ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

585

ON FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY

D. AHARONOV and S. FRIEDLAND

HELSINKI 1974
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1975.585

ISSN 0066-1953
ISBN 951-41-0199-5

Communicated 12 November 1973

KESKUSKIRJAPAINO
HELSINKI 1974

1. Introduction

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

be a locally univalent analytic function with bounded boundary rotation. Paatero [9] showed $f'(z)$ has the following integral representation

$$f'(z) = \exp\left\{ -\frac{1}{\pi} \int_0^{2\pi} \log(1 - ze^{-i\theta}) dv(\theta) \right\} \tag{1.2}$$

where $v(\theta)$ is a finite, normalized measure

$$\int_0^{2\pi} |dv(\theta)| < \infty , \tag{1.3}$$

$$\int_0^{2\pi} dv(\theta) = 2\pi . \tag{1.4}$$

Any function $f(z)$ of the form (1.1) having a representation form (1.2) with any finite non-normalized measure $v(\theta)$ is also a function of bounded boundary rotation. Denote by $V(p, q)$ the class of functions of bounded boundary rotation satisfying

$$\int_0^{2\pi} dv(\theta) = p\pi , \quad \int_0^{2\pi} |dv(\theta)| < q\pi \quad (|p| \leq q) . \tag{1.5}$$

The class $V(2, q)$ is the well known class of functions whose boundary rotation is at most $q\pi$ (see [7] for basic properties of this class).

For $p \geq 0$ and $q \geq \max(p, 2)$ we show that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $V(p, q)$ then

$$|a_n| \leq \frac{1}{n} b_{n-1} \left(\frac{q-p}{2} , \frac{q+p}{2} \right) , \quad n = 2, 3, \ldots \tag{1.6}$$

where

$$\frac{(1+z)^s}{(1-z)^t} = 1 + \sum_{n=1}^{\infty} b_n(s\,,t)z^n\,. \tag{1.7}$$

The above inequality (1.6) is sharp.

For $p = 2$ the inequality (1.6) was an open conjecture for functions of bounded boundary rotation proved recently in [3] and [1]. For $q - 2 \geq p \geq 0$ we obtain an integral representation of the closed convex hull of $V(p, q)$. Finally we show that the maximal value of $\int_0^{2\pi} |f'(re^{i\theta})|^\alpha d\theta$ for any $\alpha > 0$ in the class $V(p\,,q)$ is obtained for the function whose derivative is of the form (1.7) with $s = \frac{q-p}{2}$ and $t = \frac{q+p}{2}$.

2. Preliminary results

Let

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \tag{2.1}$$

be an analytic function in $|z| < 1$. Suppose that $\operatorname{Re}\{h(z)\} > 0$ for $|z| < 1$.

The classical Herglotz formula states that $h(z)$ has a representation

$$h(z) = \int_0^{2\pi} \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right) d\mu(\theta)\,, \tag{2.2}$$

where $\mu(\theta)$ is an increasing function with $\mu(0) = 0$ and $\mu(2\pi) = 1$. Recently Brannan, Clunie and Kirwan [3] established the following remarkable extension of (2.2).

Theorem I. *Let* $h(z)$ *be of the form* (2.1). *Assume furthermore that* $h(z)$ *is subordinate to* $\frac{1+cz}{1-z}$ *on the unit disc for* $|c| \leq 1$. *Then for any given* $\alpha \geq 1$ *there exists an increasing function* $\mu_\alpha(\theta)$ *on* $[0\,,2\pi]$ *with* $\mu_\alpha(0) = 0$ *and* $\mu_\alpha(2\pi) = 1$ *such that*

$$[h(z)]^\alpha = \int_0^{2\pi} \left[\frac{1 + cze^{-i\theta}}{1 - ze^{-i\theta}}\right]^\alpha d\mu_\alpha(\theta)\,. \tag{2.3}$$

The following result follows easily from (2.3) [3]: Let I be the set of all increasing normalized functions $\mu(\theta)$ on $[0\,,2\pi]$, i.e., $\mu(0) = 0$ and $\mu(2\pi) = 1$. Denote

$$(2.4)\qquad \mathcal{P}_\lambda = \left\{ h(z) \,\middle|\, h(z) = \int_0^{2\pi} \frac{d\mu(\theta)}{(1 - ze^{i\theta})^\lambda}\,, \quad \mu \in I \right\}.$$

Then

$$(2.5)\qquad \mathcal{P}_{\lambda_1} \mathcal{P}_{\lambda_2} \subset \mathcal{P}_{\lambda_1+\lambda_2}$$

for $\lambda_1 > 0$ and $\lambda_2 > 0$. Formula (2.5) means that if $h_1 \in \mathcal{P}_{\lambda_1}$ and $h_2 \in \mathcal{P}_{\lambda_2}$ then there exists a function h_3 in $\mathcal{P}_{\lambda_1+\lambda_2}$ such that $h_1 h_2 = h_3$.

A function (1.1) is said to be starlike of order α for $\alpha < 1$ if

$$(2.6)\qquad \operatorname{Re}\{zf'(z)/f(z)\} > \alpha$$

for all $|z| < 1$. Denote by S_α the set of all starlike functions of order α. By the Herglotz formula we obtain the well known representation of a starlike function of order α (see for example [10]):

$$(2.7)\qquad f(z) = z \exp\left\{- 2(1 - \alpha) \int_0^{2\pi} \log(1 - ze^{-i\theta}) d\mu(\theta)\right\}, \quad \mu \in I\,.$$

The following relations are easily established

(i) $S_\beta \subset S_\alpha$ for $\alpha \leq \beta$,

(ii) if $f \in S_\alpha$ then $z[f(z)/z]^{(1-\beta)/(1-\alpha)} \in S_\beta$,

(iii) if $f_1 \in S_{\alpha_1}$ and $f_2 \in S_{\alpha_2}$ then $f_1 f_2/z \in S_{\alpha_1+\alpha_2-1}$.

Let S_{α_1,α_2} denote the class of all functions $g(z)$ of the form (2.1) such that $g = f_1/f_2$ where $f_1 \in S_{\alpha_1}$ and $f_2 \in S_{\alpha_2}$. We have

Lemma 2.1. *If the class S_{α_1,α_2} is defined as above then:*

$$(2.8)\qquad S_{\alpha_1,\alpha_2} = \left\{ \begin{array}{l} g(z) \,\middle|\, g(z) = \exp\left\{- \displaystyle\int_0^{2\pi} \log(1 - ze^{-i\theta}) d\mu(\theta)\right\}, \\ \displaystyle\int_0^{2\pi} d\mu(\theta) = 2(\alpha_2 - \alpha_1)\,, \ \int_0^{2\pi} |d\mu(\theta)| < 4 - 2(\alpha_1 + \alpha_2) \end{array} \right\}.$$

Proof. Let $g(z) = f_1/f_2$ where each f_j has a representation (2.7) with $\alpha = \alpha_j$ and $\mu = \mu_j$ for $j = 1\,, 2$ respectively. Let $\mu = 2(1 - \alpha_1)\mu_1 - 2(1 - \alpha_2)\mu_2$ then $g(z)$ is of the form (2.8). Suppose now that $g(z)$ is of the form (2.8). Decompose $\mu(\theta)$ into two non-decreasing functions $\mu_1(\theta)$ and $\mu_2(\theta)$. The we obtain

$$\mu(\theta) = \mu_1(\theta) - \mu_2(\theta)$$

$$\int_0^{2\pi} d\mu_1(\theta) - \int_0^{2\pi} d\mu_2(\theta) = 2(\alpha_2 - \alpha_1)$$

$$\int_0^{2\pi} d\mu_1(\theta) + \int_0^{2\pi} d\mu_2(\theta) = \int_0^{2\pi} |d\mu(\theta)| < 4 - 2(\alpha_1 + \alpha_2) .$$

Let $\nu_i(\theta) = \mu_i(\theta) + c\theta$, where c is a nonnegative constant chosen to satisfy the equality:

$$\int_0^{2\pi} d\nu_1(\theta) + \int_0^{2\pi} d\nu_2(\theta) = 4 - 2(\alpha_1 + \alpha_2) .$$

Now $\nu_j(\theta)$ is a non decreasing function on $-[0 , 2\pi]$ with

$$\int_0^{2\pi} d\nu_j(\theta) = 2(1 - \alpha_j) , \quad j = 1 , 2 .$$

Let

$$f_j = z \exp \left\{ - \int_0^{2\pi} \log (1 - ze^{-i\theta}) d\nu_j(\theta) \right\} .$$

Then $f_j \in S_{\alpha_j}$ for $j = 1 , 2$ and $g = f_1/f_2$.

As a consequence of Lemma 1 we obtain by definition of the class $V(p , q)$ ((1.2) and (1.5)) that if $f \in V(p , q)$ then f' is a ratio of two starlike functions of the appropriate orders:

Lemma 2.2. *Let* $V'(p , q)$ *be the set of all functions* $f'(z)$ *where* $f(z)$ *belongs to* $V(p , q)$. *Then*

(2.9) $$V'(p , q) = S_{1-(q+p)/4, 1-(q-p)/4} .$$

For $p = 2$ *this was showed in* [2].

We now show that any function belonging to the class $V(p , q)$ is a function of bounded boundary rotation. Indeed, let

$$f'(z) = \exp \left\{ - \frac{1}{\pi} \int_0^{2\pi} \log (1 - ze^{-i\theta}) dv(\theta) \right\}$$

where $v(\theta)$ satisfies (1.5). Put

(2.10) $$u(\theta) = v(\theta) - c\theta .$$

Since $\int_0^{2\pi} e^{-in\theta}\, d\theta = 0$ for $n = 1, 2, \ldots,$ we obviously have

$$f'(z) = \exp\left\{-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-i\theta})du(\theta)\right\}. \tag{2.11}$$

Taking $c = \frac{p-2}{2}$ we see that $f'(z)$ has the Paatero representation (1.3), (1.4), i.e., $f(z)$ is a function of bounded boundary rotation.

We close this section by another integral representation for a starlike function of order α which was recently established in [5].

Theorem II. *Let* $f(z) \in S_\alpha$. *Then*

$$f(z) = \int_0^{2\pi} \frac{zd\mu(\theta)}{(1 - ze^{-i\theta})^{2(1-\alpha)}} \qquad (\alpha < 1) \tag{2.12}$$

for some $\mu \in I$.

3. Integral representations

Lemma 3.1. *Let* $f_1(z)$ *and* $f_2(z)$ *be two starlike functions (of order* 0*). Then the ratio* $f_1(z)/f_2(z)$ *is subordinate to* $((1+cz)/(1-z))^2$ *for some* $|c| = 1$.

Proof. Let $v_j(\theta) = \lim_{r\to 1} \arg f_j(re^{i\theta})$, $j = 1, 2$. It is well known [11] that the limit exists and $v_j(\theta)$ is an increasing function satisfying $v_j(2\pi) - v_j(0) = 2\pi$. Consider the function

$$u(\theta) = \arg[f_1(e^{i\theta})/f_2(e^{i\theta})] = u_1(\theta) - u_2(\theta).$$

We claim that there exists a real constant ψ such that

$$|u(\theta) - \psi| < \pi. \tag{3.1}$$

Indeed

$$u(\theta_2) - u(\theta_1) = u_1(\theta_2) - u_1(\theta_1) - (u_2(\theta_2) - u_2(\theta_1)).$$

Assume that $\theta_2 > \theta_1$. Now as each $u_j(\theta)$ is an increasing function with total variation 2π we obtain the inequality

$$-2\pi \leq u(\theta_2) - u(\theta_1) \leq 2\pi.$$

This proves the existence of ψ. The inequality (3.1) is equivalent to the following:

$$\left| \arg \left(\frac{f_1(e^{i\theta})}{f_2(e^{i\theta})} \right)^{1/2} - \psi/2 \right| \leq \pi/2$$

which shows that $f_1(z)/f_2(z)$ is subordinate to $\left(\dfrac{1 + e^{i\psi}z}{1 - z} \right)^2$.

For the following we need some definitions and notations: Let X be the torus $\{(x, y) : x = e^{i\theta}, y = e^{i\phi}\}$. Denote by J the set of probability measures $\mu(\theta, \phi)$ on X. Let $\mathrm{co}\,(V(p, q))$ and $\mathrm{co}\,(S_{\alpha,\beta})$ be the closed convex hulls of the sets $V(p, q)$ and $S_{\alpha,\beta}$ respectively with respect to natural topology of the analytic functions on the unit disc. Recall that a convergence in this topology means the uniform convergence on any compact subset of the open unit disc.

Theorem 3.1. *Let $S_{\alpha,\beta}$ be the set of the ratio of two starlike functions of order α and β respectively. Then for $\alpha \leq \beta \leq \frac{1}{2}$ $\mathrm{co}\,(S_{\alpha,\beta})$ is exactly the set*

$$\int_0^{2\pi}\int_0^{2\pi} \frac{(1 - ze^{-i\phi})^{2(1-\beta)}}{(1 - ze^{-i\theta})^{2(1-\alpha)}}\, d\mu(\phi, \theta), \quad \mu \in J. \tag{3.2}$$

Proof. Let $f_1 \in S$ and $f_2 \in S$. Then

$$f_1 = z\left(\frac{g_1}{z}\right)^{(1-\alpha)}, \quad f_2 = z\left(\frac{g_2}{z}\right)^{(1-\beta)}$$

where $g_1, g_2 \in S_0$. So

$$\frac{f_1}{f_2} = \left(\frac{g_1}{g_2}\right)^{(1-\beta)} \left(\frac{g_1}{z}\right)^{(\beta-\alpha)}. \tag{3.3}$$

For $\beta > \alpha$ the function $z\left(\dfrac{g_1}{z}\right)^{(\beta-\alpha)}$ is a starlike function of order $1 + \alpha - \beta$. Thus by (2.12) there exists $\nu \in I$ such that

$$\left(\frac{g_1}{z}\right)^{(\beta-\alpha)} = \int_0^{2\pi} \frac{d\nu(\psi)}{(1 - ze^{-i\psi})^{2(\beta-\alpha)}}.$$

Combining (3.3) with the equality above we obtain

$$\frac{f_1}{f_2} = \int_0^{2\pi} \left(\frac{g_1}{g_2}\right)^{(1-\beta)} \frac{d\nu(\psi)}{(1 - ze^{-i\psi})^{2(\beta-\alpha)}}. \tag{3.4}$$

By Lemma 3.1 $\left(\frac{g_1}{g_2}\right)^{(1-\beta)}$ is subordinate to the function $\left(\frac{1+cz}{1-z}\right)^{2(1-\beta)}$ for some $|c|=1$. The inequality $\beta \le \frac{1}{2}$ implies that $2(1-\beta) \ge 1$. Thus by the generalized Herglotz formula:

$$\left(\frac{g_1}{g_2}\right)^{(1-\beta)} = \int_0^{2\pi} \left(\frac{1+cze^{-i\phi}}{1-ze^{-i\phi}}\right)^{2(1-\beta)} d\omega(\phi)\ , \quad \omega \in I\ .$$

Replacing $\left(\frac{g_1}{g_2}\right)^{(1-\beta)}$ by its integral representation in (3.4) we obtain

$$\frac{f_1}{f_2} = \int_0^{2\pi}\int_0^{2\pi} \frac{(1+cze^{-i\phi})^{2(1-\beta)}}{(1-ze^{-i\phi})^{2(1-\beta)}} \frac{1}{(1-ze^{-i\psi})^{2(\beta-\alpha)}} d\nu(\psi)d\omega(\phi)\ . \tag{3.5}$$

The relation (2.5) implies

$$\frac{1}{(1-ze^{-i\phi})^{2(1-\beta)}} \frac{1}{(1-ze^{-i\psi})^{2(\beta-\alpha)}} = \int_0^{2\pi} \frac{d\sigma_{\phi,\psi}\theta}{(1-ze^{-i\theta})^{2(1-\alpha)}}\ , \ \sigma_{\phi,\psi} \in I \tag{3.6}$$

for any pair $(\phi\ ,\psi)$. Introducing (3.6) into the equality (3.5) we deduce that any function which belongs to the class $S_{\alpha,\beta}$ is of the form (3.2) for $\alpha \le \beta \le 1/2$. As the probability measures set J is convex and closed in the w^* topology we obtain that co $(S_{\alpha,\beta})$ is contained in the integrals (3.2). On the other hand any function of the form $\frac{(1-ze^{-i\phi})^{2(1-\beta)}}{(1-ze^{-i\theta})^{2(1-\alpha)}}$ clearly belongs to the set $S_{\alpha,\beta}$. This shows that the set (3.2) is exactly co $(S_{\alpha,\beta})$ for $\alpha \le \beta \le \frac{1}{2}$. This completes our proof.

Let $K_{\alpha,\beta}$ be

$$K_{\alpha,\beta} = \left\{ g \,\middle|\, g = \frac{(1-ze^{-i\phi})^{2(1-\beta)}}{(1-ze^{-i\theta})^{2(1-\alpha)}}\ , \ \ 0 \le \phi\ ,\ \ \theta \le 2\pi \right\}\ . \tag{3.7}$$

Clearly $K_{\alpha,\beta}$ is a compact set. Furthermore Theorem 3.1 states that co $(K_{\alpha,\beta})$ = co $(S_{\alpha,\beta})$ for $\alpha \le \beta \le \frac{1}{2}$. By the Milman theorem [6, p. 440] the extreme points of co $(S_{\alpha,\beta})$ are contained in $K_{\alpha,\beta}$ $(\alpha \le \beta \le \frac{1}{2})$. In what follows we partially characterize the extreme points of the compact convex set co $(K_{\alpha,\beta})$ for $\alpha \le \beta < 1$. We note first that the functions $\frac{1}{(1-ze^{-i\theta})^{2(\beta-\alpha)}}$ $(0 \le \theta \le 2\pi)$ are not extreme points. Indeed, the equation

$$1 = \int_0^{2\pi} \left(\frac{1 - ze^{-i\phi}}{1 - ze^{-i\phi}}\right)^{2(1-\beta)} \frac{d\phi}{2\pi}$$

implies

$$\frac{1}{(1 - ze^{-i\theta})^{2(\beta-\alpha)}} = \int_0^{2\pi} \frac{(1 - ze^{-i\phi})^{2(1-\beta)}}{(1 - ze^{-i\phi})^{2(1-\beta)}} \frac{d\phi}{(1 - ze^{-i\theta})^{2(\beta-\alpha)}\, 2\pi} .$$

Using the relations (3.6) we obtain an integral representation for the function in question:

$$\frac{1}{(1 - ze^{-i\theta})^{2(\beta-\alpha)}} = \int_0^{2\pi}\int_0^{2\pi} \frac{(1 - ze^{-i\phi})^{2(1-\beta)}}{(1 - ze^{-i\psi})^{2(1-\alpha)}} d\mu(\phi, \psi)$$

where $\mu \in J$ and μ is not a unit point mass. This shows that $\dfrac{1}{(1 - ze^{-i\theta})^{2(\beta-\alpha)}}$ is not an extreme point. The following theorem characterizes some extreme points of the set $\mathrm{co}\,(K_{\alpha,\beta})$:

Theorem 3.2. *Let* $\mathrm{co}\,(K_{\alpha,\beta})$ *be the closed convex hull of the set* $K_{\alpha,\beta}$ *defined by* (3.7). *Let* $\alpha \leq \beta < 1$. *If* $\beta \geq \frac{1}{2}$ *then all extreme points of* $\mathrm{co}\,(K_{\alpha,\beta})$ *are exactly the functions*

$$\frac{(1 - zx)^{2(1-\beta)}}{(1 - zy)^{2(1-\alpha)}} , \quad |x| = |y| = 1 \tag{3.8}$$

for $x \neq y$. *If* $\beta < \frac{1}{2}$ *then the functions above are extreme points at least if*

$$|\arg\,(-\,x\bar{y})| < \frac{2\pi}{3 - 2\beta} . \tag{3.9}$$

Proof. Assume that

$$\frac{(1 - zx_0)^{2(1-\beta)}}{(1 - zy_0)^{2(1-\alpha)}} = \int_X \frac{(1 - zx)^{2(1-\beta)}}{(1 - zy)^{2(1-\alpha)}} d\mu(x\,,y)\,, \quad x_0 \neq y_0\,, \quad |x_0| = |y_0| = 1$$

where X is a torus $\Gamma \times \Gamma$ and Γ is the unit circle. Decompose the integration over $\Gamma \times \Gamma$ into the integration over the set $\Gamma \times \{y_0\}$ and $\Gamma \times \Gamma \setminus \{y_0\}$:

$$\frac{(1 - zx_0)^{2(1-\beta)}}{(1 - zy_0)^{2(1-\alpha)}} = \int_{\Gamma\times\{y_0\}} \frac{(1 - zx)^{2(1-\beta)}}{(1 - zy_0)^{2(1-\alpha)}} d\mu(x\,,y) + \int_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{(1 - zx)^{2(1-\beta)}}{(1 - zy)^{2(1-\alpha)}} d\mu(x\,,y) .$$

Now multiply this equality by $(1-zy_0)^{2(1-\alpha)}$ and let z approach $\bar{y}_0$ radially from the unit disc. Then we get

$$
\begin{aligned}
(1-\bar{y}_0x_0)^{2(1-\beta)} &= \int\limits_{\Gamma\times\{y_0\}} (1-\bar{y}_0x)^{2(1-\beta)}\, d\mu(x, y) \\
&+ \lim_{z\to\bar{y}_0}\left[(1-zy_0)^{2(1-\alpha)} \int\limits_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\alpha)}}\, d\mu(x, y)\right].
\end{aligned} \tag{3.10}
$$

We claim that the limit in question exists and equals to zero. Indeed, let ε be an arbitrary positive number and let N be a corresponding neighbourhood of y_0 in Γ such that $\mu(\Gamma\times N\setminus\{y_0\})<\varepsilon$. Then

$$
\begin{aligned}
&\overline{\lim_{z\to\bar{y}_0}}\left|(1-zy_0)^{2(1-\alpha)}\int\limits_{\Gamma\times\Gamma\setminus\{y_0\}}\frac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\beta)}}\, d\mu(x, y)\right| \\
\le\ &\overline{\lim_{z\to\bar{y}_0}}\left|(1-zy_0)^{2(1-\alpha)}\int\limits_{\Gamma\times\Gamma\setminus N}\frac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\alpha)}}\, d\mu(x, y)\right| \\
+\ &\overline{\lim_{z\to\bar{y}_0}}\left|(1-zy_0)^{2(1-\alpha)}\int\limits_{\Gamma\times N\setminus\{y_0\}}\frac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\alpha)}}\, d\mu(x, y)\right|.
\end{aligned}
$$

As $\left|\dfrac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\alpha)}}\right|$ is uniformly bounded on the set $\Gamma\times\Gamma\setminus N$ for $z=r\bar{y}_0$ $(0<r<1)$ we deduce that the limit on this set is zero. On the set $\Gamma\times N$ the function $\left|\dfrac{(1-zx)^{2(1-\beta)}}{(1-zy)^{2(1-\alpha)}}\right|$ $|1-z\bar{y}_0|^{2(1-\alpha)}$ is bounded by $2^{2(1-\beta)}$ for $z=r\bar{y}_0$. Thus the limit on $\Gamma\times N\setminus\{y_0\}$ is less than $\varepsilon 2^{2(1-\beta)}$. This proves that the limit on the set $\Gamma\times\Gamma\setminus\{y_0\}$ is zero. The equation (3.10) reduces to

$$
(1-\bar{y}_0x_0)^{2(1-\beta)} = \int\limits_{\Gamma}(1-\bar{y}_0x)^{2(1-\beta)}d\mu_0(x) \tag{3.11}
$$

where $\mu_0(x)$ is an increasing function with total variation not greater than 1. In fact $\mu_0(S)=\mu(S\times\{y_0\})$ for any measurable set in Γ. The function $\dfrac{(1-zx_0)^{2(1-\beta)}}{(1-zy_0)^{2(1-\alpha)}}$ is a fortiori an extreme point of the set co $(K_{\alpha,\beta})$ if and only if a unit point mass concentrated at x_0 satisfies the equation (3.11).

(i) Suppose that $\frac{1}{2}\le\beta<1$. Then $(1+z)^{2(1-\beta)}$ is a convex function which maps the unit disc onto the convex domain D_β. Note that $w=0$

lies on the boundary of D_β. For fixed t, $0 \leq t \leq 1$ consider the following set

$$w = t \int_0^{2\pi} (1 - e^{i\theta})^{2(1-\beta)} d\mu(\theta) , \quad \mu \in I . \tag{3.12}$$

Clearly this is the domain tD_β. Thus if w is a boundary point of the domain D_β and $w \neq 0$ then this point has a unique representation (3.12) with the corresponding unit point mass measure $\mu(\theta)$ and $t = 1$. This shows that if $\bar{y}_0 x_0 \neq 1$ then (3.11) implies that $\mu_0(x)$ is a unit point mass measure which proves that $\dfrac{(1 - zx_0)^{2(1-\beta)}}{(1 - zy_0)^{2(1-\alpha)}}$ is an extreme point.

(ii) Suppose that $\beta < \frac{1}{2}$. Then the domain D_β is not convex anymore. Consider the curve $(1 + e^{i\theta})^{2(1-\beta)} = u(\theta) + iv(\theta)$ for $-\pi \leq \theta \leq \pi$:

$$\begin{aligned} u(\theta) &= 2^{2(1-\beta)} \left[\cos \frac{\theta}{2}\right]^{2(1-\beta)} \cos (1 - \beta)\theta , \\ v(\theta) &= 2^{2(1-\beta)} \left[\cos \frac{\theta}{2}\right]^{2(1-\beta)} \sin (1 - \beta)\theta . \end{aligned} \tag{3.13}$$

Clearly this curve is symmetric with respect to the real axis. Let ϕ be the first positive θ which satisfies $u'(\phi) = 0$. A straightforward calculation shows that $\phi = 2\pi/(3 - 2\beta)$. Now it is easy to see that the convex hull of D is bounded by the curves:

$$\begin{aligned} &u(\theta) + iv(\theta) \quad \text{for} \quad |\theta| \leq \phi , \\ &v(\phi) + iy \qquad \text{for} \quad |y| \leq v(\phi) . \end{aligned}$$

Therefore if $|\arg(-\bar{y}_0 x_0)| \leq \phi$ then the point $(1 - \bar{y}_0 x_0)^{2(1-\beta)}$ is an extreme point of the convex hull of D_β. In that case the equality (3.11) holds only if $\mu_0(x)$ is a unit mass point concentrated at x_0, i.e. $\dfrac{(1 - zx_0)^{2(1-\beta)}}{(1 - zy_0)^{2(1-\alpha)}}$ is an extreme point of the set co $(K_{\alpha,\beta})$. This completes the proof of Theorem 3.2.

The case $\beta = 1/2$ was proved in [4] and our proof actually extends the proof given there. Returning to the class of functions of bounded boundary rotation $V(p, q)$ we obtain

Theorem 3.3. *Suppose that* $0 \leq p \leq q - 2$. *Then the closed convex hull of the set* $V(p, q)$ *of functions of bounded boundary rotation is exactly the set*

$$\int_0^z dw \left(\int_0^{2\pi} \int_0^{2\pi} \frac{(1 - we^{-i\phi})^{\frac{q-p}{2}}}{(1 - we^{-i\theta})^{\frac{q+p}{2}}} d\mu(\phi , \theta) \right) \tag{3.14}$$

where μ *ranges over all probability measures in* J. *Furthermore if* θ *and* ϕ *satisfy the conditions*

$$|\theta - \phi + \pi| \le \frac{4\pi}{2 + q - p} \tag{3.15}$$

and $e^{i\theta} \neq e^{i\phi}$ *then the function*

$$\int_0^z \frac{(1 - we^{-i\phi})^{\frac{q-p}{2}}}{(1 - we^{-i\theta})^{\frac{q+p}{2}}} dw \tag{3.16}$$

is an extreme point in $\mathrm{co}\,(V(p\,,q))$.

Proof. By Lemma 2.2 $V'(p\,,q) = S_{\alpha,\beta}$ where $\alpha = 1 - \frac{q+p}{4}$, $\beta = 1 - \frac{q-p}{4}$. Using Theorem 3.1 we obtain the integral representation (3.14). Noting that the linear transformation $f \to f'$ of $\mathrm{co}\,(V(p\,,q))$ onto $\mathrm{co}\,(V'(p\,,q))$ is one to one we realize that the set of the extreme points of $\mathrm{co}\,(V(p\,,q))$ is transformed one to one on the set of extreme points of $\mathrm{co}\,(V'(p\,,q))$. Using now Theorem 3.2 we have that the function (3.16) which satisfies condition (3.15) is an extreme point of $\mathrm{co}\,(V(p\,,q))$.

In [3] the theorem was established for the special case $p = 2$.

4. Coefficient estimates

Let $h(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ and $H(z) = \sum_{n=0}^{\infty} \beta_n z^n$ be analytic in $|z| < 1$. By $h(z) \ll H(z)$ we mean $|\alpha_n| \le |\beta_n|$ for $n = 0\,, 1\,, \ldots$ In what follows we need the following [1]:

Theorem III. *Let* $|c| = 1$ *and* $a \ge 1$. *Then*

$$\left(\frac{1 + cz}{1 - z}\right)^a \ll \left(\frac{1 + z}{1 - z}\right)^a . \tag{4.1}$$

Theorem 4.1. *Let* $\alpha \le \beta < 1$ *and* $\alpha + \beta \le 1$. *Then any* $g(z)$ *belonging to the set* $\mathrm{co}\,(S_{\alpha,\beta})$ *satisfies sharp inequalities:*

$$g(z) \ll \frac{(1 + z)^{2(1-\beta)}}{(1 - z)^{2(1-\alpha)}} . \tag{4.2}$$

Proof. Clearly it is enough to show (4.2) for $g(z)$ belonging to $S_{\alpha,\beta}$. As $\alpha \le \beta$ by (3.4) $g(z)$ has the representation

$$g(z) = \int_0^{2\pi} \left(\frac{g_1}{g_2}\right)^{(1-\beta)} \frac{d\nu(\psi)}{(1 - ze^{-i\psi})^{2(\beta-\alpha)}}\ , \quad \nu \in I\ ,$$

where $g_1 , g_2 \in S_0$. Thus the theorem follows if (4.2) holds for

$$\left(\frac{g_1}{g_2}\right)^{(1-\beta)} \frac{1}{(1 - ze^{-i\psi})^{2(\beta-\alpha)}}\ .$$

Introducing a new variable $\xi = ze^{-i\psi}$ we reduce the inequality (4.2) to the inequality

$$\left(\frac{g_1}{g_2}\right)^{(1-\beta)} \frac{1}{(1 - z)^{2(\beta-\alpha)}} \ll \frac{(1 + z)^{2(1-\beta)}}{(1 - z)^{2(1-\alpha)}} \tag{4.3}$$

for $g_1 , g_2 \in S_0$. Noting that $\dfrac{z}{(1 - z)^{\beta-\alpha}}$ and $\dfrac{z}{(1 + z)^{\beta-\alpha}}$ belong to $S_{1-\frac{\beta-\alpha}{2}}$ we see that the functions

$$h_1(z) = \frac{1}{z}\left[z\left(\frac{g_1}{z}\right)^{(1-\beta)} \frac{z}{(1 - z)^{(\beta-\alpha)}}\right],$$

$$h_2(z) = \frac{1}{z}\left[z\left(\frac{g_2}{z}\right)^{(1-\beta)} \frac{z}{(1 + z)^{(\beta-\alpha)}}\right]$$

belong to $S_{\frac{\beta+\alpha}{2}}$. Therefore the inequality (4.3) is equivalent to

$$\frac{h_1}{h_2}\,\frac{1}{(1 - z^2)^{(\beta-\alpha)}} \ll \frac{(1 + z)^{2(1-\beta)}}{(1 - z)^{2(1-\alpha)}}\ .$$

As the function $\dfrac{1}{(1 - z^2)^{(\beta-\alpha)}}$ has nonnegative coefficients in its expansion about the origin the inequality above is certainly true if we show the inequality

$$\frac{h_1}{h_2} \ll \left(\frac{1 + z}{1 - z}\right)^{2-(\alpha+\beta)}\ . \tag{4.4}$$

By Lemma (3.1) it follows that h_1/h_2 is subordinate to the function $\left(\dfrac{1 + cz}{1 - z}\right)^{2-(\alpha+\beta)}$ for some $|c| = 1$. Now the assumption $\alpha + \beta \leq 1$ enables us to use the generalized Herglotz formula

$$\frac{h_1}{h_2} = \int_0^{2\pi} \left(\frac{1 + cze^{-i\theta}}{1 - ze^{-i\theta}}\right)^{2-(\alpha+\beta)} d\mu(\theta)\ , \quad \mu \in I\ .$$

Finally the inequality (4.1) implies (4.4). This proves (4.2). To show that this inequality is sharp we simply note that $\frac{z}{(1-z)^{2(1-\alpha)}} \in S_\alpha$ and $\frac{z}{(1+z)^{2(1-\beta)}} \in S_\beta$ and thus $\frac{(1+z)^{2(1-\beta)}}{(1-z)^{2(1-\alpha)}} \in S_{\alpha,\beta}$. This concludes the proof of the theorem.

Let

$$\frac{(1+z)^s}{(1-z)^t} = 1 + \sum_{n=1}^{\infty} b_n(s, t)z^n. \tag{4.5}$$

Combining Lemma 2.2 with Theorem 4.1 we obtain

Theorem 4.2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be a function of bounded boundary rotation belonging to the set* $V(p, q)$. *If* $p \geq 0$ *and* $q \geq \max(p, 2)$ *then*

$$|a_n| \leq \frac{1}{n} b_{n-1}\left(\frac{q-p}{2}, \frac{q+p}{2}\right), \quad n = 2, \ldots . \tag{4.6}$$

Equality holds in (4.6) *for the function*

$$f_{p,q}(z) = \int_0^z \frac{(1+w)^{\frac{q-p}{2}}}{(1-w)^{\frac{q+p}{2}}} dw \tag{4.7}$$

and its rotations.

For $p = 2$ the class $V(2, q)$ is the well known class Vq of locally univalent analytic functions that map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $q\pi$. In that case the inequalities (4.6) were established recently in [3] and [1].

5. Inequalities for integral mean values

Theorem 5.1. *Let* $g(z)$ *belong to* $S_{\alpha,\beta}$. *Then*

$$\int_{-\pi}^{\pi} |g(re^{i\theta})|^t d\theta \leq \int_{-\pi}^{\pi} \frac{|1+re^{i\theta}|^{2t(1-\beta)}}{|1-re^{i\theta}|^{2t(1-\alpha)}} d\theta \tag{5.1}$$

for any $t > 0$ *and* $0 < r < 1$. *This inequality is best possible.*

To prove this theorem we need the following lemma:

Lemma 5.1. *Let* $u_1(\theta), \ldots, u_n(\theta)$ *be* n *periodic non-negative functions* *on* $[-\pi, \pi]$. *Assume furthermore that each* $u_i(\theta)$ *is symmetric on* $[-\pi, \pi]$ *and decreases on* $[0, \pi]$. *Then*

$$\int_{-\pi}^{\pi} \prod_{i=1}^{n} u_i(\theta - \theta_i) d\theta \leq \int_{-\pi}^{\pi} \prod_{i=1}^{n} u_i(\theta) d\theta \tag{5.2}$$

for any $\theta_1, \ldots, \theta_n$.

Proof. Clearly $u_i(\theta - \theta_i)$ is equimeasurable to $u_i(\theta)$. Furthermore $u_1(\theta), \ldots, u_n(\theta)$ arranged in the same order. It now follows from a result of Lorentz [8] that

$$\int_{-\pi}^{\pi} \Phi(u_1(\theta - \theta_1), \ldots, u_n(\theta - \theta_n)) d\theta \leq \int_{-\pi}^{\pi} \Phi(u_1(\theta), \ldots, u_n(\theta)) d\theta .$$

If $\dfrac{\partial^2 \Phi}{\partial u_i \partial u_j} \geq 0$, $j = 2, \ldots, n$, $i = 1, \ldots, j-1$.

Obviously the function $\Phi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i$ satisfies these conditions. This proves (5.2)

Proof of Theorem 5.1. Let $g \in S_{\alpha,\beta}$. Then $g = f_1/f_2$ where $f_1 \in S_{\alpha_1}$ and $f_2 \in S_{\alpha_2}$ $(\alpha_1 = \alpha, \alpha_2 = \beta)$. The functions f_1 and f_2 have an integral representation:

$$f_j(z) = z \exp \left\{ -2(1-\alpha_j) \int_0^{2\pi} \log(1 - ze^{-i\theta}) d\mu_j(\theta) \right\}, \quad \mu_j \in I .$$

Each μ_j can be approximated by step functions from the set I. Therefore $g(z)$ can be approximated by the functions of the form

$$g(z) = \prod_{j=1}^{m} (1 + ze^{-i\phi_j})^{a_j} \prod_{k=1}^{n} (1 - ze^{-i\psi_k})^{-b_k} \tag{5.3}$$

where

$$a_j \geq 0, \quad \sum_{j=1}^{m} a_j = 2(1-\alpha_2)$$

$$b_k \geq 0, \quad \sum_{k=1}^{n} b_k = 2(1-\alpha_1) .$$

So

$$\int_{-\pi}^{\pi} \prod_{j=1}^{m} |1 + re^{i\theta} e^{-i\phi_j}|^{ta_j} \prod_{k=1}^{n} |1 - re^{i\theta} e^{-i\psi_k}|^{-tb_k} d\theta$$

$$= \int_{-\pi}^{\pi} \prod_{j=1}^{m} u_j(\theta - \phi_j) \prod_{k=1}^{n} v_k(\theta - \psi_k) d\theta ,$$

where

$$u_j(\theta) = |1 + re^{i\theta}|^{ta_j} , \quad v_k(\theta) = |1 - re^{i\theta}|^{-tb_k} .$$

By Lemma 5.1 we obtain the inequality (5.1) for $g(z)$ of the form (5.3). As any function in $S_{\alpha,\beta}$ can be approximated by functions of the form (5.3) we deduce the inequality (5.1). The sign of equality holds for the function $\dfrac{(1+z)^{2(1-\beta)}}{(1-z)^{2(1-\alpha)}}$ which belongs to $S_{\alpha,\beta}$.

Noting that for $t \geq 1$ the functional

$$\left(\int_{-\pi}^{\pi} |g(re^{i\theta})|^t d\theta \right)^{1/t} \tag{5.4}$$

is convex on the set co $(S_{\alpha,\beta})$ we get

Corollary 5.1. *Let* $t \geq 1$. *Then any function* $g(z)$ *from the set* co $(S_{\alpha,\beta})$ *satisfies the inequalities* (5.1).

Using the connection between $V(p, q)$ and $S_{\alpha,\beta}$ we easily obtain from Theorem 5.1

Theorem 5.2. *Let* f *belong to the set* $V(p, q)$. *Then*

$$\int_{-\pi}^{\pi} |f'(re^{i\theta})|^t d\theta \leq \int_{-\pi}^{\pi} \frac{|1 + re^{i\theta}|^{\frac{t(q-p)}{2}}}{|1 - re^{i\theta}|^{\frac{t(q+p)}{2}}} d\theta \tag{5.5}$$

for any $t > 0$ *and* $0 < r < 1$. *The equality sign holds for the function* (4.7) *and its rotations.*

Acknowledgement

The authors would like to thank Professor W. K. Hayman for reading the manuscript and giving some helpful suggestions.

Technion, Israel Institute of Technology
Haifa, Israel

Weizmann Institute Science,
Rehovot, Israel

References

[1] AHARONOV, D. and FRIEDLAND, S.: On an inequality connected with the coefficient conjecture for functions of bounded boundary rotation. - Ann. Acad. Sci. Fenn. A I 524 (1972), 1—14.

[2] BRANNAN, D. A.: On functions of bounded boundary rotation I. - Proc. Edinburgh Math. Soc. 16 (1969), 339—347.

[3] —»— CLUNIE, J. G. and KIRWAN, W. E.: On the coefficient problem for functions of bounded boundary rotation. - Ann. Acad. Sci. Fenn. A I 523 (1973), 1—18.

[4] BRICKMAN, L., MACGREGOR, T. H., and WILKEN, D. R.: Convex hulls of some classical families of univalent functions. - Trans. Amer. Math. Soc. 156 (1971), 91—107.

[5] —»— HALLENBECK, D. J., MACGREGOR, T. H. and WILKEN, D. R.: Convex hulls and extreme points of families of starlike and convex mappings (to appear).

[6] DUNFORD, N. and SCHWARTZ, J. T.: Linear operators, Part I. - Interscience, New York-London 1958.

[7] LEHTO, O.: On the distortion of conformal mappings with bounded boundary rotation. - Ann. Acad. Sci. Fenn. A I 124 (1952), 1—14.

[8] LORENTZ, G. G.: An inequality for rearrangement. - Am. Math. Monthly, 60 (1953), 176—179.

[9] PAATERO, V.: Über die konformen Abbildungen von Gebieten deren Ränder von beschränkter Drehung sind. - Ann. Acad. Sci. Fenn. Ser. A 33 (1931), 1—77.

[10] PINCHUK, B.: On starlike and convex functions of order α. - Duke Math. J. 35 (1968), 721—734.

[11] POMMERENKE, CH.: On starlike and convex functions. - J. London Math. Soc. 37 (1962), 209—224.

Printed in December 1974