight ν.

Put $D_\omega = \mathcal{E}(K)/\mathcal{I}_\omega$ and consider the tensor product $T^{(\nu)} \otimes D_\omega$, which is a K-module under the diagonal action:

$$k(x \otimes y) = [k , x] \otimes y + x \otimes ky; \quad k \in K , \quad x \in T^{(\nu)} , \quad y \in D_\omega .$$

It is known that the module $T^{(\nu)} \otimes D_\omega$ contains an irreducible submodule with maximal weight $\omega + \lambda_i$ (note that $\omega + \lambda \in \Lambda$ for any weight λ in $T^{(\nu)}$) with a multiplicity which is equal to the multiplicity $m(\lambda_i , \nu)$ of the weigth λ_i in $T^{(\nu)}$; in other words there are $m(\lambda_i , \nu)$ linearly independent vectors in $T^{(\nu)} \otimes D_\omega$ which are annihilated by $\mathcal{I}_{\omega+\lambda_i}$ (see e.g. [4, pp. 141—142]). It follows that for each $t_i \in T^{(\nu)}_{\lambda_i} = T_{\lambda_i} \cap T^{(\nu)}$ there exists a nonzero element u_0 of $T^{(\nu)} \otimes D_\omega$ of the form

$$u_0 = t_i \otimes (a \cdot \mathbf{1} + \mathcal{I}_\omega) + \sum_{\substack{j \\ \lambda_j > \lambda_i}} t_j \otimes (v_j + \mathcal{I}_\omega) ,$$

$$a \in \mathbf{C} , \quad v_j \in \mathcal{E}(K_-) ,$$

such that u_0 is annihilated by $\mathcal{I}_{\omega+\lambda_i}$. We define $v_i = a \cdot \mathbf{1}$ and let k be the smallest value of the index j for which $v_j \notin \mathcal{I}_\omega$; because of

$$e_\alpha u_0 = 0 \text{ for all } \alpha \in \Phi^+ ,$$

we have $e_\alpha v_k \in \mathcal{I}_\omega$ for all $\alpha \in \Phi^+$. Now any vector in D_ω which is annihilated by K_+ is a multiple of $\mathbf{1} + \mathcal{I}_\omega$; thus $k = i$ and $a \neq 0$. We may assume that $a = 1$ (multiply u_0 by a^{-1}).

Consider the linear mapping

$$\varphi : T^{(\nu)} \otimes D_\omega \to \mathcal{E}(G)/\mathcal{E}(G)\,\mathcal{I}_\omega$$

induced by the multiplication map $T^{(\nu)} \otimes \mathcal{E}(K) \to \mathcal{E}(G)$. This mapping is a K-module homomcrphism; in fact,

$$\begin{aligned}\varphi\,(k(t \otimes (v + \mathcal{I}_\omega))) &= \varphi\,([k\,,t] \otimes (v + \mathcal{I}_\omega) + t \otimes (kv + \mathcal{I}_\omega))\\ &= [k\,,t]v + \mathcal{E}(G)\,\mathcal{I}_\omega + tkv + \mathcal{E}(G)\,\mathcal{I}_\omega = ktv + \mathcal{E}(G)\,\mathcal{I}_\omega\\ &= k\,\varphi\,(t \otimes (v + \mathcal{I}_\omega))\,,\end{aligned}$$

for all $k \in K$, $t \in T^{(\nu)}$ and $v \in \mathcal{E}(K)$. Let $u \in \mathcal{E}(G)$,

$$u = t_i + \sum_{j, \lambda_j > \lambda_i} t_j v_j\,.$$

Then $\varphi(u_0)) = u + \mathcal{E}(G)\,\mathcal{I}_\omega$ and therefore $\mathcal{I}_{\omega+\lambda_i} u \subset \mathcal{E}(G)\,\mathcal{I}_\omega$. In other words (Lemma 4.2), $u \in A_{\omega+\lambda_i, \omega}$.

We denote by P the projection $P : \mathcal{E}(G) \to \mathcal{E}(G)$ such that $\operatorname{Ker} P = \mathcal{E}(G)\mathcal{I}_\alpha + U_1 \mathcal{E}(K_-)$ and $P(\mathcal{E}(G)) = U_1$ where U_1 consists of the elements

$$b \cdot \mathbf{1} + \sum a_{i_1 \ldots i_k} t_{i_1} \ldots t_{i_k}$$

where b, $a_{i_1 \ldots i_k} \in \mathbf{C}$ and $i_1 \leqq \ldots \leqq i_k$.

Lemma 4.4. *Let* $u_1, u_2 \in A_{\beta,\alpha}$ *such that* $P(u_1) = P(u_2)$. *Then*

$$u_1 - u_2 \in \mathcal{E}(G)\,\mathcal{I}_\alpha\,.$$

Proof. We shall again use the fact that any vector in $\mathcal{E}(K)/\mathcal{I}_\alpha$ which is annihilated by K_+ is a multiple of $\mathbf{1} + \mathcal{I}_\alpha$. First we write

$$u_1 - u_2 = w + t_{j_1} \ldots t_{j_k} v + \sum_{\{m, i_\nu\}} t_{i_1} \ldots t_{i_m} v_{i_1 \ldots i_m}$$

where each term is a sum of ordered monomials and $w \in \mathcal{E}(G)\,\mathcal{I}_\omega$, v and $v_{i_1 \ldots i_m} \in \mathcal{E}(K_-)$ and

$$\lambda_{i_1} + \ldots + \lambda_{i_m} \geqq \lambda_{j_1} + \ldots + \lambda_{j_k}\,.$$

If $m = k$ then $j_\nu \neq i_\nu$ for at least one value of the index ν. From $K_+(u_1 - u_2) \in \mathcal{E}(G)\,\mathcal{I}_\alpha$ it follows that

$$K_+\,v \subset \mathcal{I}_\alpha$$

and thus $v \in \mathcal{I}_\alpha$ ($v \notin a \cdot \mathbf{1} + \mathcal{I}_\alpha$ for any $a \neq 0$ because of $P(u_1 - u_2) = 0$). By induction it follows that the coefficient of any $t_{i_1} \ldots t_{i_m}$ belongs to $\mathcal{I}_\alpha$ and therefore $u_1 - u_2 \in \mathcal{E}(G)\,\mathcal{I}_\alpha$.

It is clear that Lemma 4.4 is valid also if we replace $A_{\beta,\alpha}$ by

$$A_\alpha = \sum_\beta A_{\beta,\alpha}\,.$$

Lemma 4.5. *Let* $\alpha \in A$ *be large. Then any* $u \in A_\alpha$ *such that* $\deg(u) \leqq n_c$ *can be written in the form*

$$u = v + a \cdot \mathbf{1} + \sum_{k,\{i_\nu\}} u_{i_1} u_{i_2} \ldots u_{i_k} \quad (a \in \boldsymbol{C}, \quad v \in \mathcal{E}(G)\,\mathcal{I}_\alpha, \quad u_{i_\nu} \in \mathcal{E}(G); \quad \nu = 1, 2, \ldots, k)$$

where $k \leqq \deg(u)$, $i_1 \leqq i_2 \leqq \ldots \leqq i_k$ *and*

(*) $u_{i_\nu} u_{i_{\nu+1}} \ldots u_{i_k} \in A_{\delta_\nu, \alpha}$; $\delta_\nu = \alpha + \lambda_{i_\nu} + \ldots + \lambda_{i_k}$ $(\nu = 1, 2, \ldots, k)$.

Proof. (1) Let S be the set consisting of finite sequences $\boldsymbol{i} = (i_1, i_2, \ldots, i_k)$ where $k \leqq n_c$ and the integers i_ν satisfy the inequalities

$$0 < i_1 \leqq i_2 \leqq \ldots \leqq i_k \leqq r = \dim T .$$

We denote by $\boldsymbol{e}$ the empty sequence. We define an order in S by putting

$$(i_1, i_2, \ldots, i_k) < (j_1, j_2, \ldots, j_m) \text{ if } \; k < m \text{ or } k = m \text{ and}$$

the first non-zero number in the sequence $i_1 - j_1$, $i_2 - j_2, \ldots$ is positive. In addition, for each $\boldsymbol{i} \in S$ we define

$$t_{\boldsymbol{i}} = t_{i_1} \ldots t_{i_k} \in \mathcal{E}(G)$$

and $t_{\boldsymbol{e}} = \mathbf{1} \in \mathcal{E}(G)$. Let V be the subspace of $\mathcal{E}(G)$ which has the set $\{t_{\boldsymbol{i}} \mid \boldsymbol{i} \in S\}$ as an ordered basis (the order is defined through the ordering of S).

(2) We put $u^e = \mathbf{1} \in A_\alpha$. From the fact that α is large and from Lemma 4.3. it follows that for each $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in S$ there exists

$$u^{\boldsymbol{i}} = u_1^{\boldsymbol{i}} u_2^{\boldsymbol{i}} \ldots u_k^{\boldsymbol{i}} \in A_\alpha \quad (u_\nu^{\boldsymbol{i}} \in \mathcal{E}(G);\ \nu = 1, 2, \ldots, k)$$

where each $u_\nu^{\boldsymbol{i}}$ is of the type described in Lemma 4.3, $P(u_\nu^{\boldsymbol{i}}) = t_{i_\nu}$ and u_{i} satisfies the relations (*). We denote by U the subspace of A_α which has the set

$$\{u^{\boldsymbol{i}} \mid \boldsymbol{i} \in S\}$$

as an ordered basis.

(3) It is clear that the operator P induces a linear mapping from U into V. Furthermore,

$$P(u^{\boldsymbol{i}}) = t_{\boldsymbol{i}} + \text{lower terms}$$

as follows easily from the properties of the $u^{\boldsymbol{i}} : s$ (see Lemma 4.3). Thus the matrix representing P is triangular in the ordered basis described above, the diagonal elements being equal to 1. It follows that the inverse of P exists and therefore for each $u \in A_\alpha$, $\deg(u) \leqq n_c$, there exists $u' \in U$ such that

$$P(u) = P(u') \, .$$

(Note that $P(u) \in V$.) From Lemma 4.4 it follows that there exists $v \in \mathcal{E}(G)\, \mathcal{I}_\alpha$ such that $u = v + u'$.

Lemma 4.6. *Let* $\alpha, \beta \in \Lambda$, *and let* V *be an irreducible G-module such that* $V_\alpha \neq 0$; *then* $V_\beta^+ = A_{\beta,\alpha} V_\alpha^+$.

Proof. It is sufficient to prove the statement for $V = \mathcal{E}(G)/\mathcal{E}(G)\, \mathcal{I}_\alpha$ (compare with the proof of Lemma 4.2). Then

$$\begin{aligned} V_\beta^+ &= \{u + \mathcal{E}(G)\, \mathcal{I}_\alpha \mid \mathcal{I}_\alpha u \subset \mathcal{E}(G)\, \mathcal{I}_\alpha\} \\ &= A_{\beta,\alpha} + \mathcal{E}(G)\, \mathcal{I}_\alpha = A_{\beta,\alpha}\, (\mathbf{1} + \mathcal{E}(G)\, \mathcal{I}_\alpha) \\ &\subset A_{\beta,\alpha}\, V_\alpha^+ \, . \end{aligned}$$

The relation $A_{\beta,\alpha} V_\alpha^+ \subset V_\beta^+$ follows from the definition of $A_{\beta,\alpha}$.

Let C be the centralizer of K in $\mathcal{E}(G)$. If V is any G-module then V_α and V_α^+ are C-modules by restriction of $\mathcal{E}(G)$ to the subalgebra C; in fact V_α is even a $\mathcal{E}(K)C$-module.

Lemma 4.7. *Let* V *be an irreducible G-module,* $V_\alpha \neq 0$. *Then the equivalence class* $[V]$ *of* V *is completely determined by the equivalence class of the C-module* V_α^+. V_α^+ *is an irreducible C-module.*

Proof. This is an easy consequence of Theorem 5.5, [6]. (Note that the action of $\mathcal{E}(K)C$ on V_α is completely determined by the action of C on V_α^+.)

Let G'_α be the set of all equivalence classes $[V]$ of irreducible G-modules V such that $V_\alpha \neq 0$ and $V_\alpha = 0$ for each $\beta < \alpha$. We call V_α the minimal component of V. Now an irreducible G-module V is K-finite if and only if $V_\alpha \neq 0$ for some weight $\alpha \in \Lambda$, [6, proposition 4.2]. It follows that V is K-finite if and only if V has a minimal component. Thus the set G' of all equivalence classes of irreducible K-finite G-modules is equal to

$$\bigcup_{\alpha \in \Lambda} G'_\alpha \, .$$

Of course $G'_\alpha \cap G'_\beta = \phi$ when $\alpha \neq \beta$.

Let $M_\alpha = \sum_{\beta < \alpha} A_{\beta,\alpha}$. If $[V] \in G'_\alpha$ then V_α^+ is in a natural way a $C/C \cap \mathcal{E}(G) M_\alpha$ -module. We denote by C'_α the set of all equivalence classes of irreducible $C/C \cap \mathcal{E}(G) M_\alpha$ -modules.

Theorem 4.8. *The mapping* $V \to V_\alpha^+$ *induces a bijection between* G'_α *and* C'_α.

Proof. If $[V], [W] \in G'_\alpha$ then it is clear that V_α^+ and W_α^+ are equivalent as C-modules if and only if they are equivalent as $C/C \cap \mathcal{E}(G) M_\alpha$-modules. The injectivity of the mapping follows now from Lemma 4.7.

Let next $[W] \in C'_\alpha$. We have to show that there exists $[V] \in G'_\alpha$ such that $V_\alpha^+ \simeq W$ as $C/C \cap \mathcal{E}(G)\, M_\alpha$ -modules. First we extend W to a C-module. Let x be a non-zero element of W, and let $\mathcal{L}$ be the annihilator of x in C so that $W = C/\mathcal{L}$. We define a left ideal of $\mathcal{E}(G)$ by

$$\mathcal{N} = \{u \in \mathcal{E}(G) | \mathcal{E}(G)u \cap C \subset \mathcal{L}\} .$$

Consider the G-module $V = \mathcal{E}(G)/\mathcal{N}$. First we show that V is irreducible i.e. the left ideal $\mathcal{N}$ is maximal. Let $\mathcal{M} \subset \mathcal{E}(G)$ be a left ideal such that $\mathbf{1} \notin \mathcal{M}$ and $\mathcal{N} \subset \mathcal{M}$. Then

$$\mathcal{L} = C \cap \mathcal{N} \subset C \cap \mathcal{E}(G)\mathcal{M} .$$

Because of the irreducibility of W, $\mathcal{L}$ is a maximal left ideal in C. Now $\mathbf{1} \notin C \cap \mathcal{E}(G)\mathcal{M}$ and therefore $\mathcal{L} = C \cap \mathcal{E}(G)\mathcal{M}$. From the definition of $\mathcal{N}$ it follows that $\mathcal{M} \subset \mathcal{N}$; thus $\mathcal{M} = \mathcal{N}$ and $\mathcal{N}$ is maximal.

Since $\mathcal{I}_\alpha \subset \mathcal{N}$ the vector $\mathbf{1} + \mathcal{N} \in V$ is annihilated by $\mathcal{I}_\alpha$, and therefore $\mathbf{1} + \mathcal{N} \in V_\alpha^+$. From Lemma 4.7 we conclude that V_α^+ consists of vectors $c + \mathcal{N}$, $c \in C$. From $C \cap \mathcal{N} = \mathcal{L}$ it then follows that the mapping

$$\varphi : V_\alpha^+ \to C/\mathcal{L} , \quad \varphi(c + \mathcal{N}) = c + \mathcal{L}$$

is a C-linear isomorphism. Thus $V_\alpha^+ \cong W$ as C-modules. Next we observe that $C \cap \mathcal{E}(G)\mathcal{M}_\alpha \subset \mathcal{L}$, so $\mathcal{M}_\alpha \subset \mathcal{N}$ and therefore $V_\beta^+ = A_{\beta,\alpha} \; V_\alpha^+ = 0$ for $\beta < \alpha$. It follows that $[V] \in G'_\alpha$.

By Lemma 3.5, rank K = rank G if and only if $T_0 = 0$.

Theorem 4.9. *Let* rank G = rank K. *Then for any large weight* $\alpha \in \Lambda$ *the set* G'_α *contains exactly one element* $[V]$ *and dim* $V_\alpha^+ = 1$.

Proof. Let c be one of the generators $c_1, \ldots, c_\varrho$ of C (see Definition 4.1). Then $c \in A_\alpha$, deg $(c) \leqq n_c$ and $[H, c] = 0$. Then c can be written in the form described in Lemma 4.5. Since $[H, c] = 0$, $\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_k} = 0$ for each of the products $u_{i_1}u_{i_2} \ldots u_{i_k}$. Now $\lambda_{i_1} \geqq \lambda_{i_2} \geqq \ldots \geqq \lambda_{i_k}$ and $\lambda_{i_\nu} \neq 0$ $(\nu = 1, 2, \ldots, k)$ $(T_0 = 0)$; thus $\lambda_{i_k} < 0$ and $u_{i_1}u_{i_2} \ldots u_{i_k} \in \mathcal{M}_\alpha$. It follows that the generators c belong to the subalgebra $\boldsymbol{C} \cdot \mathbf{1} + C \cap \mathcal{E}(G)\mathcal{M}_\alpha$ of C; hence this is true for all $c \in C$.

We conclude that the algebra $C/C \cap \mathcal{E}(G)\mathcal{M}_\alpha$ is isomorphic (when α is large) to the algebra $\boldsymbol{C}$ of complex numbers and therefore there exists exactly one equivalence class of irreducible (non-zero) $C/C \cap \mathcal{E}(G)\mathcal{M}_\alpha$-modules and the dimension of such a module is equal to one. Theorem 4.8 completes the job.

Remark 4.10. The results of this section can be easily extended to the case in which K is a reductive subalgebra of G.

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