ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

595

EXTREMAL PROBLEMS FOR FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

BY

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HELSINKI 1975
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doi:10.5186/aasfm.1975.595

ISSN 0066-1953
ISBN 951-41-0211-8

Communicated 9 April 1974

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HELSINKI 1975

1. Introduction

For $k \geq 2$ let V_k denote the class of locally univalent functions

$$f(z) = z + a_2 z^2 + \dots$$

that map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $k\pi$ (see [9] for the definition and basic properties of the class V_k).

It was shown by Paatero [9] that for $2 \leq k \leq 4$ V_k consists only of univalent functions, whereas, for each $k > 4$ V_k contains non-univalent functions. Reade [12] posed the problem of determining the radius of univalence, u_k, of the class V_k $(k > 4)$. In § 2 we give a solution to this problem. We show that $u_k = \tan(\pi/k)$. The proof of this fact consists of a remarkably simple application of the theory of linearly invariant families of functions as developed in [11].

In § 3 we recall the variational formula for V_k $(2 \leq k \leq 4)$ developed by Schiffer and Tammi [14] and apply it in § 4 in the solution of a general extremal problem for V_k. Specifically, we prove that if ζ is fixed, $|\zeta| < 1$, and $F(u, v)$ is a function analytic in a neighborhood of $\bigcup_{f \in V_k} (f(\zeta), \zeta)$ then

$$\max_{f \in V_k} \operatorname{Re} F(f(\zeta), \zeta) \quad (2 \leq k \leq 4) \tag{1.1}$$

is attained only for a function of the form

$$f(z) = \frac{2}{k}(x - y)^{-1}\left[\left(\frac{1 + xz}{1 + yz}\right)^{k/2} - 1\right] \quad (|x| = 1 = |y|, x \neq y). \tag{1.2}$$

Each function of the form (1.2) belongs to V_k and maps $|z| < 1$ onto a domain that is the complement of an infinite wedge with opening $\left(\frac{4-k}{2}\right)\pi$. In order to show that functions of the form (1.2) give the solution to (1.1), the Schiffer-Tammi variational formula alone is not sufficient. Indeed we employ this variational technique in order to show that the extremal domain for the problem (1.1) is a polygonal domain. However, using just this variational method there seems to be no way to give a bound on the number of sides in the extremal polygon. The final form of the result is achieved through an application of the Julia variational formula [6] and a method first introduced by Biernacki [3] and used subsequently in [1], [2] and [8]. As an application of our general result we show in § 5 that if $f \in V_k$, $2 \leq k \leq 4$, and $|z| < 1$, then

$$\left| \arg \frac{f(z)}{z} \right| < \left(\frac{k-1}{2} \right) \pi$$

and

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \left(\frac{k-1}{2} \right) \pi .$$

Both of these inequalities are best possible.

2. The radius of univalence

The radius of univalence, u_k, of V_k is by definition the radius of the largest circle with center at the origin in which every f in V_k is univalent. Our method for determining u_k is based on results due to Pommerenke for linearly invariant families of functions [11]. We recall that a family $\mathcal{F}$ of functions analytic in $|z| < 1$ is linearly invariant if for each $f \in \mathcal{F}$ and each bilinear map ϕ of $|z| < 1$ onto $|z| < 1$, the function

$$f_\phi(z) = \frac{f \circ \phi(z) - f \circ \phi(0)}{f'[\phi(0)]\phi'(0)} \in \mathcal{F} .$$

First we note that in [13] Robertson showed that for each $k \geq 2$, V_k is a linearly invariant family. Secondly, Pommerenke [11] showed that for any compact linearly invariant family $\mathcal{F}$, the radius of univalence $r_1 = r_1(\mathcal{F})$ of $\mathcal{F}$ and

$$r_0 = r_0(\mathcal{F}) = \inf_{f \in \mathcal{F}} \sup_{r > 0} \{r : f(z) \neq 0 \text{ for } 0 < |z| < r\}$$

are related by

$$r_1 = r_0/(1 + \sqrt{1 - r_0^2}) . \tag{2.1}$$

Hence in order to determine u_k it suffices to find the value of r_0 for the family V_k. This is the content of the following theorem.

Theorem 2.1. *Let $f \in V_k$ with $k \geq 4$. Then $f(z) \neq 0$ for*

$$0 < |z| < \sin (2\pi/k) .$$

The function

$$G_k(z\,; z_0) = \left[F_k\left(\frac{z + z_0}{1 + \bar{z}_0 z}\right) - F_k(z_0) \right] : F_k'(z_0)(1 - |z_0|^2) \tag{2.2}$$

where $z_0 = i \tan \dfrac{\pi}{k}$ *and* $F_k(z) = \dfrac{1}{k}\left[\left(\dfrac{1+z}{1-z}\right)^{k/2} - 1\right]$ *shows that this result is sharp.*

Proof. Let $r_0 = r_0(V_k)$. If $g \in V_k$ is a function with $g(r_0) = 0$ then by an argument given in the proof of [11, Satz 2.3] it follows that $\arg g'(r_0) = \pm 2\pi$, where the branch of the argument is determined by the condition $\arg g'(0) = 1$.

It was shown in [4] that if $f \in V_k$ then

$$f'(z) = \left(\frac{s_1(z)}{z}\right)^{k/4+1/2} \Big/ \left(\frac{s_2(z)}{z}\right)^{k/4-1/2} \tag{2.3}$$

where $s_i(z) = z + \ldots$ is a univalent starlike map of $|z| < 1$. For starlike maps $s(z)$, Strohhäcker [16] has shown that $\left|\arg \dfrac{s(z)}{z}\right| < 2 \operatorname{arc\,sin} |z|$. Using this estimate in (2.3) we see that if $f \in V_k$, $|\arg f'(z)| \leq k \operatorname{arc\,sin} |z|$ and equality holds when $z = r > 0$ and $f = F_k$. Since $g'(r_0) = \pm 2\pi$ we conclude that $2\pi \leq k \operatorname{arc\,sin} r_0$ or $\sin \dfrac{2\pi}{k} \leq r_0$. On the other hand we consider the value of the function (2.2) at $-i \sin \dfrac{2\pi}{k}$. Since

$$\frac{-i \sin \dfrac{2\pi}{k} + i \tan \dfrac{\pi}{k}}{1 - \sin \dfrac{2\pi}{k} \tan \dfrac{\pi}{k}} = -i \tan \frac{\pi}{k}$$

and

$$\frac{1 - i \tan \dfrac{\pi}{k}}{1 + i \tan \dfrac{\pi}{k}} = e^{-2i\pi/k}, \qquad \frac{1 + i \tan \dfrac{\pi}{k}}{1 - i \tan \dfrac{\pi}{k}} = e^{2i\pi/k}$$

it follows that $G_k\left(-i \sin \dfrac{2\pi}{k}; z_0\right) = 0$. Thus $r_0 \leq \sin\left(\dfrac{2\pi}{k}\right)$ and the proof is complete.

In view of (2.1), Theorem 2.1 is equivalent to

Theorem 2.2. *The radius of univalence of* V_k *is* $\tan(\pi/k)$.

3. Variational formulae

In [14] Schiffer and Tammi employed the method of interior variations to obtain the following variational formula for the class V_k, $2 \leq k \leq 4$.

$$
\begin{aligned}
f^*(z) = f(z) + a\varrho^2 f(z) &\left[\frac{1}{z_0^2 f'(z_0)} + \frac{d}{dz_0}\left(\frac{1}{z_0 f'(z_0)}\right)\right] \\
&+ a\varrho^2 \left[\frac{d}{dz_0}\left(\int_0^z \frac{f'(t)dt}{(t - z_0)f'(z_0)}\right) + \frac{zf'(z)}{z_0 f'(z_0)(z - z_0)}\right] \\
+ \bar{a}\varrho^2 &\left[\frac{d}{d\bar{z}_0}\left(\int_0^z \frac{tf'(t)dt}{(1 - \bar{z}_0 t)\overline{f'(z_0)}}\right) - \frac{z^2 f'(z)}{\bar{z}_0 \overline{f'(z_0)}(1 - \bar{z}_0 z)}\right] + o(\varrho^2)
\end{aligned} \tag{3.1}
$$

where, as usual, the estimate for $o(\varrho^2)$ is uniform on compact subsets of $|z| < 1$, a is a free parameter and z_0 is an arbitrary but fixed point of $|z| < 1$.

We will also require the Julia variational formula [6] in the form

$$
\begin{aligned}
f^*(z) = f(z) + &\frac{\varepsilon z f'(z)}{2\pi i} \\
&\cdot \int_\Gamma \frac{\xi + z}{\xi - z} \operatorname{Im}\left\{- \phi(w) \frac{|dw|}{dw}\right\} \frac{n(w)}{\xi^2 f'^2(\xi)}\, dw + o(\varepsilon)\,.
\end{aligned} \tag{3.2}
$$

Here $f(z) = z + a_2 z^2 + \ldots$ denotes a conformal map of $|z| < 1$ onto a domain with boundary $\Gamma : w = f(\xi)$ $(|\xi| = 1)$, $\phi(w)$ is continuous and piecewise differentiable on Γ, $n(w)$ is the unit exterior normal to Γ at w and $f^*(z) = a_1^* z + a_2^* z + \ldots$ is a conformal map of $|z| < 1$ onto the domain D^* with boundary

$$\Gamma^* : w^* = w + \varepsilon \phi(w) \quad (\varepsilon > 0)\,.$$

The expression $\operatorname{Im}\left\{- \phi(w) \dfrac{|dw|}{dw}\right\}$ represents the component of the boundary shift $w \to w + \varepsilon\phi(w)$ in the direction of $n(w)$. This form of the Julia variational formula follows, using a standard argument, from the version of the Hadamard variational formula for the Green's function of D derived in [17]. Our use of (3.2) is restricted to domains D for which the general version of Hadamard's formula proved in [17] is applicable.

The way in which we employ (3.2) is based on a method originated by Biernacki [3] and subsequently refined in [1], [2] and [8]. We outline the method as it will be used in the present context. Let ζ, $|\zeta| < 1$, be fixed. We want to characterize the extremal function for the problem (1.1). Let $f \in V_k$ and let f^* be defined by (3.2). Expanding $F(f^*(\zeta), \zeta)$ in powers of ε we obtain

$$
\begin{aligned}
&F(f^*(\zeta), \zeta) = F(f(\zeta), \zeta) + \\
&\varepsilon \zeta f'(\zeta) \frac{F_1(f(\zeta), \zeta)}{2\pi i} \int_\Gamma \frac{z + \zeta}{z - \zeta} \operatorname{Im}\left\{- \phi(w) \frac{|dw|}{dw}\right\} \frac{n(w)}{z f'^2(z)}\, d(w) + o(\varepsilon)
\end{aligned} \tag{3.3}
$$

where $F_1(u, v) = \partial F(u, v)/\partial u$.

Suppose the boundary of $f(|z| < 1)$ contains three disjoint analytic arcs. These arcs correspond to three arcs l_1, l_2 and l_3 on $|z| = 1$. As z varies on $|z| = 1$, $\zeta f'(\zeta)F_1(f(\zeta), \zeta)\dfrac{z+\zeta}{z-\zeta}$ describes a circle. Consequently, of the three arcs l_1, l_2 and l_3 there must be two, which we denote by γ_1 and γ_2, that satisfy

$$\text{(3.4)} \qquad \max_{z \in \gamma_1} \operatorname{Re}\left\{\zeta f'(\zeta)F_1(f(\zeta), \zeta)\frac{z+\zeta}{z-\zeta}\right\} < \min_{\zeta \in \gamma_2}\left\{\zeta f'(\zeta)F_1(f(\zeta), \zeta)\frac{z+\zeta}{z-\zeta}\right\}.$$

We choose $\phi(w)$ to satisfy

$$\text{(3.5)} \qquad \operatorname{Im}\left\{-\phi(w)\frac{|dw|}{dw}\right\} = \begin{cases} <0 & w \in f(\gamma_1) \\ >0 & w \in f(\gamma_2) \\ 0 & \text{elsewhere on } \gamma \end{cases}$$

and

$$\text{(3.6)} \qquad \int_{\Gamma} \operatorname{Im}\left\{-\phi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{z^2 f'^2(z)}\,dw = 0\,.$$

If we apply (3.2) to f with this choice of $\phi(w)$, then by (3.6), $f^{*\prime}(0) = 1$ and by (3.4), (3.5) and (3.6)

$$\operatorname{Re}\left\{\zeta f'(\zeta)\frac{F_1(f(\zeta), \zeta)}{2\pi i}\int_{\Gamma}\frac{z+\zeta}{z-\zeta}\operatorname{Im}\left\{-\phi(w)\frac{|dw|}{dw}\right\}\frac{n(w)}{zf'^2(z)}\,dw\right\} > 0\,.$$

It then follows from (3.3) that

$$\operatorname{Re} F(f^*(\zeta), \zeta) > \operatorname{Re} F(f(\zeta), \zeta)$$

and so if $f^* \in V_k$, f is not an extremal function for the problem $\max_{f \in V_k} \operatorname{Re} F(f(\zeta), \zeta)$. We summarize the content of these remarks in the following lemma.

Lemma 3.1. *Let* ζ, $|\zeta| < 1$, *be fixed and let* $F(u, v)$ *be analytic on* $\bigcup_{f \in V_k}(f(\zeta, \zeta))$. *Let* $f \in V_k$ *and map* $|z| < 1$ *onto a domain whose boundary contains three disjoint analytic arcs. Then*

a) *there exists two arcs* γ_1 *and* γ_2 *on* $|z| = 1$ *that satisfy* (3.4). *Further if* $\phi(w)$ *is chosen to satisfy* (3.5) *and* (3.6) *and if* f^* *defined by* (3.2) *belongs to* V_k, *then*

b) f *is not an extremal function for* $\max_{f \in V_k} \operatorname{Re} F(f(\zeta), \zeta)$.

4. A general extremal problem

We will use the variational formulae of the preceding section in order to solve the following general extremal problem.

Theorem 4.1. *Let* ζ, $|\zeta| < 1$, *be fixed and let* $F(u, v)$ *be analytic in a neighborhood of* $\bigcup_{f \in V_k} (f(\zeta), \zeta)$ *where* $2 \leq k \leq 4$. *Then*

$$\max_{f \in V_k} \operatorname{Re} F(f(\zeta), \zeta) \tag{4.1}$$

is attained only for a function of the form

$$F(z) = \frac{2}{k} (x - y)^{-1} \left[\left(\frac{1 + xz}{1 + yz} \right)^{k/2} - 1 \right] \qquad (x \neq y) \tag{4.2}$$

with $|x| = 1 = |y|$.

As noted in the introduction, functions of the form (4.2) map $|z| < 1$ onto the complement of a wedge with angular opening $((4 - k)/2)\pi$. The proof of Theorem 4.1 will require the following lemma.

Lemma 4.2. *If* $f \in V_k$ *is an extremal function for* (4.1) *then*

$$\operatorname{Im} F_1(f(\zeta), \zeta)[f(\zeta) - \zeta f'(\zeta)] = 0 .$$

Proof. Let ε be real with $|\varepsilon|$ small. The function

$$f_\varepsilon(z) = e^{i\varepsilon} f(e^{-i\varepsilon} z) = f(z) + i\varepsilon[f(z) - zf'(z)] + o(\varepsilon)$$

belongs to V_k. Expanding $F(f_\varepsilon(\zeta), \zeta)$ in powers of ε and using the fact that f is an extremal function we obtain

$$\begin{aligned} \operatorname{Re} F(f_\varepsilon(\zeta), \zeta) &= \operatorname{Re} \{F(f(\zeta), \zeta) + i\varepsilon[f(\zeta) - \zeta f'(\zeta)] F_1(f(\zeta), \zeta)\} + o(\varepsilon) \\ &\leq \operatorname{Re} F(f(\zeta), \zeta) . \end{aligned}$$

Since ε can take either positive or negative values we conclude that

$$0 = \operatorname{Re} \{iF_1(f(\zeta), \zeta)(f(\zeta) - \zeta f'(\zeta))\} = - \operatorname{Im} F_1(f(\zeta), \zeta)(f(\zeta) - \zeta f'(\zeta)) .$$

Proof of Theorem. We assume initially that $2 < k < 4$. Let $f \in V_k$ be an extremal function for (4.1). We first apply the Schiffer-Tammi variation to f and show that f maps $|z| < 1$ onto a polygonal domain. To simplify our notation we set

$$H(z_0, z) = \frac{d}{dz_0} \left(\int_0^z \frac{f'(t)}{(t - z_0) f'(z_0)} \, dt \right)$$

and

$$J(z_0, z) = \frac{d}{d\bar{z}_0} \left(\int_0^z \frac{t f'(t)}{(1 - \bar{z}_0 t) \overline{f'(z_0)}} \, dt \right)$$

in (3.1). If we expand $F(f^*(\zeta), \zeta)$ in powers of ϱ^2 using (3.1), we obtain

$$
\begin{aligned}
F(f^*(\zeta), \zeta) &= F(f(\zeta), \zeta) \\
&\quad + a\varrho^2 F_1(f(\zeta), \zeta)\left\{f(\zeta)\left[\frac{1}{z_0^2 f'(z_0)} + \frac{d}{dz_0}\left(\frac{1}{z_0 f'(z_0)}\right)\right]\right. \\
&\quad \left. + H(z_0, \zeta) + \frac{\zeta f'(\zeta)}{z_0 f'(z_0)(\zeta - z_0)}\right\} + \bar{a}\varrho^2 \overline{F_1(f(\zeta), \zeta)} \\
&\quad \cdot \left\{\overline{J(z_0, \zeta)} - \frac{\bar{\zeta}^2 \overline{f'(\zeta)}}{\bar{z}_0 \overline{f'(z_0)}(1 - \bar{z}_0 \bar{\zeta})}\right\} + o(\varrho^2) .
\end{aligned} \tag{4.3}
$$

Since f is an extremal function for (4.1),

$$\operatorname{Re} F(f^*(\zeta), \zeta) \leq \operatorname{Re} F(f(\zeta), \zeta)$$

which implies by (4.3) that

$$
\begin{aligned}
0 \leq \operatorname{Re} &\left[a\left\{F_1(f(\zeta), \zeta)\left[f(\zeta)\left[\frac{1}{z_0^2 f'(z_0)} + \frac{d}{dz_0}\left(\frac{1}{z_0 f'(z_0)}\right)\right] + H(z_0, \zeta)\right.\right.\right. \\
&\left.\left.\left. + \frac{\zeta f'(\zeta)}{z_0 f'(z_0)(\zeta - z_0)}\right] + \overline{F_1(f(\zeta), \zeta)}\left[\overline{J(z_0, \zeta)} - \frac{\overline{\zeta^2 f'(\zeta)}}{z_0 f'(z_0)(1 - z_0\bar{\zeta})}\right]\right\}\right] .
\end{aligned}
$$

Using the fact that a is arbitrary we have

$$
\begin{aligned}
&F_1(f(\zeta), \zeta)\left[f(\zeta)\left(\frac{1}{z_0^2 f'(z_0)} + \frac{d}{dz_0}\left(\frac{1}{z_0 f'(z_0)}\right)\right) + H(z_0, \zeta)\right. \\
&\left. + \frac{\zeta f'(\zeta)}{z_0 f'(z_0)(\zeta - z_0)}\right] + \overline{F_1(f(\zeta), \zeta)}\left[\overline{J(z_0, \zeta)} - \frac{\overline{\zeta^2 f'(\zeta)}}{z_0 f'(z_0)(1 - z_0\bar{\zeta})}\right] = 0 .
\end{aligned} \tag{4.4}
$$

A lengthy but straightforward calculation shows that (4.4) is equivalent to (with $F_1 = F_1(f(\zeta), \zeta)$)

$$
\begin{aligned}
&\left[F_1 \int_0^{\zeta} \frac{t}{t - z_0} f'(t)dt + \bar{F}_1 \overline{\int_0^{\zeta} \frac{\bar{z}_0 t}{1 - \bar{z}_0 t} f'(t)dt}\right]\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \\
&= F_1\left(f(\zeta) + \int_0^{\zeta} \frac{z_0 t}{(t - z_0)^2} f'(t)dt + \frac{\zeta z_0}{\zeta - z_0} f'(\zeta)\right) \\
&\quad + \bar{F}_1\left(\frac{-\bar{\zeta}^2 z_0}{1 - z_0\bar{\zeta}} \overline{f'(\zeta)} + \overline{\int_0^{\zeta} \frac{\bar{z}_0 t}{(1 - \bar{z}_0 t)} f'(t)dt}\right).
\end{aligned} \tag{4.5}
$$

Since z_0 denotes an arbitrary point of $|z| < 1$, we replace z_0 by z in (4.5) and rewrite (4.5) using the notation

$$Q(z, \zeta)\left(1 + \frac{zf''(z)}{f'(z)}\right) = R(z, \zeta) . \tag{4.6}$$

It is clear from their definitions that $Q(z, \zeta)$ and $R(z, \zeta)$ are analytic on $|z| = 1$ and are both not identically zero unless $F_1 = 0$. However by a result in [7], $F_1 = 0$ is possible only if F is identically constant, a case which is of no interest. Hence we see from the differential equation (4.6) that f has an analytic extension to $|z| = 1$ with the finite number of zeros of $Q(z, \zeta)$ on $|z| = 1$ deleted. Moreover, from (4.5) we see that on $|z| = 1$

$$Q(z, \zeta) = F_1 \int_0^\zeta \frac{t}{t-z} f'(t)dt - \bar{F}_1 \overline{\int_0^\zeta \frac{t}{t-z} f'(t)dt}$$

is pure imaginary and, using Lemma 3.1,

$$\operatorname{Im} R(z, \zeta) = \operatorname{Im} \left\{ F_1 \frac{\zeta^2}{\zeta - z_0} f'(\zeta) + \bar{F}_1 \frac{\bar{\zeta}^2}{\bar{\zeta} - \bar{z}_0} \overline{f'(\zeta)} \right.$$

$$\left. + F_1 \int_0^\zeta \frac{z_0 t}{(t-z_0)^2} f'(t)dt + \bar{F}_1 \overline{\int_0^\zeta \frac{z_0 t}{(t-z_0)^2} f'(t)dt} \right\} = 0 .$$

Thus on $|z| = 1$, $\operatorname{Re}\{1 + zf''(z)/f'(z)\} = 0$. Since $\operatorname{Re}\{1 + zf''(z)/f'(z)\}$ denotes the rate of turn of the tangent vector of $w = f(e^{i\phi})$ $(z = e^{i\phi})$ at $e^{i\phi}$, it follows that f maps $|z| < 1$ onto a polygonal domain. The vertices of the boundary polygon are the zeros on $|z| = 1$ of $Q(z, \zeta)$.

In [5] and [10] where extremal problems for V_k were considered using variational methods, it was possible to easily give an upper bound on the number of zeros of the functions corresponding to $Q(z, \zeta)$ and hence the number of sides in the extremal polygon. Because of the rather more complicated nature of the present $Q(z, \zeta)$ this does not seem to be possible and so we resort to use of Lemma 3.1 to obtain such a bound.

Our plan of attack is to first show that the extremal polygon can have only one finite vertex and then to show that the (exterior) opening at this vertex is $((4-k)/2)\,\pi$.

Suppose that there are at least 2 finite vertices P and Q formed by sides s_1 and s_2 and t_1 and t_2, respectively, of the extremal polygon and that on the positively oriented boundary s_1 precedes s_2 and t_1 precedes t_2. At first we suppose that these four sides are distinct. These sides correspond to arcs ξ_1 and ξ_2 and τ_1 and τ_2 respectively of $|z| = 1$ with ξ_1 and ξ_2 adjacent and τ_1 and τ_2 adjacent. The image of these arcs under

$$z \to \zeta f'(\zeta) F_1(f(\zeta), \zeta) \frac{z+\zeta}{z-\zeta}$$

are again arcs of a circle. An easy argument shows either the pair ξ_1, τ_1

or the pair ξ_2, τ_2 satisfy (3.4). Without loss of generality we assume (3.4) holds with $\gamma_1 = \xi_2$ and $\gamma_2 = \tau_2$.

With (3.5) in mind we perform a variation on D that has the effect of translating s_2 parallel to itself in the direction of the inner normal along s_2 and translating t_2 parallel to itself in direction of the outer normal along t_2. In order to describe more precisely the variation of s_2 we distinguish two cases according as the angular opening α formed by s_1 and s_2, as measured through the exterior of D, is greater than π or less than π.

Case 1. $\alpha < \pi$. Extend s_1 a short distance into D. Let s_1' denote the extended segment and R the endpoint of s_1' in D. Let s_2' be a line segment parallel to s_2 and emanating from R. R is chosen so that s_2' is close to s_2.

If s_2 extends to infinity, s_2' is extended to infinity. Let D' denote the domain whose boundary is the same as the boundary of D except that s_1 is replaced by s_1' and s_2 by s_2'. It is clear that the boundary rotation of D' is not larger than D.

If s_2 is finite and has T as its other endpoint (P is one endpoint of s_2), let s_3 denote the side of ∂D that together with s_2 forms the vertex T. There are two possibilities: either the angle formed by s_2 and s_3 (measured through D) is smaller than π or greater than π. If this angle is smaller than π, we choose R sufficiently close to P that s_2' intersects s_3. The part of s_3 from this point of intersection to the other endpoint (not T) of s_3 we denote by s_3' (see Fig. 1). If D' is the domain whose boundary is the same as the boundary of D except that s_1, s_2 and s_3 are replaced by s_1', s_2' and s_3' respectively, then again it is clear that D' has the same boundary rotation as D. Finally if the angle formed by s_2 and s_3 is greater than π, extend s_3 through T into D until it intersects s_2'.

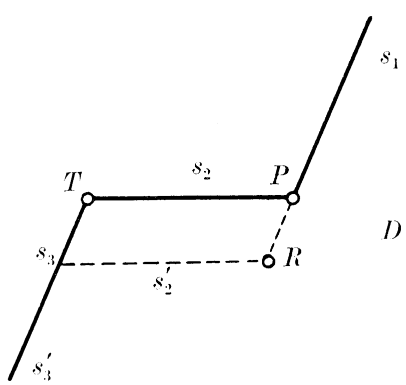

Fig. 1

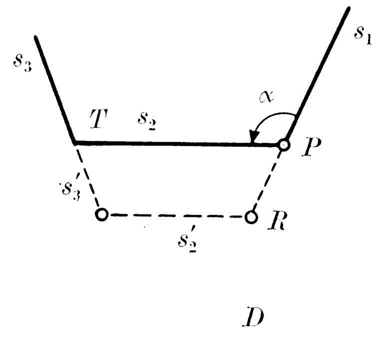

Fig. 2

Denote by s_3' this extended segment (see Fig. 2). Let D' be the domain whose boundary is the same as D except that s_1 , s_2 and s_3 are replaced by s_1' , s_2' and s_3' respectively. Once again it is clear that the boundary rotation of D' is the same as the boundary rotation of D.

Case 2. $\alpha > \pi$. Choose a point T' on s_1 near P and denote by s_1' the part of s_1 from T' to the endpoint $\neq P$ of s_1. Let s_2' emmanate from T' and be parallel to s_2. If s_2 extends to infinity, s_2' is extended to infinity and D' is the domain with the same boundary as D except that s_1 and s_2 are replaced by s_1' and s_2'. The boundary rotation of D' is the same as the boundary rotation of D. If s_2 is finite we introduce s_3' and R just as in Case 1. As before we consider the two possibilities that the angle formed by s_2 and s_3 is smaller or larger than π. Except for the fact that s_2' now emmanates from T', the construction of D' proceeds exactly as before (see Figs. 3 and 4) and D' has the same boundary rotation as D.

We now define a variation of D' by translating t_2 parallel to itself in the direction of the outer normal along t_2. Let β be the angular opening in the exterior of D' formed by t_1 and t_2. Again we have to consider the cases $\beta < \pi$ and $\beta > \pi$. The construction of D'' from D' for these two cases proceeds exactly as the construction of D' from D except that if $\beta < \pi$ we use the construction from the case $\alpha > \pi$ and if $\beta > \pi$ we use the construction from the case $\alpha < \pi$.

D'' is then a domain with boundary rotation not larger than D. Let $\phi(w)$ be a piecewise smooth function such that $w \to w + \phi(w)$ maps ∂D onto $\partial D''$. We may choose D'' and $\phi(w)$ so that (3.5) and (3.6) are satisfied with $s_2 = f(\gamma_1)$ and $t_2 = f(\gamma_2)$. Lemma 3.1 applies and we conclude that D cannot be an extremal polygon for the extremal problem (4.1).

We now must consider the case that s_1 , s_2 , t_1 and t_2 are not distinct. Since we are assuming that ∂D has at least two finite vertices we may assume without loss of generality that $s_2 = t_1$ and that ∂D consists of

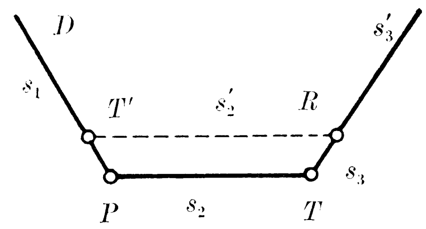

Fig. 3

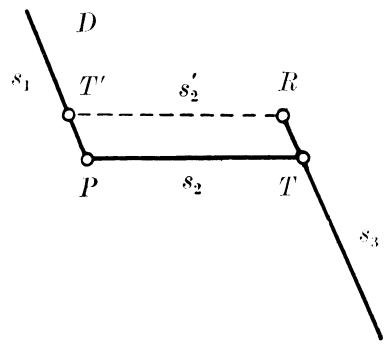

Fig. 4

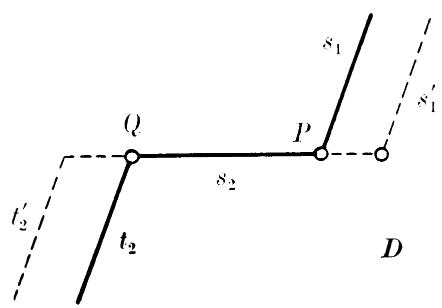

Fig. 5

the sides s_1, s_2 and t_2 and possibly a line disjoint from s_1, s_2 and t_2. If s_1 and t_2 correspond to γ_1 and γ_2 in (3.4) then we may use the type of variation that was used before to construct D'' (see Fig. 5) and apply Lemma 3.1 to conclude that such a domain could not be extremal for (4.1). If s_1 and s_2 (or equivalently s_2 and t_2) correspond to γ_1 and γ_2, we consider the cases $s_1 = f(\gamma_1)$, $s_2 = f(\gamma_2)$ and $s_1 = f(\gamma_2)$, $s_2 = f(\gamma_1)$. First suppose $s_1 = f(\gamma_1)$ and $s_2 = f(\gamma_2)$. Let s_1' be parallel to s_1, close to s_1 and on the side of s_1 determined by the inner normal along s_1 and let s_2' be parallel to s_2, close to s_2 and on the side of s_2 determined by

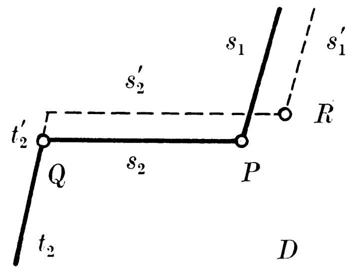

Fig. 6

the outer normal along s_2. s_1' and s_2' intersect at a point R. There are two possibilities: either s_2' intersects t_2 (this happens if the angle formed by s_2 and t_2 measured through D is larger than π) or not. If s_2' intersects t_2, say at the point T, and t_2' denotes the segment of t_2 from T to infinity then D'' is the domain whose boundary is the same as the boundary of D except that s_1 , s_2 and t_2 are replaced by s_1' , s_2' and t_2'. If s_2' does not intersect t_2 then extend t_2 through Q until the extension intersects s_2' (see Fig. 6). D'' is the domain whose boundary is the same as D except that s_1 , s_2 and t_2 are replaced by s_1' , s_2' and t_2'.

It is clear that in all of the preceding cases D'' has the same boundary rotation as D and applying Lemma 3.1 as before we conclude that D is not an extremal domain for (4.1).

If it happens that $s_1 = f(\gamma_2)$ and $s_2 = f(\gamma_1)$ the construction of D'' is similar to the preceding case and will not be repeated.

We have now shown that the boundary of the extremal domain D has at most one finite vertex and hence consists of one of the following:

a) two line segments emanating from a vertex

b) two line segments emanating from a vertex and a line, disjoint from the two segments,

c) two lines (i.e. D is a strip),

or

d) a line (i.e. D is a half plane).

We will rule out the possibilities b), c) and d). First we show that b) is impossible. Suppose that D has boundary consisting of the line segments s_1 and s_2 emanating from a vertex and a line l. These arcs correspond to arcs on $|z| = 1$. By Lemma 3.1 two of these arcs (on $|z| = 1$) will satisfy (3.4) with either $l = f(\gamma_1)$ and $s_i = f(\gamma_2)$, $s_i = f(\gamma_1)$ and $l = f(\gamma_2)$ ($i = 1$ or 2), or $s_i = f(\gamma_1)$ and $s_{i'} = f(\gamma_2)$ ($i , i' = 1 , 2$).

In either of the first two possibilities we construct D'' by translating l parallel to itself either in the direction of the inner normal or outer normal along l and s_i in the direction of the outer normal or inner normal respectively along s_i. In the third possibility we construct s_1' parallel to s_1 on the side of s_1 determined by the inner normal along s_1 and s_2' parallel to s_2 on the side of s_2 determined by the outer normal along s_2. s_1' and s_2' intersect and form a wedge which together with l forms the boundary of a domain D''. D'' has the same boundary rotation as D and we proceed by applying Lemma 3.1 as before. This shows that b) is not possible for an extremal domain.

Now consider case c) or d). Choose three segments s_1, s_2 and s_3 on a line l of the boundary. Without loss of generality we suppose $s_1 = f(\gamma_1)$ and $s_2 = f(\gamma_2)$ in (3.4). Erect a triangle with s_1 as a base that lies, except for the side s_1, in D and erect a triangle with s_2 as a base that lies in the complement of D. Let D'' be the domain whose boundary is the same as D except that s_1 and s_2 are replaced by the other two sides of the respective triangles. The boundary rotation of D is 2π and since $2 < k$ the altitudes of these two triangles may be chosen sufficiently small that D'' will have boundary rotation no larger than $k\pi$. Applying Lemma 3.1 we see that neither c) nor d) is possible for an extremal domain.

Thus we have shown that the boundary of the extremal domain consists of two segments emanating from a vertex. If the angle at this vertex (measured through the complement of D) is less than $((4 - k)/2)\pi$, the boundary rotation of D is less than $k\pi$. We can then apply the variations used in c) and d) along a segment of the boundary to show that such a domain cannot be extremal for (4.1). Thus the extremal domain must be the complement of a wedge with opening $((4 - k)/2)\pi$.

This completes the proof of the theorem when $2 < k < 4$. Let $k = 4$ and let $f \in V_4$ be extremal for (4.1). If $0 < s < 1$ then $f_s(z) = \dfrac{1}{s} f(sz) \in V_{k(s)}$ with $k(s) < 4$. By the theorem there is a function g_s of the form (4.2) (with $k = k(s)$) such that

$$\mathrm{Re}\, F(f_s(\zeta)\,,\,\zeta) \leq \mathrm{Re}\, F(g_s(\zeta)\,,\,\zeta)\,. \tag{4.7}$$

For a suitable sequence $\{s_n\}$ increasing to 1, $\{g_{s_n}\}$ converges and the limit function belongs to V_k. But the limit of a sequence of functions of the form (4.2) is again of this form. If g denotes this limit function, then by (4.7)

$$\mathrm{Re}\, F(f(\zeta)\,,\,\zeta) \leq \mathrm{Re}\, F(g(\zeta)\,,\,\zeta)\,. \tag{4.8}$$

Since f is extremal, equality must hold in (4.8).

Finally g is of the form (4.2) with $k = 4$ for if not $g \in V_k$, $k < 4$, and by the theorem could not be extremal for (4.1) in $V_{k'}$, $k < k' < 4$. The case $k = 2$ is treated in a similar way and so the proof of the theorem is complete.

5. Applications

In this section we prove the following result.

Theorem 5.1. *If* $f \in V_k$ $(2 \leq k \leq 4)$ *then*

$$\left|\arg \frac{f(z)}{z}\right| < \left(\frac{k-1}{2}\right)\pi \tag{5.1}$$

and

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \left(\frac{k-1}{2}\right)\pi\,. \tag{5.2}$$

Both of these inequalities are best possible.

Proof. By definition if a function f belongs to a linearly invariant family of functions and $|z_0| < 1$, then

$$f(z; z_0) = \frac{f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{(1-|z_0|^2)f'(z_0)}$$

also belongs to the family and

$$\frac{f(-z_0; z_0)}{-z_0} = \frac{1}{(1-|z_0|^2)} \cdot \frac{f(z_0)}{z_0 f'(z_0)}\,.$$

Hence any bound in the class for $\left|\arg \dfrac{f(z)}{z}\right|$ is also a bound for $\left|\arg \dfrac{zf'(z)}{f(z)}\right|$ and conversely. It follows therefore that we need only prove (5.1).

If $f \in V_k$, $2 \leq k \leq 4$, then $\dfrac{f(z)}{z} \neq 0$ for $|z| < 1$ and hence $\log \dfrac{f(z)}{z}$ is analytic in $|z| < 1$. Consequently we can apply Theorem 4.1 and conclude that if $|\zeta| < 1$ then

$$\max_{f \in V_k} \operatorname{Re}\left(\pm i \log \frac{f(\zeta)}{\zeta}\right) = \max_{f \in V_k} \left|\arg \frac{f(\zeta)}{\zeta}\right|$$

is attained only by a function of the form (4.2). To complete the proof we must show that for functions of the form (4.2), $\sup\limits_{|z|<1} \left|\arg \dfrac{f(z)}{z}\right| < \left(\dfrac{k-1}{2}\right)\pi$.
Since $\sup\limits_{|z|<1} \left|\arg \dfrac{f(z)}{z}\right|$ is invariant if $f(z)$ is replaced by $e^{-i\theta} f(e^{i\theta} z)$ it suffices to show that

$$\sup_{|z|<1} \left| \arg \frac{2}{k} \frac{1}{(1 - e^{i\phi})z} \left[\left(\frac{1 + e^{i\phi}z}{1 - z} \right)^{k/2} - 1 \right] \right| < \frac{k-1}{2} \pi$$

for each $\phi \in (-\pi, \pi)$.

The function

$$f(z, e^{i\phi}) = \frac{2}{k} \frac{1}{1 - e^{i\phi}} \left[\left(\frac{1 + e^{i\phi}z}{1 - z} \right)^{k/2} - 1 \right]$$

maps $|z| < 1$ onto the complement of a wedge whose vertex is the point $-\dfrac{1}{k} \dfrac{e^{-i\phi/2}}{\cos(\phi/2)}$ and whose sides have the asymptotic directions $\dfrac{k-2}{4}\phi + \dfrac{k\pi}{4}$ and $\dfrac{k-2}{4}\phi - \dfrac{k\pi}{4} + 2\pi$. Thus as $e^{i\theta}$ traverses $|z| = 1$, $\arg f(e^{i\theta}, e^{i\phi})$ varies continuously from $\dfrac{k-2}{4}\phi + \dfrac{k\pi}{4}$ to $\dfrac{k-2}{4}\phi - \dfrac{k\pi}{4} + 2\pi$. Let s_1 denote the »upper» side of the boundary wedge, s_2 the lower side and let γ_1 and γ_2 on $|z| = 1$ correspond to s_1 and s_2 respectively under the mapping f. We will show that

$$\left| \arg \frac{f(e^{i\theta}; e^{i\phi})}{e^{i\theta}} \right| \leq \frac{k-1}{2} \pi \tag{5.3}$$

on γ_1 and on γ_2. Because of the relatively simple nature of $f(z; e^{i\phi})$ it would seem that (5.3) should be easy to verify by direct computation. This does not appear to be the case and we need to invoke the following version of Loewner's lemma contained in a more general theorem of W. Schneider [15].

Theorem 5.2. *Let f and g be conformal mappings of $|z| < 1$ with g subordinate to f. Further suppose that f and g are continuous on $|z| = 1$ except at a finite number of points and locally of bounded variation on the arcs of continuity. If f maps the arc α of $|z| = 1$ continuously onto an arc γ and g maps the arc β of $|z| = 1$ continuously onto γ then the length of α is greater than or equal to the length of β.*

In order to prove (5.3) we first consider $e^{i\theta} \in \gamma_1$. If $\phi = \dfrac{4-k}{k}\pi$ then $s_1 = f(\gamma_1)$ lies on the ray from the origin through $e^{i(\pi - \phi/2)}$ and if $\dfrac{4-k}{k}\pi < \phi < \pi$, $\arg f(e^{i\theta}, e^{i\phi})$ is decreasing on γ_1. Hence for this range of ϕ

$$A(\theta, \phi) = \arg \frac{f(e^{i\theta}; e^{i\phi})}{e^{i\theta}}$$

is decreasing on γ_1 so that

$$(5.4) \qquad A(\theta\,,\phi) \leq \lim_{\theta \to 0^+} A(\theta\,,\phi) = \frac{k-2}{4}\,\phi + \frac{k\pi}{4} \qquad \left(\frac{4-k}{k}\,\pi \leq \phi < \pi\right).$$

Now consider the range $-\pi < \phi < \dfrac{4-k}{k}$. It is here that we need to invoke Loewner's lemma.

Extend s_1 to a line l_1 and let D_1 be the half-plane determined by l_1 that contains the origin. $h(z) = ae^{i\lambda}\,\dfrac{z}{1-z}$ with $\lambda = \dfrac{k-2}{4}\,\phi + \dfrac{k\pi}{4} - \dfrac{\pi}{2}$ and $a > 0$ maps $|z| < 1$ onto D_1 and is subordinate to $f(e^{i\theta}\,;\,e^{i\phi})$. Let $w_0 \in s_1$ with $f(e^{i\theta_0}\,;\,e^{i\phi}) = w_0 = g(e^{i\theta_1})$. By the version of Loewner's lemma cited above, $\theta_1 < \theta_0$. Hence

$$A(e^{i\theta_0}\,,\,e^{i\phi}) = \arg \frac{f(e^{i\theta_0}\,;\,e^{i\phi})}{e^{i\theta_0}} < \arg \frac{h(e^{i\theta_1})}{e^{i\theta_1}}$$

$$= \arg e^{i\gamma}\,\frac{1}{1-e^{i\theta_1}} < \frac{k-2}{4}\,\phi + \frac{k\pi}{4}\,.$$

This inequality together with (5.4) shows that $A(e^{i\theta}\,;\,e^{i\phi}) < \dfrac{k-2}{4} + \dfrac{k\pi}{4}$ for $e^{i\theta} \in \gamma_1$ and $-\pi < \phi < \pi$. To show that $A(e^{i\theta}\,;\,e^{i\phi}) < \dfrac{k-2}{4}\,\phi + \dfrac{k\pi}{4}$ for $e^{i\theta} \in \gamma_2$ we argue in a similar way except that we extend the side s_2 to infinity and invoke Loewner's lemma as before. The proof that $A(e^{i\theta}\,;\,e^{i\phi}) > -\left(\dfrac{k-2}{4}\,\phi + \dfrac{k\pi}{4}\right)$ is analogous and so the proof of the theorem is complete. Formula (5.4) shows that the inequality is best possible. It is interesting to note, however, that the extreme value is not attained in $|z| \leq 1$ by any function in the class.

Acknowledgement

The author would like to express his appreciation to Glenn Schober for several helpful conversations during the preparation of this paper.

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Printed in March 1975