ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

588

WHITEHEAD TORSION AND GROUP ACTIONS

BY

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HELSINKI 1974
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1975.588

ISSN 0066-1953
ISBN 951-41-0202-9

Communicated 11 March 1974

KESKUSKIRJAPAINO
HELSINKI 1974

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Introduction

Let G be a discrete group or a compact Lie group, and let X and Y denote finite equivariant CW complexes. In Section 1 of Chapter II we introduce the notion of an equivariant simple-homotopy equivalence $f: X \to Y$. The main part for this definition is the definition of an equivariant elementary expansion, see Definition 1.1 in Chapter II. Having defined equivariant elementary expansions (collapses) the notions, equivariant formal deformation, equivariant expansion (collapse), and equivariant simple-homotopy equivalence, follow in complete analogy with the corresponding notions in the ordinary non-equivariant case as defined by Whitehead [18].

We then define an equivariant Whitehead group $Wh_G(X)$ of any finite equivariant CW complex X. If G is the trivial group, and X thus is an ordinary CW complex, this simply gives us the geometrically defined Whitehead group $Wh(X)$ of X which, for connected X, is known to be isomorphic to $Wh(\pi_1(X))$, the (algebraicly defined) Whitehead group of the group $\pi_1(X)$, see Cohen [6], Eckmann-Maumary [1], and Stöcker [15]. We prove that if G acts freely on X then $Wh_G(X) \cong Wh(G\backslash X)$. Thus if G is a discrete group acting freely on X, and X is simply connected, we have $Wh_G(X) \cong Wh(G)$. In Section 1 of Chapter III we determine $Wh_G(X)$ in terms of Whitehead groups of quotient groups of subgroups of G, for the case when G is a discrete abelian group and X is such that for any subgroup H of G each component of X^H is simply connected.

Let now again G denote any discrete group or an arbitrary compact Lie group. In Section 3 of Chapter II we define the geometric equivariant Whitehead torsion $\tau_g(f) \in Wh(X)$ of any G-homotopy equivalence $f: X \to Y$, and prove that f is an equivariant simple-homotopy equivalence if and only if $\tau_g(f) = 0$. If G acts freely on X and Y, then any G-homeomorphism $f: X \to Y$ is an equivariant simple-homotopy equivalence. This follows from the recent affirmative answer by T. Chapman to the question of topological invariance of ordinary Whitehead torsion, and the isomorphism $Wh_G(X) \cong Wh(G\backslash X)$ for free actions. We also give a sum theorem for $\tau_g(f)$. In Section 4 we prove a technical result which says that any element in $Wh_G(X)$ can be represented by an element which is in »simplified form», see Corollary 4.4 in Chapter II for the precise statement. This result is the key to the results proved in Chapter III.

In Chapter III we first consider actions of discrete abelian groups and prove the already mentioned theorem about $Wh_G(X)$, see Theorem 1.4 and the discussion preceeding it. Using known results about Whitehead groups of groups this gives us information about $Wh_G(X)$. For example it follows that if $G = Z_m$, $m \geq 1$, is a finite cyclic group, and X is such that each component of any X^H, $H \subset G$, is simply connected, then $Wh_G(X)$ is a free abelian group of finite rank. If $G = Z_2$, Z_3, Z_4 or Z_6, then $Wh_G(X) = 0$, where X is as before, and hence in this situation any G-homotopy equivalence is an equivariant simple-homotopy equivalence. We also say something about the case $G = Z \oplus \ldots \oplus Z$. We conclude the first section by an example of an inclusion $i : X \to W$, in fact $X = \{x\}$, of equivariant CW complexes which is a G-homotopy equivalence but not an equivariant simple-homotopy equivalence and the induced inclusion on the orbit spaces is an ordinary simple-homotopy equivalence and moreover if we forget about the G-action then the inclusion $i : X \to W$ is an ordinary simple-homotopy equivalence.

In the final section we consider actions by an n-dimensional torus $G = T^n$, $n \geq 1$. In this case we have $Wh_G(X) = 0$, for every X satisfying the condition that for any closed subgroup H of $G = T^n$ each component of X^H is simply connected, and hence in this situation any G-homotopy equivalence is an equivariant simple-homotopy equivalence. This result applies in particular to differentiable T^n actions on compact differentiable manifolds.

Notations. By I^n we denote the n-fold product of the unit interval with itself and I^{n-1} is identified with the front $(n-1)$-face $I^{n-1} \times \{0\} \subset I^n$. By J^{n-1} we denote the union of all other $(n-1)$-faces and ∂I^n is the boundary of I^n, i.e. $\partial I^n = I^{n-1} \cup J^{n-1}$. We shall use G-spaces of the form $G/H \times I^n$, where H is some closed subgroup of G. Here G acts trivially on I^n and by the standard left action on G/H. If H is a closed subgroup of G we denote by (H) the family of all subgroups conjugate to H, and for any G-space X, we denote by X^H the set of points fixed under H.

Chapter I. Review of equivariant CW complexes

In this chapter G denotes a topological group which is either a compact Lie group or a discrete group.

Definition 1.1. Let X be a Hausdorff G-space and A a closed G-subset of X, and n a non-negative integer. We say that X is obtainable from A by adjoining equivariant n-cells if there exists a collection $\{c_j^n\}_{j\in J}$ of closed G-subsets of X such that

1) $X = A \cup (\bigcup_{j\in J} c_j^n)$, and X has the topology coherent with $\{A, c_j^n\}_{j\in J}$.

2) Denote $\dot{c}_j^n = c_j^n \cap A$, then

$$(c_j^n - \dot{c}_j^n) \cap (c_i^n - \dot{c}_i^n) = \emptyset \text{ if } i \neq j .$$

3) For each $j \in J$ there exists a closed subgroup H_j of G and a G-map

$$f_j : (G/H_j \times I^n , G/H_j \times \partial I^n) \to (c_j^n , \dot{c}_j^n)$$

such that $f_j(G/H_j \times I^n) = c_j^n$, and f_j maps $G/H_j \times (I^n - \partial I^n)$ G-homeomorphically onto $c_j^n - \dot{c}_j^n$.

Such a G-pair (X, A) in fact determines the G-subsets c_j^n uniquely, that is, any two collections of closed G-subsets of X which satisfy condition 1)—3) in the above definition are the same. Moreover X is not obtainable from A by adjoining equivariant m-cells if $m \neq n$. We call the G-subsets c_j^n for the equivariant n-cells of (X, A) and also say that X is obtained from A by adjoining the equivariant n-cells c_j^n. The G-subsets $b_j^n = c_j^n - \dot{c}_j^n$, which are open subsets of X — A, are called open equivariant n-cells of (X, A). Any G-map $f_j : G/H_j \times I^n \to c_j^n$ which satisfies the conditions in Definition 1.1 is called a characteristic G-map for c_j^n, and its restriction $f_j \mid : G/H_j \times \partial I^n \to \dot{c}_j^n \to A$ is called an attaching G-map for c_j^n. We call (H_j) for the type of c_j^n.

Definition 1.2. An equivariant relative CW complex (X, A) consists of a Hausdorff G-space X, a closed G-subset A of X, and an increasing filtration of X by closed G-subsets $(X, A)^k$, $k = 0, 1, \ldots,$ such that

1) $(X, A)^0$ is obtainable from A by adjoining equivariant 0-cells, and for $k \geq 1$ $(X, A)^k$ is obtainable from $(X, A)^{k-1}$ by adjoining equivariant k-cells.

2) $X = \bigcup_{k \geq 0} (X, A)^k$, and X has the topology coherent with $\{(X, A)^k\}_{k \geq 0}$.

The G-subset $(X, A)^k$ is called the k-skeleton of (X, A). Observe that it is part of the structure of an equivariant relative CW complex (X, A). The (open) equivariant k-cells of $((X, A)^k, (X, A)^{k-1})$ are called (open) equivariant k-cells of (X, A). Observe that the orbit space pair $(G \backslash X, G \backslash A)$ inherits the structure of an ordinary relative CW complex with k-skeleton equal to $G \backslash (X, A)^k$. We say that $\dim (X, A) = m$ if $X = (X, A)^m$ but $X \neq (X, A)^{m-1}$. If no such integer m exists we say that $\dim (X, A) = \infty$. We have $\dim (X, A) = \dim (G \backslash X, G \backslash A)$. If (X, A) is a G-pair which admits the structure of an equivariant relative CW complex then $\dim (X, A)$ is well-defined, that is, does not depend on the skeleton filtration. This follows since the corresponding statement for ordinary relative CW structures is a well-known fact.

Let (X, A) be an equivariant relative CW complex, and let X_0 be a closed G-subset of X. Then we say that $(X_0, X_0 \cap A)$ is a subcomplex of (X, A) if the filtration $X_0 \cap (X, A)^k, k = 0, 1, \ldots,$ gives $(X_0, X_0 \cap A)$ the structure of an equivariant relative CW complex. It is easy to show that then $(X, X_0 \cup A)$ is an equivariant relative CW complex with skeletons $(X, X_0 \cup A)^k = X_0 \cup (X, A)^k$. In fact $X_0 \cup (X, A)^k$ is obtained from $X_0 \cup (X, A)^{k-1}$ by adjoining all the equivariant k-cells of (X, A) which are not equivariant k-cells of $(X_0, X_0 \cap A)$.

If $A = \emptyset$ we call X an equivariant CW complex and denote its k-skeleton by X^k. An equivariant CW pair (X, X_0) consists of an equivariant CW complex X and a subcomplex X_0 of X. An equivariant CW complex X (pair (X, X_0)) is said to be finite if X has only a finite number of equivariant cells. Observe that in case G is a discrete group and X thus also automatically has the structure of an ordinary CW complex this does not mean that the ordinary CW complex X necessarily is finite.

Let (X, A) be a G-equivariant relative CW complex and (Y, B) a G'-equivariant relative CW complex. Assume that either both X and Y are locally compact or X is arbitrary and Y is compact. Then $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$ is a $(G \times G')$-equivariant relative CW complex with n-skeleton equal to $\bigcup_{k+p=n} (X, A)^k \times (Y, B)^p$. In this paper we shall only use this in the case $G' = \{e\}$, that is, (Y, B)

is an ordinary relative CW complex and the product $(X, A) \times (Y, B)$ is again a G-equivariant relative CW complex.

The following result is not explicitly stated in Chapter I of [10] although it is an immediate consequence of results given there. We shall have use of it in this paper and hence we state it here and give its proof.

Proposition 1.3. Let (X, A) be an equivariant relative CW complex. Then the inclusion $i : A \to X$ is a G-homotopy equivalence if and only if A is a strong G-deformation retract of X.

Proof. Assume that $i : A \to X$ is a G-homotopy equivalence. Then it follows that for each closed subgroup H of G the inclusion $i \mid : A^H \to X^H$ is a homotopy equivalence. Thus the pair (X^H, A^H) is n-connected for all n. By Corollary 1.11 in Chapter I of [10] this imples that (X, A) is equivariantly n-connected for all n (see Definition 1.9. in Chapter I of [10]). Thus, by Corollary 2.9 in Chapter I of [10], any G-map $f : (X, A) \to (X, A)$ is G-homotopic rel. A to a G-map from X into A. Applying this to the identity map we see that A is a strong G-deformation retract of X. The »if» part of the claim is obvious.

Now let G' be another topological group which is either a compact Lie group or a discrete group, and let $\varphi : G \to G'$ be a continuous homomorphism. Let (X, A) be a G-equivariant relative CW complex and (Y, B) a G'-equivariant relative CW complex. A φ-map $f : (X, A) \to (Y, B)$, i.e. $f(gx) = \varphi(g)f(x)$ $x \in X$, $g \in G$, is called skeletal if $f((X, A)^k) \subset (Y, B)^k$, for all $k \geq 0$. Observe that if X and Y in fact are G-equivariant and G'-equivariant, respectively, CW complexes and A and B are subcomplexes then the above condition reads $f(X^k \cup A) \subset Y^k \cup B$, for all $k \geq 0$, and hence the absolute map $f : X \to Y$ need not itself be skeletal. This »freedom» in the definition of a skeletal map between pairs is in some cases extremely convenient. The following theorem is Theorem 2.14 in Chapter I of [10].

Theorem 1.4. Let (X, A) and (Y, B) be a G-equivariant and a G'-equivariant, respectively, relative CW complex. Assume that the φ-map $f : (X, A) \to (Y, B)$ is skeletal on the subcomplex $(X_0, X_0 \cap A)$ of (X, A). Then there exists a skeletal φ-map $\tilde{f} : (X, A) \to (Y, B)$ which is φ-homotopic rel. $(X_0 \cup A)$ to f.

(In fact the statement in [10] reads »rel. X_0» but the proof gives the »rel. $(X_0 \cup A)$» version). Taking $X_0 = \emptyset$ (or $X_0 = A$, it amounts to the same) we have the following.

Corollary 1.5. Any φ-map $f : (X, A) \to (Y, B)$ is φ-homotopic rel. A to a skeletal φ-map $\tilde{f} : (X, A) \to (Y, B)$.

Applying Theorem 1.4 to a φ-map $F : (X , A) \times I \to (Y , B)$ which is assumed to be skeletal on the subcomplex $(X \times \{0\} \cup X \times \{1\}, A \times \{0\} \cup A \times \{1\})$ we get

Collary 1.6. Let $F : (X , A) \times I \to (Y , B)$ be a φ-homotopy between the skeletal φ-maps $f_0 , f_1 : (X , A) \to (Y , B)$. Then there exists a skeletal φ-homotopy $\tilde{F} : (X , A) \times I \to (Y , B)$ between f_0 and f_1 such that $\tilde{F} \mid A \times I = F \mid A \times I$.

The following facts will be used frequently in this paper without further reference. If (X , X_0) is an equivariant CW pair and $f : X_0 \to Y$ is a skeletal G-map then the adjuction space $Y \cup_f X$ is an equivariant CW complex containing Y as a subcomplex. The mapping cylinder M_f of any skeletal G-map $f : X \to Y$ is an equivariant CW complex containing X and Y as subcomplexes (X in the $\{0\}$ end and Y in the $\{1\}$ end). It follows from Proposition 1.3 that f is a G-homotopy equivalence if and only if X is a strong G-deformation retract of M_f. The subcomplex Y is always a strong G-deformation retract of M_f.

Let $\varphi : G \to G'$ be a continuous homorphism. We shall now describe the process of »changing a G-space X into a G'-space, denoted by $\varphi(X)$, through the homomorphism φ». Let X be an arbitrary left G-space. Consider the space $G' \times X$ and define a *right* G-action $\Phi : (G' \times X) \times G \to G' \times X$ by $\Phi((g' , x) , g) = (g'\varphi(g) , g^{-1}x)$, where $g' \in G$, $g \in G$ and $x \in X$. We define

$$\varphi(X) = G' \times_{\varphi} X$$

to be the orbit space of $G' \times X$ under this right G-action. Let $\pi : G' \times X \to \varphi(X)$ be the natural projection and denote $\pi(g' , x) = [g' , x]$. Thus we have $[g'\varphi(g) , x] = [g' , gx]$ for every $g \in G$. Now define a left G'-action

$$\psi : G' \times \varphi(X) \to \varphi(X)$$

by $\psi (g' , [g_0' , x]) = [g'g_0' , x]$. This completes the construction of the G'-space $\varphi(X)$. We shall use the notation

$$\eta : X \to \varphi(X)$$

for the canonical φ-map defined $\eta(x) = [e , x]$, where $e \in G'$ is the identity element. Also observe that if $\varphi : G \to \{e\} = G'$ then $\varphi(X) = G \backslash X$, and in this case $\eta : X \to \varphi(X)$ is the natural projection onto the orbit space.

Any G-map $f : X \to Y$ induces a G'-map $\varphi(f) : \varphi(X) \to \varphi(Y)$ defined by $\varphi(f)([g' , x]) = [g' , f(x)]$. It is immediately seen that $\varphi(f)$ is a well-defined continuous G'-map. If $h : Y \to Z$ is another G-map we have

$\varphi(hf) = \varphi(h)\varphi(f)$. It follows that if f is a G-homeomorphism then $\varphi(f)$ is a G'-homeomorphism.

Observe that if K is a topological space with trivial G-action then we have $\varphi(X \times K) = \varphi(X) \times K$, as G'-spaces, given by $[g', x, k] \to ([g', x], k)$. Thus a G-homotopy $F : X \times I \to Y$ between two G-maps f_0 and f_1 induces a G'-homotopy $\varphi(F) : \varphi(X) \times I \to \varphi(Y)$ between the G'-maps $\varphi(f_0)$ and $\varphi(f_1)$. If f is a G-homotopy equivalence then $\varphi(f)$ is a G'-homotopy equivalence, and if A is a strong G-deformation retract of X then $\varphi(A)$ is a strong G'-deformation retract of $\varphi(X)$.

Also observe that if Y' is a G'-space and $f : X \to Y'$ is a φ-map then f induces a G'-map

$$\varphi(f) : \varphi(X) \to Y'$$

defined by $\varphi(f)([g', x]) = g'f(x)$. The fact that we use the same notation $\varphi(f)$ in these two slightly different contexts should not cause any confusion.

Now let H be any closed subgroup of G and consider the G-space G/H (standard left G-action). We claim that the map

$$\alpha : \varphi(G/H) \to G'/\varphi(H)$$

defined by $\alpha([g', gH]) = (g'\varphi(g))\varphi(H)$ is a G'-homeomorphism. Since $\alpha([g'\varphi(g_0), g_0^{-1}gH]) = (g'\varphi(g_0)\varphi(g_0)^{-1}\varphi(g))\varphi(H) = (g'\varphi(g))\varphi(H)$ it follows that α is a well-defined continuous map. Clearly α is a G'-map. The map $\beta : G'/\varphi(H) \to \varphi(G/H)$ defined by $\beta(g'\varphi(H)) = [g', eH]$ is also immediately seen to be a well-defined continuous G'-map. Since $\alpha\beta = \mathrm{id}$ and $\beta\alpha = \mathrm{id}$ this shows that α is a G'-homeomorphism. We identify $\varphi(G/H)$ with $G'/\varphi(H)$ through α. It follows that we have

$$\varphi(G/H \times I^n) = G'/\varphi(H) \times I^n .$$

Using this fact one easily proves the following

Proposition 1.7. If (X, A) is a G-equivariant relative CW complex then $(\varphi(X), \varphi(A))$ is a G'-equivariant relative CW complex.

If $f : (X, A) \to (Y, B)$ is skeletal then also $\varphi(f) : (\varphi(X), \varphi(A)) \to (\varphi(Y), \varphi(B))$ is skeletal.

Chapter II. Foundations of equivariant simple-homotopy theory

Recall that G denotes a topological group which is either a compact Lie group or a discrete group. By »equivariant» we mean »G-equivariant» if not otherwise is specified. Only when two groups G and G' are involved in the discussion at the same time shall we be more specific and speak about G-equivariant and G'-equivariant. From now on all equivariant CW complexes are automatically assumed to be *finite* equivariant CW complexes. Thus we shall write »equivariant CW complex (pair)» when we in fact mean »finite equivariant CW complex (pair)».

§ 1. Equivariant formal deformations

Definition 1.1. An inclusion $i : X \to Y$ of equivariant CW complexes is called an equivariant elementary expansion if the equivariant CW pair $(Y, i(X))$ satisfies the following conditions.

1) There is an integer $n \geq 1$ such that

$$Y = i(X) \cup b^{n-1} \cup b^n ,$$

where b^{n-1} and b^n denote an open equivariant $(n-1)$-cell and an open equivariant n-cell, respectively, of $Y - i(X)$.

2) There exists a closed subgroup H of G and a G-map

$$\sigma : G/H \times I^n \to Y$$

such that

(a) $\sigma(G/H \times J^{n-1}) \subset (i(X))^{n-1}$.

(b) $\sigma | : G/H \times I^{n-1} \to Y$ is a characteristic G-map for the equivariant $(n-1)$-cell $\bar{b}^{n-1}$.

(c) σ is a characteristic G-map for the equivariant n-cell $\bar{b}^n$.

By definition it follows that if $i : X \to Y$ is an equivariant elementary expansion and $h : X' \cong X$ is an isomorphism of equivariant CW complexes then also $ih : X' \to Y$ is an equivariant elementary expansion. Since we also define the identity map $id : X \to X$ to be an equivariant elementary expansion it follows that any isomorphism $h : X' \cong X$ of equivariant CW complexes is an equivariant elementary expansion. It is also immediately seen that if $i : X \to Y$ is an equivariant elementary expansion and $\bar{h} : Y \cong Y'$ is an isomorphism of equivariant CW complexes then $\bar{h}i : X \to Y'$ is an equivariant elementary expansion.

We shall in the following identify X with $i(X)$ and consider X itself as a subcomplex of Y. Hence we also use the terminology »Y is an equivariant elementary expansion of X» and denote $Y = X \cup b^{n-1} \cup b^n$. Observe that the open equivariant cells b^{n-1} and b^n have the same type and we call this for the type of the equivariant elementary expansion and the integer n for its dimension. Any G-map $\sigma : G/H \times I^n \to Y$ which satisfies the conditions in Definition 1.1 will be called a characteristic simple G-map for (b^n, b^{n-1}).

The conditions (a) — (c) in Definition 1.1 are equivalent to the four conditions that, $\sigma(G/H \times J^{n-1}) \subset X^{n-1}$, $\sigma(G/H \times \partial I^{n-1}) \subset X^{n-2}$, σ maps $G/H \times (I^{n-1} - \partial I^{n-1})$ G-homeomorphically onto b^{n-1}, and σ maps $G/H \times (I^n - \partial I^n)$ G-homeomorphically onto b^n. Thus Y is an equivariant elementary expansion of X if and only if Y is the adjunction space of $G/H \times I^n$, for some closed subgroup H of G, and X by a G-map $\varphi_+ : G/H \times J^{n-1} \to X^{n-1}$ which also satisfies $\varphi_+(G/H \times \partial I^{n-1}) \subset X^{n-2}$.

We use the terminology »X is an equivariant elementary collapse of Y» to mean exactly the same thing as »Y is an equivariant elementary expansion of X». Observe that a strong G-deformation retraction $\tilde{F} : (G/H \times I^n) \times I \to G/H \times I^n$ of $G/H \times I^n$ to $G/H \times J^{n-1}$ and a characteristic simple G-map $\sigma : G/H \times I^n \to Y$ for (b^n, b^{n-1}) together give rise to a strong G-deformation retraction $F : Y \times I \to Y$ of Y to X. Let $r : Y \to X$ denote the corresponding G-retraction. Thus the inclusion $i : X \to Y$ is a G-homotopy equivalence and r is a G-homotopy inverse to i. In fact any G-retraction from Y onto X is a G-homotopy inverse to the inclusion $i : X \to Y$ and any two such G-retractions are G-homotopic rel. X. Thus regardless of the different choices of $\tilde{F}$ and σ the G-retraction $r : Y \to X$ is uniquely determined up to G-homotopy rel. X. We call $r : Y \to X$ for an equivariant elementary collapse. A G-map which is either an equivariant elementary expansion or collapse is called an equivariant elementary deformation.

Let (V, X) and (W, X) be two equivariant CW pairs. We define an equivariant formal deformation from V to W rel. X to be a finite composite $k = k_p \ldots k_1$ of equivariant elementary deformations k_j,

$$V = X_0 \xrightarrow{k_1} X_1 \xrightarrow{k_2} \cdots \xrightarrow{k_p} X_p = W$$

where each X_j contains X as a subcomplex and $k_j | X = id$, $j = 1, \ldots, p$. (Here $k_j | X = id$ means that; if $i_j : X \to X_j$ is the inclusion representing X as a subcomplex of X_j then $k_j i_{j-1} = i_j$, for $j = 1, \ldots, p$.) Let $\bar{k}_j : X_j \to X_{j-1}$ be the equivariant elementary deformation inverse to k_j. Then $\bar{k} = \bar{k}_1 \ldots \bar{k}_p$ is a G-homotopy inverse, rel X, to k, and $\bar{k}$ is an equivariant formal deformation from W to V rel. X. We say that V and W have the same equivariant simple-homotopy type rel. X if and only if there exists an equivariant formal deformation from V to W rel. X. We denote this by

$$V \text{ s } W \text{ rel. } X ,$$

adding the word »equivariantly» when we want to be very specific. If each

$k_j : X_{j-1} \to X_j$ is an equivariant elementary collapse (expansion) we say that $k = k_p \dots k_1$ is an equivariant collapse (expansion) and we also express this by saying that V collapses (expands) equivariantly to W. Observe that in these special cases we in particular have $V \mathrm{s} W$ rel. W (and rel. V, respectively).

We define a G-map $f : V \to W$, where $f|X = \mathrm{id}$, to be an equivariant simple-homotopy equivalence rel. X if and only if f is G-homotopic rel. X to an equivariant formal deformation $k : V \to W$, which thus also is rel. X.

If $G = \{e\}$, the trivial group, these definitions reduce to the corresponding definitions in the ordinary »non-equivariant» case, see Section 13 in Whitehead [18].

An equivariant simple-homotopy equivalence $f : X \to Y$ induces an ordinary simple-homotopy equivalence $f' : G \backslash X \to G \backslash Y$ on the orbit spaces. But f need not be an equivariant simple-homotopy equivalence even if the induced map on the orbit spaces is an ordinary simple-homotopy equivalence. Consider the following simple example. Let Y be the two-sphere S^2 with $G = S^1$ acting by the standard »free» rotation leaving the south pole $\{S\}$ and the north pole $\{N\}$ fixed. The orbit space is a unit interval which collapses to $\{0\}$. But the inclusion $i : \{S\} \to Y$ is not an equivariant simple-homotopy equivalence. Of course in this example the G-map i is not even a G-homotopy equivalence. We shall give a better example later on.

Lemma 1.2. Let (V, X) be an equivariant CW pair such that $V \mathrm{s} X$ rel. X. Then both the inclusion $i : X \to V$ and any G-retraction $r : V \to X$ are equivariant simple-homotopy equivalences.

Proof. Since $X \mathrm{s} V$ rel. X the inclusion $i : X \to V$ is an equivariant formal deformation and hence an equivariant simple-homotopy equivalence. Since X is a strong G-deformation retract of V it follows that any G-retraction $r : V \to X$ is a G-homotopy inverse to i and hence also an equivariant simple-homotopy equivalence.

Lemma 1.3. Let $f : X \to Y$ be an equivariant simple-homotopy equivalence, and let K be any closed subgroup of G. Then $f| : GX^K \to GY^K$ is an equivariant simple-homotopy equivalence.

Proof. Let $B = A \cup b^{n-1} \cup b^n$ be an equivariant elementary expansion of A, of say type (H). Then $GB^K = GA^K \cup b^{n-1} \cup b^n$ if $(K) \leq (H)$ and $GB^K = GA^K$ if $(K) \not\leq (H)$, that is, GB^K is in either case an equivariant elementry expansion of GA^K. It follows that if $k : X \to Y$ is an equivariant formal deformation then so is also $k| : GX^K \to GY^K$. Since any G-homotopy from f to k restricts to a G-homotopy from $f|$ to $k|$ this shows that $f| : GX^K \to GY^K$ is an equivariant simple-homotopy equivalence.

Lemma 1.4. Let Y be a G-equivariant elementary expansion of X, and let $\varphi : G \to G'$ be a continuous homomorphism. Then $\varphi(Y)$ is a G'-equivariant elementary expansion of $\varphi(X)$.

Proof. Denote $Y = X \cup b^{n-1} \cup b^n$, and assume that the type of the equivariant elementary expansion is (H). Let $\sigma : G/H \times I^n \to Y$ be a characteristic simple G-map for (b^n, b^{n-1}). We have $\varphi(Y) = \varphi(X) \cup \varphi(b^{n-1}) \cup \varphi(b^n)$, and since $\varphi(G/H \times I^n) = G'/\varphi(H) \times I^n$ it is easily seen that $\varphi(\sigma) : G'/\varphi(H) \times I^n \to \varphi(Y)$ is a characteristic simple G'-map for $(\varphi(b^n), \varphi(b^{n-1}))$. Thus $\varphi(Y)$ is a G'-equivariant elementary expansion, of type $\varphi(H)$, of $\varphi(X)$.

Corollary 1.5. Let (V, X) and (W, X) be G-equivariant CW pairs such that $V \mathrm{s} W$ rel. X G-equivariantly. Then we have $\varphi(V) \mathrm{s} \varphi(W)$ rel. $\varphi(X)$ G'-equivariantly.

Both Lemma 1.6 and Corollary 1.7 below will be used frequently in the following. We shall call both of them for the »relativity principle».

Lemma 1.6. Assume that (V, X) and (W, X) are equivariant CW pairs such that $V \mathrm{s} W$ rel. X. Let $f : X \to Y$ be a skeletal G-map. Then $(Y \cup_f V) \mathrm{s} (Y \cup_f W)$ rel. Y.

Proof. Let $V = X_0 \to X_1 \to \ldots \to X_p = W$ be an equivariant formal deformation rel. X. Denote $Y_i = Y \cup_f X_i$, $i = 0, \ldots, p$. It is then immediately seen that $Y \cup_f V = Y_0 \to Y_1 \to \ldots \to Y_p = Y \cup_f W$ is an equivariant formal deformation rel. Y.

Observe in particular the special case of Lemma 1.6. when f is an inclusion. By Corollary 1.5 and Lemma 1.6 we have

Corollary 1.7. Let (V, X) and (W, X) be as in Corollary 1.5. and let $f : X \to Y'$ be a skeletal φ-map. Then we have $(Y' \cup_{\varphi(f)} \varphi(W)) \mathrm{s} (Y' \cup_{\varphi(f)} \varphi(V))$ rel. Y' G'-equivariantly.

The following lemma and its two corollaries will be used frequently. The »same» lemma in the ordinary non-equivariant case is Lemma 11 in Whitehead [18].

Lemma 1.8. Let $f : X \to Y$ be a skeletal G-map and let X_0 be a subcomplex of X. Then M_f collapses equivariantly to $M_{f|X_0}$.

Proof. Let A be a subcomplex of X such that $X = A \cup b$, where b is an open equivariant, say, n-cell of X. We claim that $M_{f|A}$ is an equivariant elementary collapse of M_f. Assume that the type of b is (H)

and let $\alpha : G/H \times I^n \to X$ be a characteristic G-map for $\bar{b}$. We have $M_f = M_{f|A} \cup (b \times \{0\}) \cup (b \times (0, 1))$ and $\pi(\alpha \times \mathrm{id}) : G/H \times I^n \times I \to M_f$, where $\pi : X \times I \to M^n$ denotes the restriction of the natural projection, is clearly a characteristic simple G-map for $(b \times (0, 1), b \times \{0\})$. This proves the above claim. Now let $X_0 \subset X_1 \subset \ldots \subset X_m = X$ be subcomplexes of X such that $X_i - X_{i-1}$ consists of exactly one open equivariant cell, for $i = 1, \ldots, m$. By what we just showed $M_{f|X_{i-1}}$ is an equivariant elementary collapse of $M_{f|X_i}$, $i = 1, \ldots, m$. This completes the proof of the lemma.

Corollary 1.9. Let $f : X \to Y$ be a skeletal G-map. Then M_f collapses equivariantly to Y.

Corollary 1.10. Let (X, X_0) be an equivariant CW pair. Then $X \times I$ collapses equivariantly to $X_0 \times I \cup X \times \{1\}$, and hence of course also to $X_0 \times I \cup X \times \{0\}$.

§ 2. The equivariant Whitehead group $\mathrm{Wh}_G(X)$

Let (V, X) be an equivariant CW pair such that X is a strong G-deformation retract of V. (By Proposition 1.3 in Chapter I this is equivalent to the fact that the inclusion $i : X \to V$ is a G-homotopy equivalence.) Let (W, X) be another such pair. Define a relation $\sim$ by

$$(V, X) \sim (W, X) \Leftrightarrow V \text{ s } W \text{ rel. } X \text{ equivariantly}.$$

This is an equivalence relation. Since $(V, X) \sim (W, X)$ if $(V, X) \cong (W, X)$, where $\cong$ stands for an isomorphism of equivariant CW complexes which is the identity on X, it is easy to see that the equivalence classes with respect to the relation $\sim$ form a set. We denote this set by $Wh_G(X)$. Let $s(V, X)$ denote the equivalence class determined by (V, X). Now define an addition in $Wh_G(X)$ by

$$s(V_1, X) + s(V_2, X) = s(V_1 \cup_X V_2, X).$$

Since X is a strong G-deformation retract of both V_1 and V_2 it follows that X is a strong G-deformation retract of $V_1 \cup_X V_2$. Thus $s(V_1 \cup_X V_2, X) \in Wh_G(X)$ is defined. This addition is well-defined. If $(V_1, X) \sim (W_1, X)$ and $(V_2, X) \sim (W_2, X)$ then it follows from the relativity principle, Proposition 1.6, that we have $(V_1 \cup_X V_2, X) \sim (V_1 \cup_X W_2, X) \sim (W_1 \cup_X W_2, X)$. Clearly this addition is associative and commutative and the element $s(X, X) \in Wh_G(X)$ is a zero element. We shall shortly show that every element in $Wh_G(X)$ has an inverse, i.e., $Wh_G(X)$ is an abelian group. But first we establish some other results.

Let $f : X \to Y'$ be a skeletal φ-map, where $\varphi : G \to G'$ is a continuous homomorphism and Y' is a G'-equivariant CW complex. We define

$$f_* : Wh_G(X) \to Wh_{G'}(Y')$$

as follows. If $s(V, X) \in Wh_G(X)$ then we set

$$f_* s(V, X) = s(Y' \cup_{\varphi(f)} \varphi(V), Y'),$$

where $\varphi(f) : \varphi(X) \to Y'$ denotes the G'-map induced by the φ-map $f : X \to Y'$. It is easily seen that this definition makes sense and it follows from the relativity principle, Corollary 1.7, that it is well-defined. Clearly f_* is additive and takes the zero element into the zero element, i.e., f_* is a homomorphism between abelian semi-groups with zero element. If $h : Y' \to Z''$ is a φ'-map, where $\varphi' : G' \to G''$ is a continuous homomorphism, then we have $(hf)_* = h_* f_* : Wh_G(X) \to Wh_{G''}(Z'')$. In particular the canonical φ-map $\eta : X \to \varphi(X)$ induces a homomorphism which we shall denote by φ_* instead of η_*. Thus we have

$$\varphi_* : Wh_G(X) \to Wh_{G'}(\varphi(X)),$$

for any continuous homomorphism $\varphi : G \to G'$, and φ_* is defined by $\varphi_* s(V, X) = s(\varphi(V), \varphi(X))$.

Lemma 2.1. Let $f_0, f_1 : X \to Y'$ be skeletal φ-maps which are φ-homotopic. Then $(f_0)_* = (f_1)_* : Wh_G(X) \to Wh_{G'}(Y')$.

Proof. Let $F : X \times I \to Y$ be a φ-homotopy from f_0 to f_1. By the equivariant skeletal approximation theorem (Corollary 1.6 in Chapter I) we can assume that F is skeletal. Thus F induces

$$F_* : Wh_G(X \times I) \to Wh_{G'}(Y').$$

Now consider the inclusions $i_0, i_1 : X \to X \times I$, defined by $i_k = (x, k)$, $k = 0,1$. Let $s(V, X) \in Wh_G(X)$. Using Corollary 1.10 we then have $(i_0)_* s(V, X) = s(X \times I \cup V \times \{0\}, X \times I) = s(V \times I, X \times I) = s(X \times I \cup V \times \{1\}, X \times I) = (i_1)_* s(V, X)$. Thus $(i_0)_* = (i_1)_* : Wh_G(X) \to Wh_G(X \times I)$. Hence we have $(f_0)_* = F_*(i_0)_* = F_*(i_1)_* = (f_1)_*$.

Lemma 2.2. Let $X \subset V \subset W$, such that X is a strong G-deformation retract of V and V is a strong G-deformation retract of W. Then we have

$$s(W, X) = r_* s(W, V) + s(V, X),$$

where $r : V \to X$ is a skeletal G-retraction.

Proof. First we observe that it follows that X is a strong G-deformation retract of W and hence $s(W, X) \in Wh_G(X)$ is defined. Let $i : X \to V$ be the inclusion and $r : V \to X$ a skeletal G-retraction. Then $ir : V \to V$

is G-homotopic (in fact rel. X) to id_V. Thus by Lemma 2.1 we have $(ir)_* s(W, V) = s(W, V)$. This means that $V \cup_{ir} W \, s \, W$ rel. V and hence in particular rel. X. Thus $s(V \cup_{ir} W, X) = s(W, X)$ and since $s(V \cup_{ir} W, X) = s(V, X) + s(X \cup_r W, X) = s(V, X) + r_* s(W, V)$ the lemma follows.

We are now ready to prove that every element in $Wh_G(X)$ has an inverse. Let $s(V, X) \in Wh_G(X)$ and let $r: V \to X$ be a skeletal G-retraction. By Lemma 1.8. the mapping cylinder M_r collapses equivariantly to $X \times I$, and thus $M_r \, s \, X \times I$ rel. $X \times I$. Let $\pi: X \times I \to X$ denote the projection and define $\bar{M}_r = X \cup_\pi M_r$. By the relativity principle, Lemma 1.6, we have $\bar{M}_r \, s \, X$ rel. X, and hence $s(\bar{M}_r, X) = 0 \in Wh_G(X)$. Now $X \subset V \subset \bar{M}_r$, and since $X \times I \cup V \times \{0\}$ is a strong G-deformation retract of M_r it follows that V is a strong G-deformation retract of $\bar{M}_r$. Thus Lemma 2.2 applies and gives us

$$0 = r_* s(\bar{M}_r, V) + s(V, X),$$

that is, $r_* s(\bar{M}_r, V) \in Wh_G(X)$ is an inverse to $s(V, X)$. We have proved

Theorem 2.3. For every G-equivariant CW complex X $Wh_G(X)$ is an abelian group. A φ-map $f: X \to Y'$ induces a homomorphism $f_*: Wh_G(X) \to Wh_{G'}(Y')$ and any two φ-homotopic φ-maps induce the same homomorphism.

We call $Wh_G(X)$ for the equivariant Whitehead group of X. If $G = \{e\}$, the trivial group, and X hence denotes an ordinary CW complex it is clear from the way we have defined our $Wh_G(X)$ that $Wh_{\{e\}}(X) = Wh(X)$. Here $Wh(X)$ is the Whitehead group of the CW complex X as defined in Cohen [6], Eckmann-Maumary [7], and Stöcker [15]. It is also well-known that this ordinary Whitehead group $Wh(X)$, for connected X, is isomorphic to $Wh(\pi_1(X))$, the (algebraicly defined) Whitehead group of the group $\pi_1(X)$. For the proof of this see Stöcker [15] or Cohen [6].

The following type of »sum theorem» has played an important role in the ordinary simple-homotopy theory. The proof of our equivariant version of the sum theorem is exactly the »same» as the one given by Stöcker [15] for the ordinary case.

Proposition 2.4. Let (W, X) be an equivariant CW pair such that $W = W_1 \cup W_2$ and $W_0 = W_1 \cap W_2$. Denote $X_k = X \cap W_k$ and assume that X_k is a strong G-deformation retract of W_k, $k = 0, 1, 2$. Then we have

$$s(W, X) = (i_1)_* s(W_1, X_1) + (i_2)_* s(W_2, X_2) - (i_0)_* (W_0, X_0),$$

where $i_k: X_k \to X$, $k = 0, 1, 2$, denote the inclusions.

Proof. We have $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Denote $V_k = X \cup W_k$, $k = 0, 1, 2$. Clearly X is a strong G-deformation retraction of V_k, for $k = 0, 1, 2$. Since $X \subset V_0 \subset V_j$, $j = 1, 2$, it follows that also the inclusions $V_0 \to V_j$, $j = 1, 2$, are G-homotopy equivalences and hence by Proposition 1.3 in Chapter I V_0 is a strong G-deformation retract of V_j, $j = 1, 2$. Since $W = V_1 \cup V_2$ and $V_1 \cap V_2 = V_0$ we have

$$s(W, V_0) = s(V_1, V_0) + s(V_2, V_0)$$

by the definition of the sum in $Wh_G(V_0)$. Let $r : V_0 \to X$ be a retraction. Then we have by Lemma 2.2

$$s(W, X) = r_* s(W, V_0) + s(V_0, X)$$

$$s(V_j, X) = r_* s(V_j, V_0) + s(V_0, X), \quad j = 1, 2.$$

From the above four formulas we get

$$s(W, X) - s(V_1, X) - s(V_2, X) = - s(V_0, X)$$

and this is the claimed formula since $(i_k)_* s(W_k, X_k) = s(V_k, X)$, $k = 0, 1, 2$.

We shall now study the case when G acts freely on X and prove that $\varphi_* : Wh_G(X) \to Wh(G \backslash X)$ is an isomorphism. Recall that if $\varphi : G \to \{e\}$ then $\varphi(X) = G \backslash X$ and hence this φ gives us $\varphi_* : Wh_G(X) \to Wh(G \backslash X)$. We first give two lemmas.

Lemma 2.5. Let (Y, X) be an equivariant CW pair such that G acts freely on Y. Assume that $G \backslash X$ is an elementary collapse (in the ordinary sense) of $G \backslash Y$. Then X is an equivariant elementary collapse of Y.

Proof. Let us denote $(Y', X') = (G \backslash Y, G \backslash X)$ and let $p : Y \to Y'$ denote the projection onto the orbit space. By the assumption we have $Y' = X' \cup e^{n-1} \cup e^n$, where e^{n-1} and e^n denote an open $(n-1)$-cell and n-cell, respectively, of $Y' - X'$. Moreover there is a map $\sigma' : I^n \to Y'$ such that $\sigma'(J^{n-1}) \subset (X')^{n-1}$ and $\sigma'(\partial I^{n-1}) \subset (X')^{n-2}$, and furthermore σ' maps $I^{n-1} - \partial I^{n-1}$ homeomorphically onto e^{n-1} and $I^n - \partial I^n$ homeomorphically onto e^n. Since $p : Y \to Y'$ is a principal G-bundle and I^n is contractible it follows that σ' can be lifted, that is, there exists a map $\tilde{\sigma} : I^n \to Y$ such that $p\tilde{\sigma} = \sigma'$. Now define $\sigma : G \times I^n \to Y$ by $\sigma(g, x) = g\tilde{\sigma}(x)$. Then σ is a G-map such that $\sigma(G \times J^{n-1}) \subset X^{n-1}$ and $\sigma(G \times \partial I^{n-1}) \subset X^{n-2}$, and moreover σ maps $G \times (I^{n-1} - \partial I^{n-1})$ G-homeomorphically onto $p^{-1}(e^{n-1})$ and $G \times (I^n - \partial I^n)$ G-homeomorphically onto $p^{-1}(e^n)$.

Since $Y = X \cup p^{-1}(e^{n-1}) \cup p^{-1}(e^n)$ this shows that X is an equivariant elementary collapse of Y.

Lemma 2.6. Let X be an equivariant CW complex such that G acts freely on X. Let Y' be an elementary expansion in the ordinary sense of $G\backslash X$. Then there exists an equivariant elementary expansion Y of X such that $G\backslash Y = Y'$.

Proof. Denote $X' = G\backslash X$. By the assumption Y' is the adjunction space of X' and I^n by some map $\varphi'_+ : J^{n-1} \to (X')^{n-1}$ which also satisfies $\varphi'_+(\partial I^{n-1}) \subset (X')^{n-2}$, that is, $Y' = X' \cup_{\varphi'_+} I^n$. Since $p : X \to X'$ is a principal G-bundle and J^{n-1} is contractible there exists a lifting $\tilde{\sigma}_+ : I^n \to X^{n-1} \subset X$ of σ'_+. Let $\sigma_+ : G \times I^n \to X^{n-1}$ be defined by $\sigma_+(g, a) = g\sigma_+(a)$. Then we also have $\sigma_+(G \times \partial I^{n-1}) \subset X^{n-2}$. Thus $Y = X \cup_{\varphi_+} (G \times I^n)$ satisfies the conclusion of the lemma.

Theorem 2.7. Let X be an equivariant CW complex such that G acts freely on X. Then $\varphi_* : Wh_G(X) \to Wh(G\backslash X)$ is an isomorphism.

Proof. Let $s(V, X) \in Wh_G(X)$ and assume that $\varphi_* s(V, X) = (G\backslash V, G\backslash X) = 0 \in Wh(G\backslash X)$. This means that $G\backslash V \, s \, G\backslash X$ rel. $G\backslash X$. By Lemma 2.5 and 2.6 it follows that $V \, s \, X$ rel. X equivariantly, that is, $s(V, X) = 0$. Thus φ_* is a monomorphism. Now let $s(V', G\backslash X) \in Wh(G\backslash X)$. Let $r^*(p) : V \to V'$ be the principal G-bundle induced from $p : X \to G\backslash X$ by a retraction $r : V' \to G\backslash X$. Then V is an equivariant CW complex on which G acts freely and V contains X as a subcomplex. Since the inclusion $\tilde{i} : X \to V$ induces isomorphisms on all homotopy groups it follows by Corollary 5.5 in Bredon [4] in case G is a discrete group and by Proposition 2.5 in [9] in case G is a compact Lie group, that $\tilde{i}$ is a G-homotopy equivalence. Thus, by Proposition 1.3 in Chapter I, X is a strong G-deformation retract of V, and the equivariant CW pair (V, X) determines an element $s(V, X) \in Wh_G(X)$. Since $\varphi_* s(V, X) = (V', G\backslash X)$ this shows that φ_* is an epimorphism.

Let G/G_0 be the group of components of G, i.e. G_0 denotes the identity component of G. We now have

Corollary 2.8. Let X be as in Theorem 2.7 and assume moreover that X is simply connected. Then $Wh_G(X) \cong Wh(G/G_0)$.

Proof. It follows from the exact homotopy sequence of the fibration $p : X \to G\backslash X$ that $\pi_1(G\backslash X) = G/G_0$. Hence the corollary follows from Theorem 2.7 and the well-known isomorphism $Wh(G\backslash X) \cong Wh(\pi_1(G\backslash X))$, see Section 3 in Stöcker [15] or § 21 in Cohen [6].

§ 3. Geometric equivariant Whitehead torsion of a G-homotopy equivalence

In this section we first give a characterization of an equivariant simple-homotopy equivalence in terms of its mapping cylinder, see Theorem 3.6 below. The proofs of this result and of those preceeding it are completely analogous to the ones in the ordinary »non-equivariant» case, see (5.4)—(5.8) in Cohen [6] or § 3 in Eckmann-Maumary [7]. Then we define the (geometric) equivariant Whitehead torsion of a G-homotopy equivalence $f: X \to Y$. It is an element of $Wh_G(X)$, (see the remarks below concerning this choice). We denote it by $\tau_g(f)$, and we have $\tau_g(f) = 0$ if and only if f is an equivariant simple-homotopy equivalence. Recently Chapman [5] has given an affirmative answer to the outstanding problem concerning the topological invariance of ordinary Whitehead torsion for finite CW complexes. From this result it follows immediately, by Theorem 2.7, that if X and Y are equivariant CW complexes such that G-acts *freely* then any G-homeomorphism $f: X \to Y$ is an equivariant simple-homotopy equivalence.

Lemma 3.1. Assume that V collapses equivariantly to X. Let $f: V \to Y$ be a skeletal G-map. Then M_f collapses equivariantly to $V \cup M_{f|X}$, where $f|X : X \to Y$.

Proof. Let $V = X_m \to X_{m-1} \to \ldots \to X_0 = X$ be a sequence of equivariant elementary collapses. Denote $W_j = V \cup M_{f|X_j}$. We claim that W_{j-1} is an equivariant elementary collapse of W_j, $1 \leq j \leq m$. This is seen as follows. Assume that the equivariant elementary collapse from X_j to X_{j-1} is of type (H_j), and denote $X_j = X_{j-1} \cup b^{n-1} \cup b^n$, and let $\sigma : G/H_j \times I^n \to X_j$ be a characteristic simple G-map for (b^n, b^{n-1}). Then we have $W_j = W_{j-1} \cup (b^{n-1} \times (0, 1)) \cup (b^n \times (0, 1))$ and $\pi(\sigma \times id) : G/H_j \times I^n \times I \to W_j$, where $\pi : X_j \times I \to W_j$ denotes the restriction of the natural projection, is clearly a characteristic simple G-map for $(b^n \times (0, 1), b^{n-1} \times (0, 1))$. (The G-map $\pi(\sigma \times id)$ restricted to $G/H_j \times (I^{n-1} \times \{0\} \times I)$ gives a characteristic G-map for $b^{n-1} \times (0, 1)$).

Lemma 3.2. Let Y be an equivariant elementary expansion of X.
a) Let $i: X \to Y$ be the inclusion. Then M_i collapses equivariantly to X.
b) Let $r: Y \to X$ be a skeletal G-retraction. Then M_r collapses equivariantly to Y.

Proof. a) We have $M_i = X \times I \cup Y \times \{1\}$ which collapses equivariantly to $X \times I$ which again collapses equivariantly to $X \times \{0\}$.

b) By Lemma 3.1 M_r collapses equivariantly to $Y \times \{0\} \cup M_{r|X} = Y \times \{0\} \cup X \times I$ which in turn collapses equivariantly to $Y \times \{0\}$.

Lemma 3.3. Let $f : X \to Y$ and $h : Y \to Z$ be skeletal G-maps. Then $M_f \cup_Y M_h \mathrm{s} M_{hf}$ rel. $X \dot{\cup} Z$.

Proof. Define $F = hp : M_f \to Z$, where $p : M_f \to Y$ is the natural projection. Then F is skeletal. Consider the mapping cylinder M_F. First observe that by Lemma 1.8 M_F collapses equivariantly to $M_{F|X} = M_{hf}$. Secondly observe that since M_f collapses equivariantly to Y (by Corollary 1.9) and since F restricted to Y equals $h : Y \to Z$, it follows by Lemma 3.1 that M_F collapses equivariantly to $M_f \cup_Y M_h$. The claim now follows from these two facts.

By repeated use of Lemma 3.3 and the relativity principle we get.

Corollary 3.4. Let $X_0 \xrightarrow{f_1} X_1 \to \ldots \xrightarrow{f_p} X_p$ be a sequence of skeletal G-maps, and denote $f = f_p \ldots f_1$. Then $M_f \mathrm{s} M_{f_1} \cup_{X_1} M_{f_2} \cup \ldots \cup_{X_{p-1}} M_{f_p}$.

Proposition 3.5. Let $f_0, f_1 : X \to Y$ be G-homotopic skeletal G-maps. Then $M_{f_0} \mathrm{s} M_{f_1}$ rel. $X \dot{\cup} Y$. Thus, if f_0 and f_1 furthermore are G-homotopy equivalences, we have $s(M_{f_0}, X) = s(M_{f_1}, X) \in Wh_G(X)$.

Proof. Let $F : X \times I \to Y$ be a G-homotopy from f_0 to f_1. By the equivariant skeletal approximation theorem, Corollary 1.6 in Chapter I, we can assume that F is skeletal. Since $X \times I$ collapses equivariantly to $X \times \{0\}$ and to $X \times \{1\}$ it follows by Proposition 3.1 that we in particular have

$$X \times I \cup M_{f_0} \mathrm{s} M_F \mathrm{s} X \times I \cup M_{f_1} \text{ rel. } X \times I \dot{\cup} Y .$$

Let $q : X \times I \dot{\cup} Y \to X \dot{\cup} Y$ be the map defined by $q(x, t) = x$ and $q(y) = y$, and denote $\bar{M}_F = (X \dot{\cup} Y) \cup_q M_F$. By the relativity principle, Lemma 1.6, we then have $M_{f_0} \mathrm{s} \bar{M}_F \mathrm{s} M_{f_1}$ rel. $X \dot{\cup} Y$.

Theorem 3.6. Let $f : X \to Y$ be a G-map. Then the following three statements are equivalent:

(a) f is an equivariant simple-homotopy equivalence.

(b) There exists an equivariant skeletal approximation $\hat{f}$ to f such that $M_{\hat{f}} \mathrm{s} X$ rel. X.

(c) For any equivariant skeletal approximation $\tilde{f}$ to f we have $M_{\tilde{f}} \mathrm{s} X$ rel. X.

Proof. By Proposition 3.5 (and the fact that equivariant skeletal approximations exist) statements (b) and (c) are equivalent. We shall show that (a) and (b) are equivalent. Assume that $f : X \to Y$ is an equivariant simple-homotopy equivalence. This means, by definition, that f is G-homotopic to an equivariant formal deformation $k = k_p \dots k_1 : X = X_0 \to X_1 \to \dots \to X_p = Y$. Then k is an equivariant skeletal approximation to f, and by Corollary 3.4 and repeated use of Lemma 3.2 a) and b) we have $M_k \operatorname{s} X$ rel. X.

It remains to prove that (b) implies (a). Let $\hat{f} : X \to Y$ be an equivariant skeletal approximation to f such that $M_{\hat{f}} \operatorname{s} X$ rel. X. We have $\hat{f} : pi : X \to M_{\hat{f}} \to Y$, where i denotes the inclusion and p is the natural projection. By Lemma 1.2 i is an equivariant simple-homotopy equivalence. By Corollary 1.9 and Lemma 1.2 p is an equivariant simple-homotopy equivalence. Since f is G-homotopic to $\hat{f} = pi$ it follows that f is an equivariant simple-homotopy equivalence.

Now let $f : X \to Y$ be a skeletal G-homotopy equivalence. We define

$$\tau_g(f) = s(M_f, X) \in Wh_G(X) .$$

(Here the »g» in τ_g stands for »geometric».) We call $\tau_g(f)$ for the (geometric) equivariant Whitehead torsion of f. If $f_0, f_1 : X \to Y$ are G-homotopic skeletal G-homotopy equivalences then, by Proposition 3.5, we have $\tau_g(f_0) = \tau_g(f_1)$.

Thus we can extend the above definition to any G-homotopy equivalence $f : X \to Y$ by defining

$$\tau_g(f) = \tau_g(\hat{f})$$

where $\hat{f}$ is any equivariant skeletal approximation to f. We can now reformulate Theorem 3.6. in the following form.

Theorem 3.6′. A G-homotopy equivalence $f : X \to Y$ is an equivariant simple-homotopy equivalence if and only if $\tau_g(f) = 0$.

In the »classical» ordinary case the question whether a homeomorphism between two CW complexes is a simple-homotopy equivalence has until very recently been an open problem ever since Whitehead posed the question (see the Introduction in Whitehead [18]). An affirmative answer to this »classical» question has now been given by Chapman [5]. He has proved the following theorem.

Theorem. (Chapman). Let K, L be finite connected CW complexes and let $f : |K| \to |L|$ be a map. Then f is a simple-homotopy equivalence if and only if the map

$$f \times \mathrm{id} : |K| \times Q \to |L| \times Q$$

is homotopic to a homeomorphism of $|K| \times Q$ onto $|L| \times Q$.

(We have used Chapmans notation.) Here Q denotes the Hilbert cube. The »only if» part is due to West [16].

In the case when G is assumed to act *freely* we have

Theorem 3.7. Let X and Y be equivariant CW complexes such that G acts freely on X and Y. Then any G-homeomorphism $f : X \to Y$ is an equivariant simple-homotopy equivalence.

Proof. Let $f : X \to Y$ be a G-homeomorphism. Denote the induced map on the orbit spaces by $f' : G \backslash X \to G \backslash Y$. Then f' is a homeomorphism and hence by Chapmans result a simple-homotopy equivalence. (Observe that we have not assumed that $G \backslash X$ and $G \backslash Y$ are connected, but of course the conclusion still holds.) Thus $\tau_g(f') = 0 \in Wh(G \backslash X)$. By Theorem 2.7 φ_*: $Wh_G(X) \to Wh(G \backslash X)$ is an isomorphism. Since $M_{f'} = G \backslash M_f$ it follows that we have $\varphi_*(\tau_g(f)) = \tau_g(f') = 0$. Thus $\tau_g(f) = 0$ and hence f is an equivariant simple-homotopy equivalence by Theorem 3.6′.

Proposition 3.8. Let $f : X \to Y$ and $h : Y \to Z$ be G-homotopy equivalences. Then we have

$$\tau_g(hf) = \tau_g(f) + f_*^{-1}\tau_g(h) \, .$$

Proof. By Lemma 3.3. we have $\tau_g(hf) = s(M_{hf} , X) = s(M_f \cup_Y M_h , X)$. Consider the inclusions $X \subset M_f \subset M_f \cup_Y M_h$ and let $r : M_f \to X$ be a G-retraction. By Lemma 2.2 we have

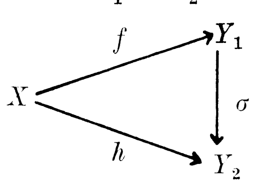
$$\begin{aligned} s(M_f \cup_Y M_h , X) &= r_* s(M_f \cup_Y M_h , M_f) + s(M_f , X) \\ &= r_* j_* s(M_h , Y) + s(M_f , X) \, , \end{aligned}$$

where $j : Y \to M_f$ denotes the inclusion. But since $r_j : Y \to M_f \to X$ is a G-homotopy inverse to f we have $r_* j_* = f_*^{-1}$. This completes the proof.

Corollary 3.9. Let $\bar{f} : Y \to X$ be a G-homotopy inverse to $f : X \to Y$. Then $\tau_g(\bar{f}) = - f_* \tau_g(f)$.

Proof. We have $0 = \tau_g(\bar{f}f) = \tau_g(f) + f_*^{-1}\tau_g(\bar{f})$, and hence $\tau(\bar{f}) = - f_* \tau_g(f)$.

Corollary 3.10. Let $f : X \to Y_1$ and $h : X \to Y_2$ be G-homotopy equivalences. Then $\tau_g(f) = \tau_g(h)$ if and only if there exists an equivariant simple-homotopy equivalence $\sigma : Y_1 \to Y_2$ such that the diagram



is G-homotopy commutative.

Proof. Assume that such a σ exists. Then $\tau_g(\sigma) = 0$, and hence $\tau_g(h) = \tau_g(\sigma f) = \tau_g(f) + f_*^{-1}\tau_g(\sigma) = \tau_g(f)$.

Now assume on the other hand that $\tau_g(f) = \tau_g(h)$. Define $\sigma = h\bar{f} : Y_1 \to Y_2$, where $\bar{f} : Y_1 \to X$ is a G-homotopy inverse to f. Then

$$\tau_g(\sigma) = \tau_g(h\bar{f}) = \tau_g(\bar{f}) + (\bar{f})_*^{-1}\tau_g(h) = -f_*\tau_g(f) + f_*\tau_g(h) = 0$$

and hence σ is an equivariant simple-homotopy equivalence.

Lemma 3.11. Let (V, X) be an equivariant CW pair such that the inclusion $i : X \to V$ is a G-homotopy equivalence. Then we have $\tau_g(i) = s(V, X) \in Wh_G(X)$.

Proof. By definition we have

$$\tau_g(i) = s(M_i, X) = s(X \times I \cup V \times \{1\}, X \times \{0\}) .$$

It follows from Corollary 1.10 that $(X \times I \cup V \times \{1\})\, s\, V \times I\, s\, V \times \{0\}$ rel. $X \times \{0\}$, and hence $s(X \times I \cup V \times \{1\}, X \times \{0\}) = s(V, X)$.

It should be observed that we could as well have defined the geometric equivariant Whitehead torsion of a G-homotopy equivalence $f : X \to Y$ to be $\bar{\tau}_g(f) = f_* s(M_f, X) \in Wh_G(Y)$. Since f_* is an isomorphism $\bar{\tau}_g(f) = 0$ if and only if $\tau_g(f) = 0$. In fact we have by Corollary 3.9 that $\bar{\tau}_g(f) = -\tau_g(\bar{f})$, where $\bar{f} : Y \to X$ is a G-homotopy inverse to f. Taking $\bar{\tau}_g(f)$ as the definition would be in complete analogy with the definition given in Cohen [6] in the standard non-equivariant case. Our choice is by Proposition 3.8 in agreement with the point of view taken in Eckmann-Maumary [7], see 2.2 in [7].

We conclude this section by observing that a restatement of the »sum theorem», i.e. Proposition 2.4., gives us the following important result (compare again with Stöcker [15] and Cohen [6]).

Theorem 3.12. Assume that $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$. Let $f : X \to Y$ be a G-map which restricts to G-homotopy equivalences $f_k : X_k \to Y_k$ for $k = 0, 1, 2$. Then f is a G-homotopy equivalence and

$$\tau_g(f) = (i_1)_*\tau_g(f_1) + (i_2)_*\tau_g(f_2) - (i_0)_*\tau_g(f_0) ,$$

where $i_k : X_k \to X$, $k = 0, 1, 2$, denote the inclusions. In particular if each f_k, $k = 0, 1, 2$, is an equivariant simple-homotopy equivalence so is f.

§ 4. Simplified form

We shall in this section show that every element in $Wh_G(X)$ can be represented by a pair (W, X) which is in simplified from. This means that the equivariant cells of $W - X$ are concentrated in two consecutive dimensions $n - 1$ and n, where $n - 1 \geq 2$, plus some further purely technical conditions on the attaching G-maps for these equivariant cells. As the case is in the ordinary non-equivariant theory the »simplified form» result is the clue to the transition from the geometric side of the theory to the algebraic side. We shall study this transition for our equivariant Whitehead theory in Chapter III in the two cases that G is either a discrete abelian group or G is a torus group T^n, $n \geq 1$. For Whiteheads original treatment of »simplified form» in the ordinary non-equivariant case see Lemmas 13—15 in Whitehead [18]. See also (7.4) in Cohen [6].

Lemma 4.1. Let X be an equivariant CW complex and let

$$f_0, f_1 : G/H \times \partial I^n \to X$$

be G-homotopic G-maps such that $f_i(G/H \times \partial I^n) \subset X^{n-1}$, $i = 0, 1$. Let $Y_i = X \cup_{f_i} (G/H \times I^n)$, the equivariant CW complex obtained by adjoining $G/H \times I^n$ to X by f_i, $i = 0, 1$. Then Y_0 s Y_1 rel. X.

Proof. Let $F : (G/H \times \partial I^n \times I, G/H \times \partial I^n \times \{0, 1\}) \to (X, X^{n-1})$ be a G-homotopy from f_0 to f_1. It follows from the equivariant skeletal approximation theorem, Corollary 1.5 in Chapter I, that we can assume that F is skeletal, i.e. that $F(G/H \times \partial I^n \times I) \subset X^n \cup X^{n-1} = X^n$. Define $Y = X \cup_F (G/H \times I^n \times I)$, the equivariant CW complex obtained by adjoining $G/H \times I^n \times I$ to X by $F : G/H \times \partial I^n \times I \to X^n \to Y$. Since the obvious inclusions $i_0 : Y_0 \to Y$ and $i_1 \to Y$ both are equivariant elementary expansions the lemma follows.

Let (V, V_0) be an equivariant CW pair. By $\alpha_n(H)(V - V_0)$ we denote the number of equivariant n-cells of type (H) in $V - V_0$.

Lemma 4.2. Let (V_0, X) and (W_0, X) be equivariant CW pairs and let $k : V_0 \to W_0$ be an equivariant formal deformation rel. X. Let V be an equivariant CW complex containing V_0 as a subcomplex. Then there exists an equivariant CW complex W containing W_0 as a subcomplex, such that $\alpha_n(H)(W - W_0) = \alpha_n(H)(V - V_0)$ for every n and (H), and an equivariant formal deformation $\tilde{k} : V \to W$ rel. X.

Proof. By induction on the number of equivariant cells in $V - V_0$ and on the number of equivariant elementary deformations in k it follows that it is enough to prove the lemma in the case when $V - V_0$ consists of one equivariant cell and $k: V_0 \to W_0$ is an equivariant elementary deformation.

If $k: V_0 \to W_0$ is an equivariant elementary expansion then its natural extension $\tilde{k}: V \to W_0 \cup_k V = W_0 \cup_{V_0} V$ is also an equivariant elementary expansion and $W = W_0 \cup_{V_0} V$ is of the required form.

Now let $k: V_0 \to W_0$ be an equivariant elementary collapse. Assume that the type of the open equivariant cell $e^p = V - V_0$ is (H) and let $f: G/H \times \partial I^p \to V_0^{p-1} \to V_0$ be an attaching G-map for $\bar{e}^p$. Thus $V = V_0 \cup_f (G/H \times I^p)$. The G-map $kf: G/H \times \partial I^p \to W_0^{p-1} \to V_0$ is G-homotopic to f and hence by Lemma 4.1 there exists an equivariant formal deformation $h: V \to V_0 \cup_{kf} (G/H \times I^p)$ rel. V_0. The natural extension $\hat{k}: V_0 \cup_{kf} (G/H \times I^p) \to W_0 \cup_{kf} (G/H \times I^p)$ of $k: V_0 \to W_0$ is an equivariant elementary collapse. Thus $\tilde{k} = \hat{k}h: V \to W_0 \cup_{kf} (G/H \times I^p)$ is an equivariant formal deformation rel. X and $W = W_0 \cup_{kf} (G/H \times I^p)$ is of the required form.

Recall from [10] (see Definition 1.8 in Chapter I of [10]) that we say that a G-pair (Y, B) satisfies condition π_m, $m \geq 0$, if for any closed subgroup H of G every G-map $f: (G/H \times I^m, G/H \times \partial I^m) \to (Y, B)$ is G-homotopic rel. $G/H \times \partial I^m$ to a G-map from $G/H \times I^m$ into B.

Lemma 4.3. Let (V, X) be an equivariant CW pair which satisfies condition π_m, for some $m \geq 0$, and assume that

$$V = X \cup \bigcup b_i^m \cup \bigcup b_i^{m+1} \cup \ldots \cup \bigcup b_i^r .$$

Then there exists an equivariant CW complex W of the form

$$W = X \cup \bigcup e_j^{m+1} \cup \bigcup e_j^{m+2} \cup \ldots \cup \bigcup e_j^{\max(r,m+2)}$$

such that $V \mathrel{s} W$ rel. X. In fact the number of open equivariant cells b_i^s and e_j^t is such that

$$\alpha_k(H)(W - X) = \begin{cases} 0 & , \ k \leq m , \\ \alpha_{m+1}(H)(V - X) & , \ k = m + 1 , \\ \alpha_m(H)(V - X) + \alpha_{m+2}(H)(V - X) & , \ k = m + 2 , \\ \alpha_k(H)(V - X) & , \ k \geq m + 3 , \end{cases}$$

for every orbit type (H).

Proof. Let $f_i: (G/H_i \times I^m, G/H_i \times \partial I^m) \to (V^m, X^{m+1}) \to (V, X)$ be a charachteristic G-map for b_i^m. Using the fact that (V, X) satisfies condition π_m and the equivariant skeletal approximation theorem, Corollary 1.5 in Chapter I, it follows that there exists a G-homotopy rel. $(G/H_i \times \partial I^m)$

$$F_i' : (G/H_i \times I^m , G/H_i \times \partial I^m) \times I \to (V , X^{m-1})$$

such that $(F_i')_0 = f_i$ and $(F_i')_1(G/H_i \times I^m \times \{1\}) \subset X^m$. It now follows by applying the equivariant skeletal approximation theorem once more, this time Corollary 1.6 in Chapter I, that we can assume that F_i' is skeletal, i.e. that $F_i'(G/H_i \times I^m \times I) \subset V^{m+1}$. Observe that

$$F_i(G/H_i \times \partial I^{m+1}) \subset V^m$$

and $F_i'(G/H_i \times J^m) \subset X^m$. Now define

$$F_i : G/H_i \times J^{m+1} \to V^{m+1} \to V$$

by $F_i(a , 1) = F_i'(a)$, for $(a , 1) \in G/H_i \times I^{m+1} \times \{1\}$, and $F_i(b , t) = F_i'(b)$, for $(b , t) \in G/H_i \times \partial I^{m+1} \times I , t \in I$. For each F_i adjoin $G/H_i \times I^{m+2}$ to V by F_i, thus forming

$$\tilde{V} = V \cup_{F_i} (\dot{\cup}\, G/H_i \times I^{m+2}) .$$

Since F_i also satisfies $F_i(G/H_i \times \partial I^{m+1}) \subset V^m$ it follows that $\tilde{V}$ is an equivariant expansion of V. Let

$$h_i : G/H_i \times I^{m+2} \to \tilde{V}$$

be the restriction of the natural projection. Denoting $h_i(G/H_i \times \mathring{I}^{m+1}) = B_i^{m+1}$ and $h_i(G/H_i \times \mathring{I}^{m+2}) = B_i^{m+2}$ we can write

$$\tilde{V} = V \cup \bigcup B_i^{m+1} \cup \bigcup B_i^{m+2} .$$

Since $\dot{B}_i^{m+1} \subset V^m \subset X \cup \bigcup b_i^m$ it follows that

$$\tilde{V}_0 = X \cup \bigcup b_i^m \cup \bigcup B_i^{m+1}$$

is a subcomplex of $\tilde{V}$. Let $\bar{h}_i : G/H_i \times I^{m+1} \to \tilde{V}_0$ denote the G-map obtained by restricting h_i to $G/H_i \times I^{m+1} \times \{0\}$. Then $\bar{h}_i$ is a characteristic G-map for $\bar{B}_i^{m+1}$ and its restriction $\bar{h}_i \mid : G/H_i \times I^m \to \tilde{V}_0$ is a characteristic G-map for $\bar{b}_i^m$, and moreover $\bar{h}_i(G/H_i \times J^m) = F_i(G/H_i \times J^m \times \{0\}) = F_i'(G/H_i \times J^m) \subset X^m$. Thus it follows directly from the definitions that $\tilde{V}_0$ collapses equivariantly to X. By Lemma 4.2 there exists an equivariant CW complex W containing X as a subcomplex, with $\alpha_n(H)(W - X) = \alpha_n(H)(\tilde{V} - \tilde{V}_0)$ for every n and (H), such that $\tilde{V}$ s W rel. X. Since, as we already noted, $\tilde{V}$ is an equivariant expansion of V it follows that V s W rel. X. Clearly the number of equivariant cells in $W - X$ with a specific dimension and type is as required.

Corollary 4.4. Let (V, X) be an equivariant CW pair such that X is a strong G-deformation retract of V. Then there exists an equivariant CW complex W of the form

$$W = X \cup \bigcup b_i^{n-1} \cup \bigcup b_i^n \,, \text{ where } n-1 \geq 2 \,,$$

such that $V \mathrm{s} W$ rel. X and such that there are characteristic G-maps $y_i : G/H_i \times I^{n-1} \to \bar{b}_i^{n-1}$ satisfying $h_i(\{eH_i\} \times \partial I^{n-1}) = \{y_i\}, y_i \in X$, and $f_i : G/H_i \times I^n \to \bar{b}_i^n$ satisfying $f_i(\{eH_i\} \times J^{n-1}) = \{x_i\}, x_i \in X$.

Moreover, for any closed subgroup H of G and any G-component GW_1^H of GW^H, we have $\alpha_{n-1}(H)(GW_1^H - GX_1^H) = \alpha_n(H)(GW_1^H - GX_1^H)$ where GX_1^H denotes G-component corresponding to GW_1^H.

Proof. Since X is a strong G-deformation retract of V it follows that (V, X) satisfies condition π_m for all $m \geq 0$. Thus, by repeated use of Lemma 4.3., there exists W' such that $W' = X \cup \bigcup e_i^{n-1} \cup \bigcup e_i^n$, with $n-1 \geq 2$, and $V \mathrm{s} W'$ rel. X. Let $h_i' : G/H_i \times I^{n-1} \to \bar{e}_i^{n-1}$ be characteristic G-maps. Since $rh_i' : G/H_i \times I^{n-1} \to X$, where $r : W' \to X$ is a G-retraction, is an extension of $h_i'| : G/H_i \times \partial I^{n-1} \to X$ it follows that $h_i'|$ is G-homotopic the G-map which maps $\{eH_i\} \times \partial I^{n-1}$ to some point $y_i \in X$. Adjoining $G/H_i \times I^{n-1}$ to X by these »equivariantly constant» maps we obtain an equivariant CW complex $X \cup \bigcup b_i^{n-1}$, and it follows from Lemma 4.1 that $(X \cup \bigcup e_i^{n-1}) \mathrm{s} (X \cup \bigcup b_i^{n-1})$ rel. X. By Lemma 4.2 there exists $\widetilde{W} = X \cup \bigcup b_i^{n-1} \cup \bigcup d_i^n$ such that $W' \mathrm{s} \widetilde{W}$ rel. X. Let $\tilde{f}_i : G/H_i \times \partial I^n \to \dot{d}_i^n \to X \cup \bigcup b_i^{n-1}$ be attaching G-maps. Since J^{n-1} is contractible it follows that $\tilde{f}_i$ is G-homotopic to some $f_i : G/H_i \times \partial I^n \to X \cup \bigcup b_i^{n-1}$ satisfying $f_i(\{eH_i\} \times J^{n-1}) = \{x_i\} \in X$. Adjoining $G/H_i \times I^n$ to $X \cup \bigcup b_i^{n-1}$ by the f_i:s we obtain $W = X \cup \bigcup b_i^{n-1} \cup \bigcup b_i^n$, and by Lemma 4.1 it follows that $\widetilde{W} \mathrm{s} W$ rel. X. (Now denote the corresponding characteristic G-maps for $\bar{b}_i^n$ also by f_i).

To prove the claim about the number of the equivariant cells b_i^{n-1} and b_i^n observe that the G-homotopy equivalence $i : X \to W$ induces a one-to-one correspondence between the G-components of GX^H and GW^H, and the restriction $i| : GX_1^H \to GW_1^H$ is again a G-homotopy equivalence. Let, for convenience, $\hat{i} : A \to B$ denote the inclusion induced by $i|$ on the orbit spaces. Since $B - A$ consists of $(n-1)$-cells and n-cells and since $\hat{i} : A \to B$ is a homotopy equivalence it follows that the number of $(n-1)$-cells and n-cells in $B - A$ are the same. This being true for any G-component GW_1^H, the claim follows by induction starting from $H = G$, i.e. from the components of the fixed point set.

Chapter III. Actions of abelian groups

§ 1. Actions of discrete abelian groups

In this section the transformation group G is assumed to be a *discrete abelian group*. First we recall the definition of the Whitehead group $Wh(\pi)$ of a group π and some facts about torsion of acyclic chain complexes of based modules, see Whitehead [18], Milnor [14], Maumary [13] and Cohen [6].

Let π be a group and $R = Z[\pi]$ the group ring of π over the integers. Denote the group of all non-singular $n \times n$ matrices over R by $GL(n, R)$. We have the natural inclusion of $GL(n+1, R)$ into $GL(n, R)$ given by

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The direct limit $GL(R) = \varinjlim GL(n, R)$ is called the infinite general linear group of R. A matrix is called elementary if it agrees with the identity matrix except for one off-diagonal entry. Let $E(R)$ denote the subgroup of $GL(R)$ generated by all elementary matrices. Whitehead proved that $E(R)$ is the commutator subgroup of $GL(R)$. Let $F(R)$ be the subgroup of $GL(R)$ generated by $E(R)$ and all matrices obtained by replacing one diagonal entry in an identity matrix by $\pm \alpha$, where $\alpha \in \pi$. Define

$$Wh(\pi) = GL(R)/F(R).$$

This is the Whitehead group of π. Since $F(R)$ contains the commutator subgroup of $GL(R)$ it follows that $Wh(\pi)$ is in fact a group and moreover that it is abelian. We write $Wh(\pi)$ additively. Denote the natural projection by $\tau : GL(R) \to Wh(\pi)$, and identifying a non-singular $n \times n$ matrix A with its image in $GL(R)$ we write $\tau(A) \in Wh(\pi)$ and call $\tau(A)$ for the torsion of the matrix A. Thus we have $\tau(AB) = \tau(A) + \tau(B)$.

Let A be an $n \times n$ matrix over $R = Z[\pi]$. Then $\tau(A) = 0 \in Wh(\pi)$ if and only if A can be transformed into an identity matrix I^{n+p} by a finite sequence of operations of the following type.

(*)

(1) Multiply a row by -1.
(2) Multiply a row on the left by an element of π.
(3) Change a row by adding to it some other row.
(4) Expand to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1, R)$.

Observe that a permutation of the rows can be performed by using operations of type (1) and (3). Also observe that the operation of changing a row by adding to it a left group ring multiple of some other row is the composite of operations of type (1), (2) and (3). In fact this operation is the result of multiplying the matrix A on the left by an elementary matrix, but it is convenient to have the above four operations as the basic ones.

Let M be a free $Z[\pi]$-module. (Here and in the following we always mean by »free module» a »finite dimensional free module» if not otherwise is explicitly stated.) Let $\{e_1, \ldots, e_m\}$ be a basis for M. The family of preferred bases generated by $\{e_1, \ldots, e_m\}$ is the family of all bases $\{e'_1, \ldots, e'_m\}$ such that the change of bases matrix $A = (a_{ij})$, where $e'_i = \sum_j a_{ij} e_j$, $a_{ij} \in Z[\pi]$, satisfies $\tau(A) = 0 \in Wh(\pi)$. (Any group ring $Z[\pi]$ has the property that any two bases of a free $Z[\pi]$-module contain the same number of elements, i.e. the dimension is well-defined.) A free $Z[\pi]$-module together with a family of preferred bases is called a based $Z[\pi]$-module.

Now let M and N be based $Z[\pi]$-modules and

$$f : M \to N$$

an isomorphism of $Z[\pi]$-modules. Let B denote the matrix of f with respect to some bases for M and N from the respective families of preferred bases. The element $\tau(B) \in Wh(\pi)$ is independent of which bases from the families of preferred bases one chooses and is called the torsion of the isomorphism $f : M \to N$ and denoted by

$$\tau(f) \in Wh(\pi) .$$

Finally let us recall the definition of the torsion of an acyclic chain complex of based $Z[\pi]$-modules. Let

$$C : 0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0$$

be an acyclic chain complex over $Z[\pi]$ where each C_i is a based $Z[\pi]$-module. Since C is acyclic and each C_i is a free module there exists a chain homotopy $\delta : C \to C$ from the identity homomorphism to the zero homomorphism (such a δ is also called a chain contraction), that is, homomorphisms $\delta : C_i \to C_{i+1}$ such that $\delta\partial + \partial\delta = \text{id}$. (The homomorphisms δ can be chosen such that $\delta^2 = 0$, which is a convenient but not necessary choice.) Denote

$$C_{\text{odd}} = C_1 \oplus C_3 \oplus \ldots$$
$$C_{\text{even}} = C_0 \oplus C_2 \oplus \ldots$$

and consider C_{odd} and C_{even} as based $Z[\pi]$-modules in the obvious way. Now the homomorphisms ∂ and δ define an isomorphism

$$(\partial + \delta) : C_{\text{odd}} \to C_{\text{even}}$$

and $\tau(\partial + \delta) \in Wh(\pi)$ depends only on C and not on the different choices made. The torsion of C is now defined to be

$$\tau(C) = \tau(\partial + \delta) \in Wh(\pi) .$$

Observe that if C is of the form

$$C : 0 \to C_n \overset{\partial}{\underset{\to}{\cong}} C_{n-1} \to 0$$

then we have

$$\tau(C) = (-1)^{n-1}\tau(\partial) .$$

This fact is of great importance for us as well as the following result. If

$$0 \to C' \overset{i}{\to} C \overset{j}{\to} C'' \to 0$$

is a short exact sequence of acyclic chain complexes of based $Z[\pi]$-modules such that in each degree m the short exact sequence

$$0 \to C'_m \overset{i}{\to} C_m \to C''_m \to 0$$

splits as a sequence of based $Z[\pi]$-modules (i.e. the union of a preferred basis for C'_m and one for C''_m is a preferred basis for C_m) then

$$\tau(C) = \tau(C') + \tau(C'').$$

With this much about torsion of acyclic chain of based modules in mind we are now ready to proceed.

Let Y be an equivariant CW complex and H a subgroup of G. Consider Y^H, i.e. the set of points fixed under H. Since G is assumed to be abelian we have $GY^H = Y^H$. Let Y_1^H be a component of Y^H. Denote $\alpha = Y_1^H$ and define

$$G_\alpha = \{g \in G \mid gY_1^H = Y_1^H\} .$$

We call G_α for the group of Y_1^H. Observe that G_α is a proper subgroup of G if and only if the G-component GY_1^H contains more than one connected component. Since every point in Y_1^H is fixed under H it follows that Y_1^H is a (G_α/H)-equivariant CW complex. Also observe that if Y_2^H is a component of Y^H belonging to the same G-component as Y_1^H, i.e. $GY_2^H = GY_1^H$, then the group of Y_2^H equals G_α and Y_2^H and Y_1^H are isomorphic as (G_α/H)-equivariant CW complexes. Now define

$$Y_1^{>H} = \bigcup Y_i^K$$

where the union is over all subgroups K such that $H \subset K$ and $H \neq K$, and for fixed K the union is over all components Y_i^K of Y^K which are subsets of Y_1^H. We have $G_\alpha Y_1^{>H} = Y_1^{>H}$, (the groups of all the components Y_i^K may well be proper subgroups of G_α). Thus $Y_1^{>H}$ is a (G_α/H)-equivariant subcomplex of Y_1^H and G_α/H acts freely on $Y_1^H - Y_1^{>H}$. (Of course $Y_1^H - Y_1^{>H}$ is empty for all but a finite number of subgroups H but an action on the empty set is free.)

Now let (V, X) be an equivariant CW pair such that $i : X \to V$ is a G-homotopy equivalence. Then i induces an one-to-one correspondence between the components of X^H and V^H for every subgroup H. Let X_1^H be a component of X^H and let V_1^H denote the corresponding component V^H. Denote the group of X_1^H by G_α and observe that G_α is also the group of V_1^H. Thus we have the (G_α/H)-equivariant CW pair $(V_1^H, X_1^H \cup V_1^{>H})$. If X_2^H is a component of X^H belonging to the same G-component as X_1^H then the (G_α/H)-equivariant CW pairs $(V_2^H, X_2^H \cup V_2^{>H})$ and $(V_1^H, X_1^H \cup V_1^{>H})$ are isomorphic. Let

$$\cdots \to C_n(V_1^H, X_1^H \cup V_1^{>H}) \xrightarrow{\partial} C_{n-1}(V_1^H, X_1^H \cup V_1^{>H}) \to \cdots$$

be the cellular chain complex of $(V_1^H, X_1^H \cup V_1^{>H})$. That is, $C_n(V_1^H, X_1^H \cup V_1^{>H}) = H_n((V_1^H)^n \cup X_1^H \cup V_1^{>H}, (V_1^H)^{n-1} \cup X_1^H \cup V_1^{>H})$, where $H_n(,)$ denotes singular homology with integer coefficients, and ∂ is the boundary homomorphism in the exact homology sequence of the corresponding triple. Since G_α/H acts freely on $V_1^H - (X_1^H \cup V_1^{>H})$ it follows that each $C_n(V_1^H, X_1^H \cup V_1^{>H})$ is a free $Z[G_\alpha/H]$-module and a basis is obtained by choosing one ordinary n-cell from each (G_α/H)-equivariant n-cell of $V_1^H - (X_1^H \cup V_1^{>H})$. Since any two bases obtained in this way differ from each other only in the order of the bases elements and by multiplication of the bases elements by $\pm$ elements from the group G_α/H, it follows that they both generate the same family of preferred bases. Thus each $C_n(V_1^H, X_1^H \cup V_1^{>H})$ becomes a based $Z[G_\alpha/H]$-module in this way. Since the homology of the chain complex $C(V_1^H, X_1^H \cup V_1^{>H})$ is isomorphic to $H_*(V_1^H, X_1^H \cup V_1^{>H})$ it follows from the lemma below that $C(V_1^H, X_1^H \cup V_1^{>H})$ is an acyclic chain complex.

Lemma 1.1. Let (Y, B) be an equivariant CW pair and let $\{Y_1, \ldots, Y_m\}$ be a finite collection of equivariant subcomplexes of Y which is closed under intersection. If the inclusions $i : B \to Y$ and $i_k : B \cap Y_k \to Y_k$, $k = 1, \ldots, m$, are G-homotopy equivalences then so is the inclusion $j : B \cup (\bigcup_{k=1}^m Y_k) \to Y$.

Proof. We shall prove by induction on m that $\hat{i} : B \to B \cup (\bigcup_{k=1}^m Y_k)$ is a G-homotopy equivalence, and since $i : B \to Y$ is a G-homotopy equivalence the claim follows from this. Let $m = 1$. Since $i_1 : B \cap Y_1 \to Y_1$

is a G-homotopy equivalence $B \cap Y_1$ is a strong G-deformation retract of Y_1 and hence B is a strong G-deformation retract of $B \cup Y_1$ which shows that $\hat{i} : B \to B \cup Y_1$ is a G-homotopy equivalence.

Now assume that our claim is true for the value m-1 and denote $\tilde{Y}_{m-1} = \bigcup_{k=1}^{m} Y_k$. Since the family of the Y_k: s is closed under intersection it follows that $\tilde{Y}_{m-1} \cap Y_m$ is the union of $m-1$ sets Y_k. Thus by the induction hypothesis the inclusions $\hat{i} : B \to B \cup (\tilde{Y}_{m-1} \cap Y_m)$ and $\hat{i} : B \to B \cup \tilde{Y}_{m-1}$ are G-homotopy equivalences and so is also $\hat{i} : B \to B \cup Y_m$. Thus it follows that the inclusions of $B \cup (\tilde{Y}_{m-1} \cap Y_m)$ into both $B \cup \tilde{Y}_{m-1}$ and $B \cup Y_m$ are G-homotopy equivalences. It now follows that $B \cup (\tilde{Y}_{m-1} \cap Y_m)$ is a strong G-deformation retract of $B \cup (\tilde{Y}_{m-1} \cup Y_m)$, and therefore the composite $\hat{i} : B \to B \cup (\tilde{Y}_{m-1} \cap Y_m) \to B \cup (\tilde{Y}_{m-1} \cup Y_m)$ is G-homotopy equivalence.

We now apply Lemma 1.1 as follows. Since V^H is a finite equivariant CW complex and G is abelian there exists a finite number of subgroups $K_1, \ldots, K_m$ such that if K is any subgroup of G such that $H \subset K$ and $H \neq K$ then $Y^K = Y^{K_i}$ for some $i = 1, \ldots, m$. In other words let $K_1, \ldots, K_m$ be the family of all isotropy groups for points in V^H except the group H, which of course may or may not occur as the isotropy group of a point in V^H. Since $V^{K_i} \cap V^{K_j} = V^{(K_i+K_j)}$ it follows that the family $\{V^{K_1}, \ldots, V^{K_m}\}$ is closed under intersection. Since $i : X \to V$ is a G-homotopy equivalence it follows that also $i| : X^H \to V^H$ and $i| : X^H \cap V^{K_i} = X^{K_i} \to V^{K_i}$ are G-homotopy equivalences. Thus by Lemma 1.1. the inclusion $j : X^H \cup (\bigcup_{i=1}^{m} V^{K_i}) \to Y^H$ is a G-homotopy equivalence, and hence in particular a homotopy equivalence. Since the component of $X^H \cup (\bigcup_{i=1}^{m} V^{K_i})$ which contains X_1^H equals $X_1^H \cup V_1^{>H}$ the restriction of j gives a homotopy equivalence $j| : X_1^H \cup V_1^{>H} \to V_1^H$. Thus $H_*(V_1^H, X_1^H \cup V_1^{>H}) = 0$ and the chain complex $C(V_1^H, X_1^H \cup V_1^{>H})$ is acyclic. We have proved.

Proposition 1.2. Let (V, X) be an equivariant CW pair such that $i : X \to V$ is a G-homotopy equivalence and let H be any subgroup of G and X_1^H a component of X^H. Denoting the group of X_1^H by G_α we have that $C(V_1^H, X_1^H \cup V_1^{>H})$ is an acyclic chain complex of based $Z[G_\alpha/H]$-modules.

We denote the torsion of the chain complex $C(V_1^H, X_1^H \cup V_1^{>H})$ by

$$\tau(C(V, X)_1^H) \in Wh(G_\alpha/H) .$$

Next we show that this torsion is an invariant of equivariant simple-homotopy type.

Proposition 1.3. Assume that $V \text{ s } W$ rel. X. Then we have

$$\tau(C(V, X)_1^H) = \tau(C(W, X)_1^H)$$

for every subgroup H of G and any component X_1^H of X^H.

Proof. It is enough to prove this for the case that W is equivariant elementary expansion of V. Thus assume that this is the case and denote

$$W = V \cup b^{n-1} \cup b^n .$$

Assume that the type of this equivariant elementary expansion is K and let

$$\sigma : G/K \times I^n \to W$$

be a characteristic simple G-map for (b^n, b^{n-1}). Since the set $\sigma(\{eK\} \times J^{n-1})$ is connected it lies in one component, say X_0^K, of X^K.

If $K \subset H$ then $W^H = V^H$ and the claim is obvious. If $H \subset K$, where $H \neq K$, and $GX_0^K \subset GX_1^H$, then by excision we have an isomorphism of chain complexes $i : C(V_1^H, X_1^H \cup V_1^{>H}) \xrightarrow{\cong} C(W_1^H, X_1^H \cup W_1^{>H})$ which in each degree is an isomorphism of based modules and hence the wanted conclusion follows. In case $GX_0^K \not\subset GX_1^H$ we have again the situation that $GW_1^H = GV_1^H$ and this is also true for $K = H$.

Thus it only remains to prove our claim in the case $H = K$ and for the component X_0^K (or any other component of the G-component GX_0^K). Denote $C' = C(V_0^K, X_0^K \cup V_0^{>K})$ and $C = C(W_0^K, X_0^K \cup W_0^{>K})$, and let G_β be the group of X_0^K. Then we have a short exact sequence of chain complexes

$$0 \to C' \to C \to C'' \to 0 ,$$

where C'' is of the form

$$0 \to C''_n \overset{\partial}{\underset{\cong}{\to}} C''_{n-1} \to 0$$

and C''_n and C''_{n-1} are based $Z[G_\beta/K]$-modules of rank 1 and $\tau(\partial) = 0$. Since moreover the above short exact sequence of chain complexes in each dimension splits as a sequence of based $Z[G_\beta/K]$-modules we have

$$\tau(C) = \tau(C') + \tau(C'') = \tau(C') + (-1)^{n-1}\tau(\partial) = \tau(C') ,$$

which is what we wanted to prove.

Thus we have a well-defined map

$$\Phi : Wh_G(X) \to \sum_H \sum_{i=1}^{m(H)} \oplus \; Wh(G_{\alpha(i,H)}/H)$$

by defining

$$\Phi(s(V, X)) = \{\tau(C(V, X)_i^H\}_{1 \le i \le m(H)}, \text{ all } H \subset G .$$

Here $m(H)$ denotes the number of G-components of X^H and $G_{\alpha(i,H)}$ is the group of a component X_i^H representing the G-component GX_i^H, $1 \le i \le m(H)$. The direct sum is over all subgroups H of G and for each fixed H over a set of one representing component from each G-component of X^H. This definition is independent on which component of a G-component we choose to represent the G-component. Moreover for any (V, X) we have that $\tau(C(V, X)_i^H) \neq 0$ for only a finite number of subgroups H.

It follows from the appropriate short exact sequence of chain complexes that $\tau(C(V \cup_X W, X)_i^H) = \tau(C(V, X)_i^H) + \tau(C(W, X)_i^H)$, for any subgroup H and any component X_i^H of X^H. Thus

$$\Phi(s(V \cup_X W, X)) = \Phi(s(V, X)) + \Phi(s(W, X)) ,$$

that is, Φ is a homomorphism. We are now ready for

Theorem 1.4. Assume that X is an equivariant CW complex such that for any subgroup H of G each component X_i^H of X^H is simply connected. Then Φ is an isomorphism.

Proof. Let us first make a general remark. Let (X, A) be a pair with X and A connected and assume that A is simply connected. Then any map $f: (I^n, \partial I^n) \to (X, A)$ with $f(J^{n-1}) = \{a\}$, some a $\in A$, determines a unique element $[f] \in \pi_n(X, A)$, in other words we need not consider any base point in A. It follows that if (X, A) is a π-pair where π is a discrete group then $\pi_n(X, A)$ is a left $Z[\pi]$-module, and moreover the Hurewicz homomorphism $\phi: \pi_n(X, A) \to H_n(X, A)$ is a homomorphism of $Z[\pi]$-modules.

Let $s(W, X) \in Wh_G(X)$ be such that $\Phi(s(W, X)) = 0$. By Corollary 4.4 in Chapter II (and the fact that Φ is well-defined, i.e. Proposition 1.3.) we can assume that (W, X) is in simplified form. Thus we have

$$W = X \cup \bigcup b_i^{n-1} \cup \bigcup b_i^n , \quad \text{where } n - 1 \geq 2 .$$

We denote

$$Y = X \cup \bigcup b_i^{n-1} .$$

Now let H denote a subgroup of G which occurs as the type of some of the equivariant cells b_i^n. Let X_1^H be a component of X^H and let Y_1^H and W_1^H be the corresponding components of Y^H and W^H, respectively. By the second part of Corollary 4.4. in Chapter II the number of equivariant n-cells b_i^n in GW_1^H which have type H equals the number of equivariant

$(n-1)$-cells b_i^{n-1} in GW_1^H (and hence in GY_1^H) which have type H. Let us denote these by b_s^n and b_s^{n-1}, $s = 1, \ldots, m$. Thus we have

$$GW_1^H = G(Y_1^H \cup W_1^{>H}) \cup \bigcup_{s=1}^{m} b_s^n ,$$

$$GY_1^H = G(X_1^H \cup Y_1^{>H}) \cup \bigcup_{s=1}^{m} b_s^{n-1} .$$

Now let G_α be the group of X_1^H, and hence also the group of Y_1^H and W_1^H, and consider the (G_α/H)-equivariant CW pair $(W_1^H, X_1^H \cup W_1^{>H})$. We have the commutative diagram

$$\begin{array}{ccc} 0 \to H_n(W_1^H, Y_1^H \cup W_1^{>H}) & \overset{\partial}{\cong} & H_{n-1}(Y_1^H \cup W_1^{>H}, X_1^H \cup W_1^{>H}) \to 0 \\ \uparrow \phi & & \uparrow \phi \\ \pi_n(W_1^H, Y_1^H \cup W_1^{>H}) & \to & \pi_{n-1}(Y_1^H \cup W_1^{>H}, X_1^H \cup W_1^{>H}) \end{array}$$

where ϕ denotes the Hurewicz homomorphism. First observe that the upper row equals the cellular chain complex of $(W_1^H, X_1^H \cup W_1^{>H})$. Thus we have

$$(-1)^{n-1}\tau(\partial) = \tau(C(W, X)_1^H) = 0 \in Wh(G_\alpha/H) .$$

Secondly observe that since X_1^H is simply connected by assumption and $X_1^H \cup W_1^{>H}$ is obtained from X_1^H by adjoining (ordinary) $(n-1)$-cells and n-cells and since $n-1 \geq 2$ it follows that $X_1^H \cup W_1^{>H}$ is simply connected, and for the same reason $Y_1^H \cup W_1^{>H}$ is simply connected. The pair $(W_1^H, Y_1^H \cup W_1^{>H})$ is $(n-1)$-connected since W_1^H is obtained from $Y_1^H \cup W_1^{>H}$ by adjoining n-cells and similarly $(Y_1^H \cup W_1^{>H}, X_1^H \cup W_1^{>H})$ is $(n-1)$-connected. Thus by the Hurewicz theorem the homorphisms ϕ are isomorphisms and hence by the remarks made at the beginning of the proof each ϕ is an isomorphism of $Z[G_\alpha/H]$-modules.

The homology modules are based $Z[G_\alpha/H]$-modules, i.e. free $Z[G_\alpha/H]$ modules together with a family of preferred bases. Thus the homotopy groups are free $Z[G_\alpha/H]$-modules and with corresponding preferred bases given as follows. Let

$$f_s : (G/H \times I^n, G/H \times \partial I^n) \to (\bar{b}_s^n, \dot{b}_s^n) \to (GW_1^H, G(Y_1^H \cup W_1^{>H}))$$

$$h_s : (G/H \times I^{n-1}, G/H \times \partial I^{n-1}) \to (\bar{b}_s^{n-1}, \dot{b}_s^{n-1}) \to (G(Y_1^H \cup W_1^{>H}), G(X_1^H \cup W_1^{>H}))$$

be characteristic G-maps for $\bar{b}_s^n$ and $\bar{b}_s^{n-1}$, $s = 1, \ldots, m$, as in Corollary 4.4 in Chapter II, and moreover chosen such that

$$\left.\begin{array}{l} f_s(\{eH\} \times I^n) \subset W_1^H \\ h_s(\{eH\} \times I^{n-1}) \subset Y_1^H \subset Y_1^H \cup W_1^H \end{array}\right. \qquad s = 1, \ldots, m .$$

Define

$$\bar{f}_s : (I^n, \partial I^n) \to (W_1^H, Y_1^H \cup W_1^{>H})$$
$$\bar{h}_s : (I^{n-1}, \partial I^{n-1}) \to (Y_1^H \cup W_1^{>H}, X_1^H \cup W_1^{>H})$$

by $\bar{f}_s = f_s|\{eH\} \times I^n$ and $\bar{h}_s = h_s|\{eH\} \times I^{n-1}$, $s = 1, \ldots, m$. Since $\bar{f}_s(J^{n-1}) = \{x_s\}$, $x_s \in X_1^H \subset X$ and $\bar{h}_s(\partial I^{n-1}) = \{\tilde{x}_s\}$, $\tilde{x}_s \in X_1^H \subset X$, we have

$$[\bar{f}_s] \in \pi_n(W_1^H, Y_1^H \cup W_1^{>H}),$$
$$[\bar{h}_s] \in \pi_{n-1}(Y_1^H \cup W_1^H, X_1^H \cup W_1^{>H}),$$

for $s = 1, \ldots, m$. The images of the elements $[\bar{f}_s]$ and $[\bar{h}_s]$, $s = 1, \ldots, m$, under the Hurewicz isomorphisms form preferred bases for the respective homology modules.

Now let

$$\bar{\partial}[\bar{f}_s] = \sum_{t=1}^{m} a_{st}[\bar{h}_t], \; a_{st} \in Z[G_\alpha/H],$$

and denote $A = (a_{st})$. Thus we have

$$\tau(A) = 0 \in Wh(G_\alpha/H),$$

and hence the matrix A can be transformed into an identity matrix by a finite sequence of operations of the four types given in (*). By a different choice of characteristic G-maps f_s' for b_s^n, $s = 1, \ldots, m$, (but still satisfying $f_s'(\{eH\} \times I^n) \subset W_1^H$ and $\bar{f}_s'(J^{n-1}) = \{x_s'\}$, $x_s' \in X_1^H$) the matrix of $\bar{\partial}$ can be made into any matrix obtained from A by multiplying rows by $\pm$ elements from G_α/H.

Now let $1 \leq r \leq m$ and $1 \leq p \leq m$, where $r \neq p$. Let

$$\bar{v} : (I^n, \partial I^n) \to (W_1^H, Y_1^H \cup W_1^{>H}),$$

where $\bar{v}(J^{n-1}) = \{x\}$ for some $x \in X_1^H$, be such that

$$[\bar{v}] = [\bar{f}_r] + [\bar{f}_p] \in \pi_n(W_1^H, Y_1^H \cup W_1^{>H}).$$

Denote $\bar{f}_p(\mathring{I}^n) = c_p^n$. Thus we have $c_p^n \subset b_p^n \cap W_1^H$ and $\dot{c}_p^n \subset Y_1^n$ and moreover $Gc_p^n = b_p^n$. Since $\bar{f}_p| : \partial I^n \to Y_1^H \cup W_1^{>H} \cup c_p^n$ is homotopic to a contant map it follows that the maps

$$\bar{f}_r|, \bar{v}| : \partial I^n \to Y_1^H \cup W_1^{>H} \cup c_p^n$$

are homotopic. Define the G-map

$$v| : G/H \times \partial I^n \to G(Y_1^H \cup W_1^{>H}) \cup b_p^n$$

by $(v|)(gH, y) = g\bar{v}(y)$, $y \in \partial Y^n$. It follows that $v|$ is G-homotopic to $f_r| : G/H \times \partial I^n \to G(Y_1^H \cup W_1^{>H}) \cup b_p^n$. Now define

$$\widetilde{W} = (W - b_r^n) \cup_{v|} (G/H \times I^n) ,$$

(where $v|$ is considered as a G-map into $W - b_r^n$). By Lemma 4.1 in Chapter II we have $\widetilde{W} \mathrm{s}\, W$ rel. $(W - b_r^n)$. Moreover the matrix of the boundary homomorphism

$$\bar{\partial} : \pi_n(\widetilde{W}_1^H , Y_1^H \cup W_1^{>H}) \to \pi_{n-1}(Y_1^H \cup W_1^{>H} , X_1^H \cup W_1^{>H})$$

is the one obtained from A by changing the r : th row by adding to it the p : th row.

An expansion of the matrix A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is realized geometrically by performing an equivariant elementary expansion of type H, that is, adjoin $G/H \times I^n$ to W by a G-map

$$\sigma_+ : G/H \times J^{n-1} \to GX_1^H \subset W$$

defined by $\sigma_+(gH , y) = gx$, for some fixed $x \in X_1^H$.

Thus it now follows that there exists an equivariant CW complex V such that

$$V \mathrm{s}\, W \text{ rel. } W - (\bigcup_{s=1}^{m} b_s^n)$$

and (V , X) is in simplified form and there are $m + q$, where $q \geq 0$, equivariant n-cells $e_1^n , \ldots , e_{m+q}^n$ and q equivariant $(n - 1)$-cells $e_{m+1}^{n-1} , \ldots , e_{m+q}^{n-1}$ in $V - (W - (\bigcup_{s=1}^{m} b_s^n))$, and characteristic G-maps $u_s : G/H \times I^n \to \bar{e}_s^n , s = 1 , \ldots , m + q$, and $h_s : G/H \times I^{n-1} \to \bar{e}_s^{n-1}$, $s = m + 1, \ldots, m + q$, such that

$$\bar{\partial}[\bar{u}_s] = [\bar{h}_s] , s = 1 , \ldots , m + q .$$

Here

$$\bar{\partial} : \pi_n(V_1^H , U_1^H \cup W_1^{>H}) \to \pi_{n-1}(U_1^H \cup W_1^{>H} , X_1^H \cup W_1^{>H})$$

where we have denoted $U = Y \cup e_{m+1}^{n-1} \cup \ldots \cup e_{m+1}^{n-1}$. Observe that $W_1^{>H} = V_1^{>H}$.

Thus the maps $\bar{u}_s| , \bar{h}_s : (I^{n-1} , \partial I^{n-1}) \to (U_1^H \cup W_1^{>H} , X_1^H \cup W_1^{>H})$ are homotopic. It follows from this that there exist

$$\bar{w}_s : \partial I^n \to U_1^H \cup W_1^{>H} , s = 1 , \ldots , m + q ,$$

such that $\bar{w}_s$ is homotopic to $\bar{u}_s| \partial I^n$ and $\bar{w}_s| I^{n-1} = \bar{h}_s$, and moreover $\bar{w}_s(J^{n-1}) \subset X_1^H$. Thus the corresponding G-map $w_s : G/H \times \partial I^n \to G(U_1^H \cup W_1^{>H}$ is G-homotopic to $u_s|G/H \times \partial I^n$. Now form the equivariant CW

complex $\tilde{V}$ by attaching equivariant n-cells $G/H \times I^n$ to $V - (\bigcup_{s=1}^{m+q} e_s^n)$ by the attaching G-maps w_s, $s = 1, \ldots, m+q$. It follows from Lemma 4.1 in Chapter II that

$$\tilde{V} \text{ s } V \text{ rel. } V - (\bigcup_{s=1}^{m+q} e_s^n) .$$

Moreover it follows directly from the properties of attaching G-maps w_s that $\tilde{V}$ collapses equivariantly to $V - (\bigcup_{s=1}^{m} b_s^{n-1} \cup \bigcup_{s=m}^{m+q} e_s^{n-1} \cup \bigcup_{s=1}^{m+q} e_s^n) = W - (\bigcup_{s=1}^{m} b^{n-1} \cup \bigcup_{s=1}^{m} b^n)$.

We have shown that

$$W \text{ s } W - (\bigcup_{s=1}^{m} b_s^{n-1} \cup \bigcup_{s=1}^{m} b_s^n) \text{ rel. } W - (\bigcup_{s=1}^{m} b_s^{n-1} \cup \bigcup_{s=1}^{m} b_s^n) .$$

That is, all equivariant cells in $W - X$ which have type H and belong to the G-component GW_1^H have been »removed». Applying this procedure for every subgroup that occurs as the type of some equivariant cell in $W - X$ and to each G-component of W^K we get

$$W \text{ s } X \text{ rel. } X .$$

That is $s(W, X) = 0 \in Wh_G(X)$ and we have proved that Φ is injective.

The surjectivity of Φ is proved as follows. Let H be a subgroup of G for which $m(H) > 0$, and let GX_1^H be a G-component of X^H. Choose some $x \in (X_1^H)^0$. Let G_α be the group of X_1^H, and let $A = (a_{st})$ be any non-singular $m \times m$ matrix over $Z[G_\alpha/H]$. Now define

$$h| : G/H \times \partial I^2 \to GX_1^H \subset X$$

to be the G-map determined by the condition $(h|)(\{eH\} \times \partial I^2) = \{x\}$. Let $GY_1^H = GX_1^H \cup b_1^2 \cup \ldots \cup b_m^2$ be the equivariant CW complex obtained by adjoining m different equivariant 2-cells $G/H \times I^2$ to GX_1^H by the attaching G-map $h|$, and let h_t denote the corresponding characteristic G-map for b_t^2. Then $[\bar{h}_t] \in \pi_2(Y_1^H, X_1^H)$, $t = 1, \ldots, m$, form a bases for the free $Z[G_\alpha/H]$-module $\pi_2(Y_1^H, X_1^H)$.

Now let $v_s : (I^2, \partial I^2) \to (Y_1^H, X_1^H)$ be such that $[v_s] = \sum_{t=1}^{m} a_{st}[\bar{h}_t]$. It follows since $\bar{h}_t(\partial I^2) = \{x\}$, that we can choose v_s such that $v_s(\partial I^2) = \{x\}$. Now let $\bar{f}_s| : \partial I^3 \to Y_1^H$ be the extension of v_s defined by $(\bar{f}_s|)(J^{n-1}) = \{x\}$, and let

$$f_s| : G/H \times \partial I^3 \to GY_1^H$$

be the corresponding G-map. Let $GW_1^H = GY_1^H \cup b_1^3 \cup \ldots \cup b_m^3$ be obtained by adjoining equivariant 3-cells $G/H \times I^3$ to GY_1^H by $f_s|$, $s = 1, \ldots, m$, and let f_s denote the corresponding characteristic G-maps. Now the boundary homomorphism

$$\bar{\partial} : \pi_3(W_1^H, Y_1^H) \to \pi_2(Y_1^H, X_1^H)$$

is given by $\bar{\partial}[\bar{f}_s] = \sum_{t=1}^{m} a_{st}[\bar{h}_t]$, i.e. has matrix A in these bases. Thus $\bar{\partial}$ is an isomorphism and it follows easily that $\pi_n(W_1^H, X_1^H) = 0$ for all n. Thus the inclusion $i : X_1^H \to W_1^H$ is a homotopy equivalence and using Corollary 5.5 in Bredon [4] we see that $i : GX_1^H \to GW_1^H$ is a G-homotopy equivalence. Thus X is a strong G-deformation retract of $GW_1^H \cup X$ and hence $s(GW_1^H \cup X, X) \in Wh_G(X)$. We now have $\Phi(s(GW_1^H \cup X, X)) = \tau(A) \in Wh(G_\alpha/H)$. Since Φ is a homomorphism this shows that Φ is surjective.

Known facts about Whitehead groups $Wh(\pi)$ together with Theorem 1.4 now gives us the following information about $Wh_G(X)$.

Theorem 1.5. Let G be a finite abelian group and X an equivariant CW complex such that for any subgroup H of G each component of X^H is simply connected. Then $Wh_G(X)$ is a finitely generated abelian group.

Proof. This follows from Corollary (20.3) in Bass [1] and Theorem 1.4.

Theorem 1.6. Let $G = Z_m$, $m \geq 1$, be a finite cyclic group and let X be as above. Then $Wh_G(X)$ is a finitely generated free abelian group.

Proof. It is known that if $\pi = Z_n$, $n \geq 1$, then $Wh(\pi)$ is a free abelian group on $[n/2] + 1 - \delta(n)$ generators, where $[n/2]$ denotes the integral part of $n/2$ and $\delta(n)$ is the number of divisors of n, see Example 3 on page 54 in Bass [1] and Proposition 4.14 in Bass-Milnor-Serre [3]. Since Φ is an isomorphism onto a finite direct sum of such Whitehead groups, the theorem follows.

Observe that the result quoted in the above proof implies that $Wh(Z_n) = 0$ if $n = 1, 2, 3, 4$ or 6, and that for all other finite cyclic groups π we have $Wh(\pi) \neq 0$. (The fact that $Wh(Z_n) = 0$ for $n = 2, 3$ and 4 is due to Higman [8], and the case $n = 1$ is elementary.) Since any quotient group of a subgroup of one of the groups $\{e\}, Z_2, Z_3, Z_4$ and Z_6 is again one of these groups we have.

Theorem 1.8. Let $G = Z_m$, where $m = 1, 2, 3, 4$ or 6, and let X be as before. Then $Wh_G(X) = 0$.

Corollary 1.9. Let G and X be as in Theorem 1.8, and let Y denote an equivariant CW complex. Then any G-homotopy equivalence $f: X \to Y$ is an equivariant simple-homotopy equivalence.

Proof. This follows from Theorem 3.6′ in Chapter II and Theorem 1.8.

The case $G = \{e\}$ in Corollary 1.9 is just the standard fact that a homotopy equivalence between simply connected CW complexes is a simple-homotopy equivalence.

Now let G again denote an arbitrary discrete abelian group and let K be a subgroup of G. Then the equivariant Whitehead group of the discrete G-space G/K is given by

$$Wh_G(G/K) \cong \sum_{all\ H \subset K} \oplus\ Wh(K/H)\ .$$

Thus in particular

$$Wh_G(G) \cong Wh(\{e\}) = 0\ , \text{ and}$$
$$Wh_G(\{x\}) \cong \sum_{\text{all } H} \oplus\ Wh(G/H)\ .$$

In many cases it is convenient and natural to restrict the attention to the subgroup of $Wh_G(X)$ consisting of all elements $s(W, X)$ such that the isotropy groups of points in W belong to some family F of subgroups of G. Denote this group by $Wh_G(X\,;F)$. It is clear from the proof of Theorem 1.4 that Φ gives an isomorphism

$$Wh_G(X\,;F) \cong \sum_{H \in F} \sum_{i=1}^{m(H)} \oplus\ Wh(G_{\alpha(i,H)}/H)\ .$$

If for example the G-action on X is semi-free, that is, the only possible isotropy groups are the trivial group $\{e\}$ and the whole group G, then it is in many cases natural to only consider pairs (W, X) where the action on W also is semi-free.

Theorem 1.10. Assume that X is a connected and simply-connected equivariant CW complex such that the G-action is semi-free and each component of the fixed points set is simply connected. Then we have $Wh_G(X\,;\{\{e\}, G\}) \cong Wh(G)$.

Corollary 1.11. Let $G = Z \oplus \ldots \oplus Z$ and let X be as in Theorem 1.10. Then $Wh_G(X\,;\{\{e\}, \{G\}\}) = 0$.

Proof. By a Corollary to Theorem 2 in Bass-Heller-Swan [2] we have $Wh(Z \oplus \ldots \oplus Z) = 0$. The case $Wh(Z) = 0$ is due to Higman [8].

Corollary 1.12. Let $G = Z \oplus \ldots \oplus Z$ and let X and Y be equivariant CW complexes such that the G-action is semi-free and every component of the fixed point sets is simply-connected. Then any G-homotopy equivalence $f : X \to Y$ is an equivariant simple-homotopy equivalence.

Proof. By Theorem 3.6. in Chapter II, f is an equivariant simple-homotopy equivalence if and only if $s(M_{\tilde{f}}, X) = 0 \in Wh_G(X)$, where $M_{\tilde{f}}$ is the mapping cylinder of some equivariant skeletal approximation of f. But the G-action on $M_{\tilde{f}}$ is semi-free and hence $s(M_{\tilde{f}}, X) = Wh_G(X; \{\{e\}, G\}) = 0$.

Both Corollary 1.11 and 1.12 still hold with the family $\{\{e\}, G\}$ replaced by any family $F = \{H\}$ of subgroups of the form $H = H_1 \oplus \ldots. \oplus H_m$, where each H_i either equals $\{0\}$ or Z, here m denotes the number of summands in $G = Z \oplus \ldots \oplus Z$.

Example 1.13. We conclude this section by the following example. Let $G = Z_5$ and consider the element $a = (-1 + t + t^4) \in Z[Z_5]$. Since $(-1 + t + t^4)(-1 + t^2 + t^3) = 1$ the element a is a unit and it is known that the 1×1 matrix $[a]$ represents a generator of $Wh(Z_5) \cong Z$. Let $X = \{x\}$ and let W be the equivariant CW complex, constructed as in the proof of the surjectivity of Φ, such that $\Phi(s(W, \{x\})) = \tau[a] \in Wh(Z_5)$. Since $s(W, \{x\}) \neq 0 \in Wh_G(\{x\})$ the inclusion $i : \{x\} \to W$ is not an equivariant simple-homotopy equivalence. The orbit space $W' = Z_5 \diagdown W$ is a CW complex obtained by adjoining a 3-cell to S^2 by a degree one map and hence W' s D^3 rel. S^2, by Lemma 13 in Whitehead [18]. Thus $W' \sim \{x\}$ rel. $\{x\}$ and the induced inclusion on the orbit spaces $i' : \{x\} \to W'$ is an ordinary simple-homotopy equivalence. This is the »better example» promised in the discussion preceeding Lemma 1.2 in Chapter II. Observe moreover that Z_5 acts freely on $W - \{x\}$, and that if we forget the Z_5-action then the inclusion $i : \{x\} \to W$ is an ordinary simple-homotopy equivalence.

§ 2. Toral actions

In this section we consider actions by the n-dimensional torus T^n, $n \geq 1$. (The case $T^0 = Z_2$ was already treated in Section 1).

Theorem 2.1. Let $G = T^n$, $n \geq 1$, and let X be an equivariant CW complex such that each component of X^H is simply connected for every closed subgroup H of G. Then we have $Wh_G(X) = 0$.

The proof is similar to the part of the proof of Theorem 1.4 which proves that Φ is injective. Since G is connected any component of X^H is a G-component, i.e. the group of every component of X^H is G. Since

G/H is connected the group of components of G/H is the trivial group $\{e\}$ and we know that $Wh(\{e\}) = 0$. We omit the details.

Corollary 2.2. Let $G = T^n$, $n \geq 1$, and let X and Y be equivariant CW complexes as in Theorem 2.1. Then any G-homotopy equivalence $f: X \to Y$ is an equivariant simple-homotopy equivalence.

Proof. Follows from Theorem 3.6′ in Chapter II and Theorem 2.1.

By Theorem 2.6 in [9] (or Lemma 4.4 in Matumoto [11]), Corollary 2.2 in particular applies when X and Y are compact differentiable G-manifolds. Moreover the assumption that $f: X \to Y$ is a G-homotopy equivalence can by the equivariant Whitehead theorem be expressed in non-equivariant homotopy terms, see Proposition 2.5 in [9].

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Printed in December 1974