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BOUNDARY MAPPINGS OF GEOMETRIC ISOMORPHISMS OF FUCHSIAN GROUPS

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Boundary mappings of geometric isomorphisms of Fuchsian groups

The object of the present paper is to apply certain ergodic theoretical results of E. Hopf ([2], [3]) to the study of boundary mappings of geometric isomorphisms of Fuchsian groups.

1. An isomorphism $j : G_1 \to G_2$ of two Fuchsian groups acting in the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ is said to be *geometric* if there exists a homeomorphism $\Phi : D \to D$ inducing the isomorphism j, i.e. if we have

$$\Phi \circ g = j(g) \circ \Phi \tag{1}$$

for all $g \in G_1$. If both groups G_1, G_2 are the first kind, then Φ has a unique homeomorphic extension $\hat{\Phi} : \bar{D} \to \bar{D}$, so that also the *boundary mapping* $\varphi = \hat{\Phi}|_{BdD}$ satisfies

$$\varphi \circ g = j(g) \circ \varphi , \quad g \in G_1 . \tag{2}$$

Unlike Φ, the homeomorphism $\varphi : \mathbf{T} \to \mathbf{T}$ of the unit circle $\mathbf{T} = Bd\, D$ is uniquely determined by the isomorphism j ([5] §3, [6] 3.B). In the following, all Fuchsian groups are supposed to be of the first kind.

Occasionally we may study Fuchsian groups which act in the upper half plane H instead of D. In that case we assume that the boundary mapping ψ fixes the point ∞, so that ψ will be a strictly monotone mapping $\psi : \mathbf{R} \to \mathbf{R}$.

2. We normalize the Lebesgue measure τ_1 on $\mathbf{T}$ by $\tau_1(\mathbf{T}) = 1$, and the torus $\mathbf{T} \times \mathbf{T}$ has the product measure $\tau_2 = \tau_1 \times \tau_1$.

As a homeomorphism of the unit circle a boundary mapping $\varphi : \mathbf{T} \to \mathbf{T}$ has a derivative $\varphi' \in \mathbf{C}$ a.e. on $\mathbf{T}$. Similarly a real-valued boundary mapping $\psi : \mathbf{R} \to \mathbf{R}$ which corresponds to Fuchsian groups acting in H has a finite derivative $\psi' \in \mathbf{R}$ a.e. on $\mathbf{R}$. Because ψ is monotone, the derivative ψ' cannot change its sign.

Since the cross ratio $[z_1, z_2, z_3, z_4]$ is preserved under Moebius transformations it follows that also the differential

$$dz_1\, dz_2\, (z_1 - z_2)^{-2} = -[z_1, z_2, z_1 + dz_1, z_2 + dz_2] \tag{3}$$

remains invariant. Let now $\varphi : \mathbf{T} \to \mathbf{T}$ be the boundary mapping corresponding to a geometric isomorphism $j : G_1 \to G_2$. The invariance of (3) implies that also the expression

$$\chi_\varphi(z_1 , z_2) = \varphi'(z_1)\, \varphi'(z_2) \left[\frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2}\right]^{-2} \tag{4}$$

is invariant under Moebius transformations. Thus if h, k are two Moebius transformations, we have

$$\chi_\xi(k(z_1) , k(z_2)) = \chi_\varphi(z_1 , z_2) \tag{5}$$

for $\xi = h \circ \varphi \circ k^{-1} : k\mathbf{T} \to h\mathbf{T}$. Since G_1 and G_2 have conjugate groups acting in H, we see that $\chi_\varphi : \mathbf{T} \times \mathbf{T} \to \mathbf{R}$ is a non-negative measurable function. Further it follows from (2) that χ_φ is *automorphic* with respect to G_1; that is,

$$\chi_\varphi(gz_1 , gz_2) = \chi_\varphi(z_1 , z_2) \tag{6}$$

for all $g \in G_1$.

3. *The class* O_{HB}. Suppose that the Riemann surface $S = D/G$ corresponding to a Fuchsian group G is of class O_{HB}, i.e. S does not have non-constant bounded harmonic functions, or equivalently that there is no non-constant G-automorphic bounded harmonic function in D. Using the Poisson representation we see that all G-automorphic bounded harmonic functions are constant if and only if the action of G on $\mathbf{T}$ is metrically transitive, i.e. if and only if a measurable G-invariant subset $E \subset \mathbf{T}$ has either measure $\tau_1(E) = 0$ or $\tau_1(E) = 1$.

Theorem 1. Let φ be the boundary mapping of a geometric isomorphism $j : G_1 \to G_2$. If one of the Riemann surfaces $S_i = D/G_i$, $i = 1, 2$, is of class O_{HB}, then the mapping φ is either absolutely continuous or completely singular.

Proof. Suppose that S_2 is of class O_{HB}. If φ is not absolutely continuous, there exists a Borel set $E \subset \mathbf{T}$ such that $\tau_1(E) = 0$, $\tau_1(\varphi(E)) > 0$. The set $F_1 = G_1 E$ is invariant under G_1, and $F_2 = \varphi(F_1) = G_2\, \varphi(E)$ under G_2. Now $\tau_1(F_1) = 0$, and $\tau_1(F_2) = 1$ since G_2 is metrically transitive. Thus both φ and φ^{-1} are completely singular.

4. *The Hopf classification.* Let S be a hyperbolic Riemann surface, $T(S)$ the tangent manifold of S, and $\sigma_x(v , w)$, $x \in S$, $v, w \in T_x(S)$, the hyperbolic metric of S. Since S is a complete Riemannian manifold with respect to the hyperbolic metric, the *geodesic flow* β_t determined

by the Lagrangian $L(x, \dot{x}) = \sigma_x(\dot{x}, \dot{x})$ is globally defined on $T(S)$, i.e. $\beta_t : T(S) \to T(S)$, $t \in \mathbf{R}$, is a one-parameter transformation group. The surfaces $\mathcal{E}_c \subset T(S)$ of constant energy, $L(x, v) = c$, are invariant under the geodesic flow, and since the flow β_t is essentially similar on every $\mathcal{E}_c$, $c > 0$, we can consider only $\mathcal{E} = \mathcal{E}_1$. The geodesic flow β_t restricted to $\mathcal{E}$ is simply the flow of unit speed along geodesics.

E. Hopf has shown that the geodesic flow β_t of a hyperbolic Riemann surface S always is either ergodic or dissipative on $\mathcal{E}$ ([2], [3]). The surface S is said to be *of the first class* in the ergodic case, and *of the second class* in the dissipative case. Suppose now that the surface S is represented by a Fuchsian group G acting in D, $S = D/G$. It follows then further that S is of the first class if and only if the action

$$\{g, (x, y)\} \mapsto (gx, gy), \quad g \in G, \quad (x, y) \in \mathbf{T} \times \mathbf{T}, \tag{7}$$

of G on the torus $\mathbf{T} \times \mathbf{T}$ is metrically transitive, i.e. if and only if each measurable G-invariant subset $E \subset \mathbf{T} \times \mathbf{T}$ has either measure $\tau_2(E) = 0$ or $\tau_2(E) = 1$ ([2] 8.1). It follows immediately that every surface of the first class is always of class O_{HB}.

Theorem 2. Suppose that one of the Riemann surfaces $S_i = D/G_i$, $i = 1, 2$, is of the first class. Then for each geometric isomorphism $j : G_1 \to G_2$ either the boundary mapping φ is completely singular or the isomorphism is induced by a Moebius transformation on $\mathbf{T}$.

Proof. Let S_1 be of the first class, so that the boundary mapping is either absolutely continuous or completely singular by the preceding theorem. Since χ_φ is G_1-automorphic by (6), it is equal to a constant a.e. on $\mathbf{T} \times \mathbf{T}$. Obviously we must have $\chi_\varphi = 1$ a.e. in the case of absolute continuity, and $\chi_\varphi = 0$ a.e. in the singular case.

Suppose now that φ is absolutely continuous. Using appropriate Moebius transformations h, k we can find groups $G_1' = hG_1h^{-1}$, $G_2' = kG_2k^{-1}$ acting in H with a real-valued boundary mapping

$$\psi = k \circ \varphi \circ h^{-1} : \mathbf{R} \to \mathbf{R}.$$

We may further suppose that $\psi(0) = 0$, $\psi'(0) = 1$, so that ψ satisfies on $\mathbf{R}$ the differential equation

$$\psi'(x) = \psi(x)^2 / x^2 \tag{8}$$

because $\chi_\psi = 1$ a.e. on $\mathbf{R} \times \mathbf{R}$. But given the initial value $\psi(0) = 0$, $\psi(x) = x$ is the only solution of (8) continuous on all of $\mathbf{R}$. Thus $\varphi = k^{-1} \circ h$, so that the isomorphism j is induced on $\mathbf{T}$ by a Moebius transformation.

5. A Riemann surface $S = D/G$ can obviously be of the first class only if G is a Fuchsian group of the first kind, but this condition is by far insufficient. If $S \subset \hat{\mathbf{C}}$ is a hyperbolic planar surface, the covering group of S is of the first kind if the complement $\hat{\mathbf{C}} \setminus S$ is totally disconnected, but S is of class O_{HB} if and only if $\hat{\mathbf{C}} \setminus S$ has vanishing logarithmic capacity.

If A is the hyperbolic area of a hyperbolic Riemann surface S, the volume of $\mathscr{E}$ is $2\pi A$ (cf. n:o 4), so that all Riemann surfaces of finite hyperbolic area are of the first class by Poincaré's recurrence theorem ([2] 7.1, [3]). Now the hyperbolic area of a Riemann surface $S = D/G$ is finite if and only if G is a finitely generated group of the first kind ([4], Theorem 5). Thus the Riemann surface $S = D/G$ is of the first class for all *finitely generated Fuchsian groups* G *of the first kind.*

Theorem 3. Suppose that the geometric isomorphism $j : G_1 \to G_2$ of two finitely generated Fuchsian groups of the first kind acting in H has an increasing boundary mapping $\psi : \mathbf{R} \to \mathbf{R}$. Then ψ is either affine or a completely singular quasisymmetric function.

Proof. If G is a finitely generated Fuchsian group of the first kind, the Riemann surface $S = (S_G, n_G) = H/G$ is a pointed surface of finite type, i.e. S is a compact surface S' with finitely many punctures; further, the support of n_G is finite. Thus in the case of finitely generated groups of the first kind there always exists a quasiconformal mapping $\Phi : H \to H$ inducing the given isomorphism j (cf. [5] Theorem 2.1, [6] 2.B), so that the boundary mapping $\psi : \mathbf{R} \to \mathbf{R}$ must be quasisymmetric, and the conclusion follows now from theorem 2.

Recently Sorvali has obtained results of a similar kind (cf. [5] Theorem 5.1). For quasisymmetric functions, cf. also Beurling — Ahlfors [1], for singular functions especially §7.

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