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ESTIMATES FOR UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSIONS

BY

ZERRİN GÖKTÜRK

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Zerrin Göktürk

Introduction

A representation formula for normalized quasiconformal homeomorphisms f of the plane whose complex dilatation μ has bounded support provides an efficient tool for studying the dependence of $f(z)$ on μ. If μ depends analytically on a complex parameter, then the same is also true of $f(z)$. Making use of this result and resorting to fundamental theorems of the classical theory of analytic functions, Lehto [7] presented a general inequality which provides a method for studying the properties of normalized univalent functions with quasiconformal extensions.

Another approach to these problems is based on variational techniques introduced by Belinskii [1] and Schiffer [11]. Such methods have later been used by Kruškal [5], Kühnau [6], and Schiffer and Schober [12] for solving a variety of extremal problems.

We also mention the work of Blevins [2] who studied conformal homeomorphisms mapping the unit disc onto domains bounded by quasiconformal circles.

In this paper we study the class of conformal homeomorphisms f of the unit disc which are normalized by the conditions $f(0) = 0$, $f'(0) = 1$, and have k-quasiconformal extensions to the whole plane such that the point at infinity remains fixed.

After some preliminary remarks in section 1, we summarize Lehto's results in sections 2 and 3. In section 4 we apply the general inequalities and obtain estimates for $|f(z)|$ and $|f''(z)/f'(z)|$. The latter es imate is needed in section 5 where we study the power series coefficients a_n of f. Modifying a method of Clunie and Pommerenke [3] we obtain an upper bound for $|a_n|$ in terms of n and k.

1. The families S and $S_k(\infty)$

1.1. *Definitions.* Let S be the class of functions f that are analytic and univalent in the unit disc $D = \{ z \mid |z| < 1\}$ with the normalization $f(0) = 0$, $f'(0) = 1$. We denote by $S_k(\infty)$ the class of k-quasiconformal homeomorphisms f of the extended plane whose restrictions to D are in S and which leave ∞ fixed. The k-quasiconformality means that f is a homeomorphic L^2-solution of a Beltrami differential equation $f_{\bar z} = \mu f_z$, where the complex dilatation μ satisfies the condition $\|\mu\|_\infty \leq k < 1$.

Conversely, from the existence and uniqueness theorems for the Beltrami equation it follows that a measurable function μ whose support lies in $|z| \geq 1$ and which has the property $\|\mu\|_\infty \leq k$ determines uniquely the element of $S_k(\infty)$ whose complex dilatation agrees a.e. with μ. In particular, $S_0(\infty)$ consists of the identity mapping only.

1.2. *Approximation of functions of* S *by functions of* $S_k(\infty)$. The union of the classes $S_k'(\infty) = \{f|D \mid f \in S_k(\infty)\}$ is dense in the class S, i.e., every $f \in S$ can be approximated by functions f_n belonging to some $S_k'(\infty)$. This is seen as follows: Let $\{r_n\}$, $r_n < 1$, be a sequence of numbers with $\lim r_n = 1$, and set $f_n(z) = f(r_n z)/r_n$ for $z \in D$. Then every f_n admits a quasiconformal extension g_n to the rest of the plane, since the image of the unit circle under f_n is an analytic curve. Suppose that g_n takes the value ∞ at the point z_n. Let h_n be a quasiconformal self-mapping of the outside of the unit disc which carries ∞ into the point z_n and keeps every boundary point fixed. Then $g_n \circ h_n$ is a quasiconformal extension of f_n which keeps ∞ fixed. Clearly $\lim f_n(z) = f(z)$ in D.

1.3. $S_k(\infty)$ *is a closed normal family.* Every $S_k(\infty)$, $0 \leq k < 1$, is a normal family. To prove this, let $z_0 \neq 0$ be a point in the unit disc. By the well-known distortion theorem we have for each $f \in S_k(\infty)$

$$\frac{r}{(1+r)^2} \leq |f(z_0)| \leq \frac{r}{(1-r)^2}, \quad |z_0| = r .$$

Hence, there is a constant $d > 0$, depending on z_0 but not on f, such that the points $f(0) = 0$, $f(\infty) = \infty$, and $f(z_0)$ have a spherical distance $> d$ from each other. It follows that $S_k(\infty)$ is a normal family ([9], Theorem II.5.1).

Moreover, $S_k(\infty)$ is closed under uniform convergence in the spherical metric. Indeed, if f is the uniform limit of a sequence $f_n \in S_k(\infty)$, then f is either a k-quasiconformal mapping, a mapping of the plane onto two points, or a constant ([9], Theorem II.5.3). Since for each n, the function f_n takes the values 0 and ∞, and satisfies $f_n'(0) = 1$, the limit function can be neither a constant nor a mapping onto two points.

As a consequence, the functions f in $S_k(\infty)$ are uniformly bounded on every compact subset E of the finite plane. For if not, there is a sequence of functions $f_n \in S_k(\infty)$ and points $z_n \in E$, such that $\lim z_n = a \in E$, $\lim f_n = f \in S_k(\infty)$, and

(1.1) $$\lim f_n(z_n) = \infty \ .$$

Being a normal family, $S_k(\infty)$ is equicontinuous. Therefore, $\lim f_n(z_n) = f(a)$, which contradicts (1.1).

In particular,

$$C_k = \sup_{f \in S_k(\infty)} \left(\max_{|z| \leq 1} |f(z)|\right)$$

is finite. In section 4 we shall give an upper estimate for C_k.

2. Analytic correspondence

In this section we shall review results of the dependence of a quasiconformal mapping on its complex dilatation. For technical reasons, we prefer to consider for a moment a class of quasiconformal mappings whose complex dilatation has bounded support.

2.1. *Class Σ_k and representation formula.* Let Σ_k denote the class of quasiconformal homeomorphisms f of the extended plane which have complex dilatation μ with $\|\mu\|_\infty \leq k < 1$, are conformal in

$$D^* = \{z \mid |z| > 1\} ,$$

and satisfy the normalization condition

(2.1) $$\lim_{z \to \infty} (f(z) - z) = 0 .$$

By a result of Bojarski ([9], p. 218), a function $f \varepsilon \Sigma_k$ can be expressed in terms of μ,

(2.2) $$f(z) = z + \sum_{i=1}^{\infty} T\phi_i(z) .$$

Here the functions ϕ_i are defined by means of the two-dimensional Hilbert transformation S : $\phi_1 = \mu$, $\phi_i = \mu S \phi_{i-1}$, $i = 2, 3, \ldots$, and

$$T\phi_i(z) = -\frac{1}{\pi} \iint \frac{\phi_i(\zeta)}{\zeta - z} \, d\xi d\eta .$$

The series $\sum_{i=1}^{\infty} T\phi_i(z)$ is uniformly convergent in the whole plane.

2.2. *Dependence of $f \in \Sigma_k$ on its complex dilatation.* Suppose that every point w of a finite domain G determines a unique measurable function $\mu(\ , w)$ in the plane which has the support in $|z| \leq 1$ and satisfies

$$\|\mu(\ ,w)\|_\infty < 1 .$$

For each $\mu(\ ,w)$ there is a unique quasiconformal homeomorphism $f(\ ,w)$ of the plane which has complex dilatation equal to $\mu(\ ,w)$ a.e., and the normalization (2.1). Thus, for every fixed finite z, $w \mapsto f(z,w)$ is a complex valued function in G. Let $f(z,w) = z + \Sigma\, b_n(w)\, z^{-n}$ for $z \in D^*$, and let $f^{(n)}(z,w)$ denote the value of the nth derivative of the function $f(\ ,w)$ at the point $z \in D^*$. Using (2.2) Lehto [7] proved the following result.

Theorem 2.1. *Let $\mu(z,\ )$ be analytic in G for every z. Then the function $w \mapsto f(z,w)$ is analytic in G for every finite z. Moreover, the functions $w \mapsto f^{(n)}(z,w)$, $n = 1, 2, \ldots$, $z \in D^*$, and $w \mapsto b_n(w)$, $n = 1, 2, \ldots$, are analytic in G.*

2.3. *Dependence of $f \in S_k(\infty)$ on its complex dilatation.* Theorem 2.1 can easily be carried over to functions of $S_k(\infty)$. To do this, consider a family of complex dilatations $\mu(\ ,w)$ which are defined as before except that they have support outside the unit disc D. Again it follows from the existence and uniqueness theorems for the Beltrami equation that for each $\mu(\ ,w)$ there is a unique quasiconformal homeomorphism $f(\ ,w)$ of the plane which has complex dilatation $\mu(\ ,w)$ a.e., a power series expansion

$$f(z,w) = z + \sum_{n=2}^{\infty} a_n(w) z^n$$

in $|z| < 1$, and which fixes the point at infinity.

Theorem 2.2. *If $\mu(z,\ )$ is analytic in a domain G for every z, then the function $w \mapsto f(z,w)$ is analytic in G for every finite z. Furthermore, the functions $w \mapsto f^{(n)}(z,w)$, $n = 1, 2, \ldots$, are analytic in G for every $z \in D$.*

Proof: Consider the function $\psi(\ ,w)$, defined by

$$\psi(z,w) = \frac{1}{f(1/z,w)} - a_2(w),$$

which is quasiconformal in the plane, conformal in D^*, and satisfies the normalization condition (2.1). If ν is the complex dilatation of ψ, then $\nu(z,w) = (z/\bar{z})^2 \mu(1/z,w)$. Hence, we first conclude from Theorem 2.1 that the function $a_2 = \psi(0,\ )$ is analytic in G. From Theorem 2.1 it then follows that the functions $w \mapsto f(z,w) = (\psi(1/z,w) - a_2(w))^{-1}$ and $w \mapsto f^{(n)}(z,w)$ are analytic in G. The coefficients a_n, $n = 3, 4, \ldots$, are analytic in G, since $a_n = f^{(n)}(0,\ )/n!$.

3. General inequalities

3.1. *Analytic functionals.* Let f be a function of S. We define an analytic functional ϕ on S to be a complex-valued function which depends analytically on finitely many power series coefficients of f, and on the values of f and its derivatives $f^{(k)}$, $k = 1, 2, \dots, n$, at finitely many given points. An analytic functional is continuous, i.e., $\lim \phi(f_n) = \phi(f)$ whenever f is the uniform limit of the functions f_n on compact subsets of the unit disc.

An analytic functional ϕ defined on S is defined on every class $S'_k(\infty)$ (see 1.2); to simplify notation we write $S_k(\infty)$ instead of $S'_k(\infty)$ in the rest of the paper. Since ϕ is continuous, and S and $S_k(\infty)$ are closed normal families, there are functions which maximize $|\phi(f)|$ in $S_k(\infty)$ and S. We set

$$M(k) = \max_{f \in S_k(\infty)} |\phi(f)| ,$$

and denote by $M(1)$ the maximum of $|\phi(f)|$ in S. Then M is a non-decreasing function on the closed interval [0,1].

3.2. *Continuity of* M. The function M is continuous on [0,1]. To prove this, choose an arbitrary k_0, $0 < k_0 < 1$. Because M is non-decreasing, the left and right limits $\lim\limits_{k \to k_0-} M(k)$ and $\lim\limits_{k \to k_0+} M(k)$ exist, and

$$\lim_{k \to k_0-} M(k) \leq M(k_0) , \quad \lim_{k \to k_0+} M(k) \geq M(k_0) . \tag{3.1}$$

Suppose first that $k < k_0$. Let f_0 be extremal in $S_{k_0}(\infty)$, with complex dilatation μ. Consider the functions f_k with complex dilatation $k\mu/k_0$, $0 < k < k_0$, so normalized that $f_k \in S_k(\infty)$. Since $\{f_k\}$ is a normal family, there is a sequence k_i, $i = 1, 2, \dots$, so that $\lim k_i = k_0$ and the mappings f_{k_i} converge uniformly (in the spherical metric) to a limit mapping g. Then the mapping g has complex dilatation μ a.e. ([9], Theorem IV.5.2). Hence, because of normalization $g = f_0$. From the continuity of ϕ it follows that $\phi(f_{k_i}) \to \phi(f_0)$. Consequently,

$$\lim_{i \to \infty} M(k_i) \geq \lim_{i \to \infty} |\phi(f_{k_i})| = |\phi(f_0)| = M(k_0) .$$

In conjunction with the first inequality (3.1) this shows that

$$\lim_{k \to k_0-} M(k) = M(k_0) ,$$

i.e., M is continuous to the left at k_0.

Suppose next that $k > k_0$. Let f_k now denote the extremal mapping in $S_k(\infty)$. Again, there is a sequence k_i, $i = 1, 2, \ldots$, so that $\lim k_i = k_0$ and the mappings f_{k_i} converge uniformly to a limit g. Then, by Theorem I.5.2 in [9], the maximal dilatation of g is not greater than the limit of the maximal dilatations of f_{k_i}. Consequently, $g \in S_{k_0}(\infty)$. It follows that

$$\lim_{i\to\infty} M(k_i) = \lim_{i\to\infty} |\phi(f_{k_i})| = |\phi(g)| \leq M(k_0) .$$

Together with the second inequality (3.1) this yields $\lim\limits_{k\to k_0+} M(k) = M(k_0)$, i.e., M is continuous to the right at k_0.

Continuity to the right at 0 is proved similarly. Finally, let f be extremal in S, and $f_n(z) = f((1 - 1/n)\, z)/(1 - 1/n)$. Then f_n admits a quasiconformal extension so that the extended mapping is in a class $S_{k_n}(\infty)$, where k_n tends increasingly to 1. Since $f_n(z) \to f(z)$, uniformly in every compact subset of the unit disc, $\phi(f_n) \to \phi(f)$. Hence,

$$M(1) = |\phi(f)| = \lim_{n\to\infty} |\phi(f_n)| \leq \lim_{n\to\infty} M(k_n) ,$$

and left continuity at 1 follows.

3.3. *Majorant principle.* The following inequality, which has wide applications in transferring many of the results known to hold for univalent functions to the classes $S_k(\infty)$ and Σ_k, $k < 1$, is due to Lehto.[1]

Theorem 3.1. *Let ϕ be an analytic functional defined on S, which vanishes for the identity mapping. Then*

$$M(k) \leq k M(1) . \tag{3.2}$$

If equality holds in (3.2) for one value of k, $0 < k < 1$, then it holds for all values of k, and if μ is an extremal complex dilatation, then all dilatations $w\mu$, $|w| < 1/\|\mu\|_\infty$, are extremal. For the proofs we refer to [7].

3.4. *Real part of an analytic functional.* An analogue of Theorem 3.1 is obtained if one considers $\operatorname{Re}\phi$ instead of ϕ. Again, the extremal problems $\max \operatorname{Re}\phi(f)$, $\min \operatorname{Re}\phi(f)$ have solutions in $S_k(\infty)$. We shall first consider the minimum, and write

[1] Communicated in the Complex Analysis Seminar at Case Western Reserve University in 1971.

$$m(k) = \min_{f \in S_k(\infty)} \operatorname{Re} \phi(f) .$$

The following theorem is due to Lehto (unpublished). We denote

$$m(1) = \min_{f \in S} \operatorname{Re} \phi(f) \quad \text{and} \quad m(0) = \operatorname{Re} \phi(id) .$$

Theorem 3.2. *Let ϕ be an analytic functional defined on S. Then for every $f \in S_k(\infty)$*

$$\frac{2k}{1+k}(m(1) - m(0)) \leq \operatorname{Re} \phi(f) - m(0) \leq \frac{2k}{1-k}(m(0) - m(1)). \tag{3.3}$$

Proof: Let f be an arbitrary mapping in $S_k(\infty)$, $0 \leq k < 1$, and μ its complex dilatation. Consider the subclass of S whose functions are restrictions to D of quasiconformal homeomorphisms $\hat{f}$ of the plane with the complex dilatation $w\mu$, where $|w| < 1/k$. By Theorem 2.2, the functional $\phi(\hat{f})$ depends analytically on w in the disc $|w| < 1/k$. Therefore, $u = \operatorname{Re} \phi(\hat{f})$ is harmonic in $|w| < 1/k$. Applying Poisson's formula for $|w| \leq \varrho < 1/k$ we have

$$u(w) - m(1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varrho^2 - r^2}{|\varrho e^{i\theta} - w|^2} (u(\varrho e^{i\theta}) - m(1))\, d\theta ,$$

where $|w| = r < \varrho$. Since $u - m(1)$ is non-negative and

$$\frac{\varrho - r}{\varrho + r} \leq \frac{\varrho^2 - r^2}{|\varrho e^{i\theta} - w|^2} \leq \frac{\varrho + r}{\varrho - r}, \tag{3.4}$$

this yields the upper estimate

$$u(w) - m(1) \leq \frac{1}{2\pi} \frac{\varrho + r}{\varrho - r} \int_0^{2\pi} (u(\varrho e^{i\theta}) - m(1))\, d\theta .$$

The arithmetic mean of $u(\varrho e^{i\theta}) - m(1)$ over the interval $0 \leq \theta \leq 2\pi$ equals $u(0) - m(1) = m(0) - m(1)$, and it follows that

$$u(w) - m(1) \leq \frac{\varrho + r}{\varrho - r}(m(0) - m(1)) . \tag{3.5}$$

Letting ϱ tend to $1/k$ and rearranging the terms, we obtain

$$u(w) - m(0) \leq \frac{2rk}{1 - rk}(m(0) - m(1)) . \tag{3.6}$$

Taking $w = 1$ we have $u(1) = \operatorname{Re} \phi(f)$, and the right-hand side of (3.3) follows from (3.6). Similarly, using the left-hand inequality (3.4), we obtain the lower estimate in (3.3).

Remark. Since the inequalities (3.3) hold for every function $f \in S_k(\infty)$, it follows that

$$m(k) \geq \frac{1-k}{1+k} m(0) + \frac{2k}{1+k} m(1) . \tag{3.7}$$

Denoting by $M_R(k)$ the maximum of $\operatorname{Re} \phi$ in $S_k(\infty)$, we also conclude that

$$M_R(k) \leq \frac{1+k}{1-k} m(0) - \frac{2k}{1-k} m(1) . \tag{3.8}$$

3.5. *Equality in the estimates.* Let us assume that equality holds in (3.8) [or in (3.7)] for one value of k, $0 < k < 1$. Then it holds for all values of k, and if μ is an extremal complex dilatation, then all dilatations $w\mu$ with $0 < w < 1/\|\mu\|_\infty$ are extremal. To prove this, suppose that (3.8) holds as an equality for a k, $0 < k < 1$; let f_0 be the extremal function and μ_0 its complex dilatation. For functions with complex dilatation $w\mu_0$, $|w| < 1/k$, the Poisson formula yields the inequality (3.5). For brevity, let us write $U(w) = (u(w) - m(1))/(m(0) - m(1))$. Thus, letting ϱ tend to $1/k$ in (3.5) we obtain

$$U(w) \leq \frac{1 + k|w|}{1 - k|w|} . \tag{3.9}$$

Because $U(1) = (M_R(k) - m(1))/(m(0) - m(1))$, we conclude from (3.8) that

$$U(1) = \frac{1+k}{1-k} . \tag{3.10}$$

Since U is a non-negative harmonic function in $|w| < 1/k$, we can apply the Poisson-Stieltjes formula to U, the integral being extended along the boundary $|w| = 1/k$. For $w = re^{i\varphi}$ we get

$$U(w) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1 - k^2 r^2}{1 + k^2 r^2 - 2kr \cos(\theta - \varphi)} \, d\psi(\theta) . \tag{3.11}$$

Here ψ is a non-decreasing function determined up to an additive constant. We normalize ψ so that $\psi(-\pi) = 0$. Since $U(0) = 1$, we have

$\int_{-\pi}^{+\pi} d\psi(\theta) = 2\pi$ ([10], pp. 191—201). Comparison of (3.10) and (3.11) gives

$$U(1) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1 - k^2}{1 + k^2 - 2k \cos\theta} \, d\psi(\theta) = \frac{1 + k}{1 - k}.$$

Since the function

$$\theta \mapsto \frac{1 - k^2}{1 + k^2 - 2k \cos\theta}$$

is continuous on the interval $[-\pi, \pi]$ and strictly less than $(1 + k)/(1 - k)$ for $\theta \neq 0$, it follows that the function ψ must be of the form

$$\psi(\theta) = \begin{cases} 0 & \text{for } -\pi \leq \theta < 0, \\ 2\pi & \text{for } \quad 0 < \theta \leq \pi. \end{cases}$$

Hence, again from (3.11), it follows that

$$U(w) = \frac{1 - k^2 r^2}{1 - k^2 r^2 - 2kr \cos\varphi} = \operatorname{Re}\left[\frac{1 + kw}{1 - kw}\right].$$

Thus for every $w > 0$,

$$U(w) = \frac{1 + kw}{1 - kw}. \tag{3.12}$$

If $k' \in [0,1]$ is arbitrarily given, then for $w = k'/k$ the function f is in $S_{k'}(\infty)$. From (3.12) we conclude that (3.8) holds as an equality with k replaced by k'. The function f is extremal in $S_{k'}(\infty)$ and $w\mu_0$ is the extremal complex dilatation.

4. Maximum modulus estimates

4.1. *Estimates for* $|f|$. Making use of Theorem 3.2 we shall first give estimates for $|f(z)|$ in the case when $f \in S_k(\infty)$ and z lies in the closure of the unit disc. In what follows, $K = (1 + k)/(1 - k)$.

Theorem 4.1. *Let* $f \in S_k(\infty)$. *Then for* $|z| = r < 1$

$$r(1 + r)^{2(1/K-1)} \leq |f(z)| \leq r(1 + r)^{2(K-1)}. \tag{4.1}$$

Proof: For z fixed, $\phi(f) = \log (f(z)/z)$ is an analytic functional, and

$\operatorname{Re}\phi(f) = \log|f(z)/z|$, $\operatorname{Re}\phi(id) = 0$. From the theory of univalent functions it is well-known that

$$m(1) = \min_{f \in S} \operatorname{Re}\phi(f) = \log\frac{1}{(1+r)^2}.$$

Thus, (4.1) follows directly from Theorem 3.2.

Remark. Application of Theorem 3.1 to the functional $\phi(f) = \log(f(z)/z)$ yields the upper bound

$$|f(z)| \leq \frac{r}{(1-r)^{2k}}.$$

For small values of r this is sharper than the upper estimate in (4.1).

Letting r tend to 1, we obtain from (4.1) simple upper and lower bounds for $|f(z)|$ on the unit circle.

Corollary 4.1. *If* $f \in S_k(\infty)$, *then for* $|z| = 1$

$$\left(\frac{1}{4}\right)^{1-1/K} \leq |f(z)| \leq 4^{K-1}. \tag{4.2}$$

For $k = 0$, the lower and upper bound both take the value 1. As $k \to 1$, the lower bound tends to the sharp limit 1/4. For further discussion of (4.2), let us introduce the modified Koebe functions

$$f(z) = \begin{cases} \dfrac{z}{(1+ke^{i\varphi}z)^2} & \text{if } |z| \leq 1, \\[2ex] \dfrac{z\bar{z}}{(\sqrt{\bar{z}}+ke^{i\varphi}\sqrt{z})^2} & \text{if } |z| > 1. \end{cases} \tag{4.3}$$

Direct computation shows that $f \in S_k(\infty)$. For this function

$$\min_{|z|=1} |f(z)| = \frac{1}{(1-k)^2}.$$

It follows that the lower bound $4^{1/K-1}$ in (4.2) cannot be replaced by 4^{-k}.

Let us again consider $C_k = \sup_{f \in S_k(\infty)} (\max_{|z| \leq 1} |f(z)|)$, for which Corollary 4.1 yields the upper bound $C_k \leq 4^{K-1}$. The function (4.3) tells us also that $C_k \geq (1-k)^{-2}$. It is an interesting open problem to determine the exact value of C_k.

4.2. *Estimates for* $|f''/f'|$. We shall use the following consequence of Theorem 3.1 later in estimating the coefficients of the functions $f \in S_k(\infty)$.

Theorem 4.2. *Let* $f \in S_k(\infty)$. *Then for* $|z| = r < 1$

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 2k \frac{r+2}{1-r^2}. \tag{4.4}$$

Proof: Consider the analytic functional $\phi(f) = f''(z)/f'(z)$, $z \in D$, which vanishes for the identity mapping. In S

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 2 \frac{r+2}{1-r^2}$$

([4], p. 50). Hence (4.4) follows from the inequality (3.2).

This estimate is sharp for $z = 0$, equality holding for the functions (4.3).

For our applications we need (4.4) when r is close to 1. We shall show that this estimate is essentially sharp in the sense that for every $k > 0$ there is a function $f \in S_k(\infty)$ and a boundary point $e^{i\theta}$ so that

$$\lim_{r\to 1} (1-r^2)\left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right| > 0. \tag{4.5}$$

To construct a mapping for which (4.5) holds we map the unit disc by $z \mapsto (1+z)/(1-z)$ onto the right half-plane $\operatorname{Re} \zeta > 0$. We note that the desired function f must have singularity on $|z| = 1$ since $f''(z)/f'(z)$ must tend to ∞ as $r \to 1$. Therefore, we map $\operatorname{Re} \zeta > 0$ onto the angular domain $|\arg(\omega + 1/(2+2k))| \leq \pi(1+k)/2$ by the function

$$\omega(\zeta) = (\zeta^{k+1} - 1)/2(1+k),$$

which has the k-quasiconformal extension $\omega(\zeta) = [\zeta(-\bar{\zeta})^k - 1]/2(1+k)$. If $\omega(f(\infty)) = \omega_0$, then this function is transformed into the class $S_k(\infty)$ by the Möbius transformation $h(\omega) = \omega_0\omega/(\omega_0 - \omega)$. In the unit disc the function $f = h\circ\omega\circ\zeta$, which is in $S_k(\infty)$, has the expression

$$f(z) = \frac{(1+z)^{1+k} - (1-z)^{1+k}}{(1+k)\,[(1+k)\,(1+z)^{1+k} - k(1-z)^{1+k}]}.$$

Hence

$$\frac{f''(z)}{f'(z)} = k\left(\frac{1}{1+z} - \frac{1}{1-z}\right) + O(1),$$

so that

$$\lim_{r\to 1}(1-r^2)\left|\frac{f''(\pm r)}{f'(\pm r)}\right| = 2k .$$

5. Coefficient estimates

5.1. *Preliminary remarks.* Let f be a function in $S_k(\infty)$. Then the area of the image of the unit disc under the mapping f is at most πC_k^2 . It follows that

$$\sum_1^\infty n|a_n|^2 < C_k^2 .$$

This yields the estimate

$$|a_n| = O(n^{-1/2}) . \tag{5.1}$$

For bounded univalent functions it was shown by Clunie and Pommerenke [3] that the estimate $|a_n| = O(n^{-1/2})$ is not the best possible.

Application of the inequality (3.2) to the functional $\phi(f) = a_n$, which vanishes for the identity mapping, gives the estimate

$$\max_{S_k(\infty)} |a_n| \leq k \max_S |a_n| . \tag{5.2}$$

This inequality is sharp only if $n = 2$ ([8], Corollary 4.2). Since $\max_S |a_n| \geq n$, this estimate becomes very inaccurate for large values of n , in view of (5.1).

In the present section we shall estimate the coefficients of the functions in $S_k(\infty)$. It turns out that the existence of a k-quasiconformal extension not only gives a k-contraction to the coefficient estimate but has a marked effect on the order of magnitude: $|a_n| \leq k A_k n^{-1/2-\alpha(k)}$, where A_k is finite for every $k < 1$ and $\alpha(k)$ decreases from $1/2$ to a value > 0 as k grows from 0 to 1.

5.2. *Mean value estimate for* $|f'|^2$. We find it convenient first to establish the following lemma.

Lemma 5.1. *If* $f \in S_k(\infty)$, *then for* $r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \leq \frac{C_k^2}{1-r^4} . \tag{5.3}$$

Proof: An easy computation gives

(5.4) $$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta = \sum_1^\infty n^2 |a_n|^2 r^{2n-2} .$$

We set $n|a_n|^2 = b_n$, $a_1 = 1$. Then

$$\sum_1^\infty n|a_n|^2 = \sum_1^\infty b_n < C_k^2 .$$

Consider the analytic function $\varphi(z) = \sum_1^\infty b_n z^n$ in the unit disc $|z| < 1$. Then $|\varphi(z)| < C_k^2$ for $|z| < 1$. From Schwarz's Lemma it follows that

$$|\varphi'(z)| \le \frac{C_k^2 - |\varphi(z)|^2/C_k^2}{1-|z|^2} \le \frac{C_k^2}{1-|z|^2} .$$

Hence, for $z = r$

$$|\varphi'(z)| = \sum_1^\infty n b_n r^{n-1} \le \frac{C_k^2}{1-r^2} .$$

Replacing r by r^2 we obtain

$$\sum_1^\infty n b_n r^{2n-2} = \sum_1^\infty n^2 |a_n|^2 r^{2n-2} \le \frac{C_k^2}{1-r^4} ,$$

and (5.3) follows from (5.4).

5.3. *Mean value estimate for* $|f'|$. The proof of the following lemma is carried out by the method of Clunie and Pommerenke [3], with the difference that Theorem 4.2 is taken into consideration.

Lemma 5.2. *If* $f \in S_k(\infty)$, *then for* $r < 1$

(5.5) $$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \le (1 + C_k)\,(1-r)^{-1/2+\alpha(k)} ,$$

where $\alpha(k) = 1/2 - 8k(\sqrt{1+64k^2} - 8k)$.

Proof: Let $\delta > 0$. By Schwarz's inequality

(5.6) $$J(r) = \left(\int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta} \, d\theta\right)^2 \le \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \int_0^{2\pi} |f'(re^{i\theta})|^{2\delta} d\theta .$$

To estimate the last integral in (5.6) we write

$$[f'(z)]^\delta = \sum_{m=0}^{\infty} c_m z^m \, , \qquad c_0 = 1 \, .$$

Then

$$F(r) = \frac{1}{2\pi} \int_0^{2\pi} | \, [f'(re^{i\theta})]^\delta |^2 \, d\theta = \sum_{m=0}^{\infty} |c_m|^2 r^{2n} \, .$$

Direct calculation gives

$$F''(r) \leq 4 \sum_{m=1}^{\infty} m^2 |c_m|^2 r^{2m-2} = \frac{2}{\pi} \int_0^{2\pi} \left| \frac{d}{dz} [f'(re^{i\theta})]^\delta \right|^2 d\theta$$

$$= \frac{2\delta^2}{\pi} \int_0^{2\pi} \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right|^2 |f'(re^{i\theta})|^{2\delta} d\theta \, .$$

By Theorem 4.2,

$$\left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| \leq 2k \frac{r+2}{1-r^2} \leq \frac{4k}{1-r} \, .$$

Combining this with the preceding inequality we get

$$F''(r) \leq 64\delta^2 k^2 \frac{F(r)}{(1-r)^2} \, .$$

Integrating by parts we obtain

$$F'(r) \leq 64\delta^2 k^2 \left[\frac{F(r)}{1-r} - F(0) - \int_0^r \frac{F'(t)}{1-t} \, dt \right] .$$

Dropping the last two negative terms and dividing by $F(r)$ we deduce

$$\frac{F'(r)}{F(r)} \leq \frac{64\delta^2 k^2}{1-r} \, .$$

Hence

$$F(r) \leq (1-r)^{-64\delta^2 k^2} \, . \tag{5.7}$$

By Lemma 5.1,

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \leq \frac{C_k^2}{1-r} \, .$$

From the inequalities (5.6) and (5.7) it thus follows that

$$(5.8) \qquad \int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta}\, d\theta \leq 2\pi\, C_k(1-r)^{-1/2-32\delta^2k^2} .$$

For a $\beta > 0$, let

$$E_1 = \{\theta \mid |f'(re^{i\theta})| \leq (1-r)^{-\beta}\} ,$$
$$E_2 = \{\theta \mid |f'(re^{i\theta})| > (1-r)^{-\beta}\} .$$

Then by (5.8)

$$\int_0^{2\pi} |f'(re^{i\theta})| d\theta = \int_{E_1} |f'(re^{i\theta})|\, d\theta + \int_{E_2} |f'(re^{i\theta})|\, d\theta$$

$$\leq 2\pi\,(1-r)^{-\beta} + (1-r)^{\beta\delta} \int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta} d\theta$$

$$\leq 2\pi\,(1-r)^{-\beta} + 2\pi\, C_k(1-r)^{-1/2+\beta\delta-32\delta^2k^2} .$$

The choice $\beta = \beta_0 = (1 + 64\delta^2k^2)/2(1+\delta)$ gives

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leq 2\pi\,(1 + C_k)\,(1-r)^{-\beta_0} .$$

Finally, in order to minimize β_0 we take $\delta = -1 + \sqrt{1 + 64k^2}\,/\,8k$, and (5.5) follows.

5.4. *An estimate for* a_n. Using the above lemma and once more Theorem 4.2, we now obtain an estimate for a_n.

Theorem 5.1. *Let* $f \in S_k(\infty)$. *Then*

$$(5.9) \qquad |a_n| \leq kA_k n^{-1/2-\alpha(k)} ,$$

where $A_k = 4e(1 + C_k)$ *and* $\alpha(k)$ *is defined as in Lemma* 5.2.

Proof: Applying the Cauchy integral formula to f'' we get

$$n(n-1)|a_n| \leq \frac{1}{2\pi r^{n-2}} \int_0^{2\pi} |f''(re^{i\theta})|\, d\theta , \; r < 1 .$$

From Theorem 4.2 and Lemma 5.2 it follows, therefore, that

$$n(n-1)\,|a_n| \leq 4k\, \frac{1 + C_k}{r^{n-2}} (1-r)^{-3/2+\alpha(k)} .$$

Taking $r = 1 - 1/n$, we obtain

$$|a_n| \leq 4k\left(\frac{n}{n-1}\right)^{n-1}(1 + C_k)n^{-1/2-\alpha(k)},$$

and (5.9) follows.

As $k \to 0$, the exponent $1/2 + \alpha(k)$ tends to 1. This order of magnitude cannot be improved in the sense that an estimate of the form (5.9) with an exponent tending to a limit > 1 as $k \to 0$ is not possible. A counterexample is provided by the function f defined by

$$f(z) = \begin{cases} z(1 + kz^{n-1})^{2/(1-n)} & \text{for } |z| < 1, \\ z\bar{z}(\bar{z}^{\frac{1}{2}(n-1)} + kz^{\frac{1}{2}(n-1)})^{2/(1-n)} & \text{for } |z| \geq 1. \end{cases}$$

This function belongs to $S_k(\infty)$, and $|a_n| = 2k/(1-n)$ ([8]).

5.5. *Estimates of Clunie and Pommerenke.* Let $f \in S$ and assume that $|f(z)| \leq M$. For $|z| < 1$ write $f_m(z) = f((1 - 1/m)z)/(1 - 1/m)$, $m = 2, 3, \ldots$. We showed in 1.2 that f_m admits a quasiconformal extension so that the extended mapping belongs to a class $S_k(\infty)$ for some $k < 1$. By Schwarz's Lemma, $|f((1 - 1/m)z)| \leq M(1 - 1/m)$ in $|z| < 1$. Hence $|f_m(z)| \leq M$. If we take $C_k = M$ and $k = 1$ in (5.9), then the inequality holds for the nth coefficient of f_m for every m. Therefore, it also holds for the coefficient a_n of f, and we obtain

$$|a_n| \leq 4e(1 + M)n^{-(1/2+1/517)}.$$

Essentially, this is the estimate of Clunie and Pommerenke [3] for bounded functions.

Department of Mathematics
University of Helsinki

Current address:
Department of Mathematics
University of Istanbul

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