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ON A DENSITY THEOREM OF H. L. MONTGOMERY FOR *L*-FUNCTIONS

BY

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1. Introduction

1.1. The density theorem. For $\frac{1}{2} \leqq \alpha < 1$, $T > 0$, and for integral $D \geqq 1$ let $N(\alpha, T, D)$ stand for the number of zeros of the function

$$\prod_{\chi \bmod D} L(s, \chi)$$

in the rectangle

$$R(\alpha, T): \qquad \alpha \leqq \sigma \leqq 1, \quad |t| \leqq T.$$

The quantity $N(\alpha, T, D)$ has been extensively investigated; the best known general upper estimate (apart from a logarithmic factor) is the following result of Montgomery:

$$N(\alpha, T, D) \ll (DT)^{\min\left(\frac{3}{2-\alpha}, \frac{2}{\alpha}\right)(1-\alpha)} l^{14} \tag{1.1}$$

with $l = \log DT$, uniformly for $\frac{1}{2} \leqq \alpha < 1$, $T \geqq 2$, $D \geqq 1$. (For history and reference, see Montgomery [7], Chapter 12; in particular, (1.1) is essentially theorem 12.1.)

The result (1.1) is obtained using certain theorems on Dirichlet polynomials involving Dirichlet characters, and a mean-value estimate for $|L(\frac{1}{2} + it, \chi)|^4$. The object of the present paper is to show that the method of Montgomery can be refined to yield the following improvement of (1.1).

Theorem. *For any fixed $\varepsilon > 0$ there exist (calculable) numbers $C = C(\varepsilon)$, $B = B(\varepsilon)$ such that uniformly for $\frac{1}{2} \leqq \alpha < 1$, $T \geqq 2$, $D \geqq 1$*

$$N(\alpha, T, D) \leqq C(DT)^{(\omega(\alpha)+\varepsilon)(1-\alpha)} l^B \tag{1.2}$$

with

$$\omega(\alpha) = \frac{3}{2-\alpha} \textit{ for } \tfrac{1}{2} \leqq \alpha \leqq \frac{\sqrt{17}-1}{4} = 0.78077\ldots, \tag{1.3}$$

$$\omega(\alpha) = \frac{6\alpha - 3}{6\alpha - 4} \textit{ for } \frac{\sqrt{17}-1}{4} \leqq \alpha \leqq \tfrac{5}{6}, \tag{1.4}$$

(1.5) $\quad \omega(\alpha) = 2$ *for* $\frac{5}{6} \leqq \alpha < 1$.

Consequently, in any case,

$$N(\alpha\,, T\,, D) \leqq C(DT)^{(\omega_0+\varepsilon)(1-\alpha)}\, l^B \tag{1.6}$$

with $\omega_0 = \frac{3}{16}(9 + \sqrt{17}) = 2.46058\ldots$

1.2. Remarks on special cases. In the case of the zeta-function ($D = 1$) , better results have been obtained by Huxley [4] and Bombieri (announced in a lecture in Moscow in September 1971). For example, they find in (1.6) $\omega_0 = 2.4$ for $D = 1$. Also the range of validity of the density hypothesis is found to be wider than our range $\alpha \geqq \frac{5}{6}$.

Another interesting special case is $T \ll 1$. In this case it can be proved that $\omega(\alpha)$ is *strictly* less than 2 for α near 1. Indeed, one may take in (1.2)

$$\omega(\alpha) = \frac{3}{4\alpha - 2}$$

for $\frac{3}{4} \leqq \alpha < 1$, so that $\omega(\alpha) < 2$ for $\alpha > \frac{7}{8}$. A proof of this requires estimates of Burgess for L-functions (see [3]). However, we do not go into details here.

1.3. Method of the proof. For the proof of the theorem we shall introduce two new ideas into the Dirichlet polynomial method of Montgomery.

Firstly, given a point $s_0 = \sigma_0 + it_0$ in $R(\alpha\,, T)$ and a character χ (mod D) , there exists a region

$$\sigma > \sigma(s_0\,, \chi)\,,\ |t - t_0| \leqq 2l^2\,, \tag{1.7}$$

free of zeros of $L(s\,, \chi)$. If $s = \sigma + it$ lies in (1.7), and, moreover,

$$\sigma \geqq \sigma(s_0\,, \chi) + \varepsilon\,,\ |t - t_0| \leqq l^2\,, \tag{1.8}$$

then we have by function-theoretic arguments (the Borel-Carathéodory theorem and the three-circles theorem of Hadamard applied to the function $\log\,(L(s\,, \chi))$ the estimate

$$|L(s\,, \chi)| \leqq C_1(\varepsilon)(DT)^{\varepsilon}\,, \tag{1.9}$$

provided in the case $\chi = \chi_0$ we also have $|t_0| \geqq 2l^2$ (a principle, used by Bombieri for the zeta-function in [2]).

Secondly, we shall apply the Halász-Montgomery method to a set of auxiliary Dirichlet polynomials of the form $B^u(s\,, \chi)$, where $B(s\,, \chi)$ are »short» Dirichlet polynomials, serving as indicators of zeros, and u is a suitable positive integer. The estimate (1.9) appears to be useful in sharpening the Halász-Montgomery method.

1.4. Arithmetical applications. The quantity ω_0 in (1.6) plays an important role in prime number theory. Two examples:

(i) Let p be a fixed prime $\geqq 3$, and let D run over the sequence $D = p^n$, $n = 1, 2, \ldots$ Let $p(D, k)$ stand for the least prime $\equiv k (\mathrm{mod}\, D)$. In [1] it is proved that $p(D, k) \leqq C(p, \varepsilon) D^{8/3+\varepsilon}$ if $(D, k) = 1$. Our theorem gives a similar result with $\frac{8}{3}$ replaced by ω_0.

(ii) Let q be an odd prime, $(q, k) = 1$, and let $G(q, k)$ be the least Goldbach's number (a number of the form $p_1 + p_2$ with p_1, p_2 primes) which is $\equiv k (\mathrm{mod}\, q)$. Then we have by the method of [5] the estimate

$$G(q, k) \leqq C_2(\varepsilon) q^{\omega_0/2+\varepsilon} .$$

1.5. Notation. Throughout, ε will be a fixed number, $0 < \varepsilon < \frac{1}{100}$, and the constants $B_1, B_2, \ldots$ will depend on ε. The constants in the symbols $\ll$ are absolute (numerical), in $\ll_\varepsilon$ they depend on ε.

As usual, $\mu(n)$ and $\varphi(n)$ stand for Möbius's and Euler's functions, and $\tau(n)$ for the number of positive divisors of n.

2. Classification of the zeros

2.1. The class of the »good» zeros. Let B_1 be a positive number to be specified from certain conditions later. Let B_2 and B_3 be (sufficiently large) integers such that

$$N(\alpha, T, D) \leqq B_2 (DT)^{2(1-\alpha)} l^{B_3} \tag{2.1}$$

for all $\frac{1}{2} \leqq \alpha < 1$, $T \geqq 2$, $D \geqq 1$, $DT \leqq B_1$. Let $\xi \geqq 2 + 73\varepsilon$ be a real number which will be fixed during the following construction. *All subsequent constants will be independent of* B_1, B_2, B_3, *and* ξ.

Using the estimates (1.1) and (2.1), and supposing $B_2 \geqq 1$ and $B_3 \geqq 1$ to be sufficiently large, we have the following alternatives: either

$$N(\alpha, T, D) \leqq B_2 (DT)^{\xi(1-\alpha)} l^{B_3} \tag{2.2}$$

for all $\frac{1}{2} \leqq \alpha < 1$, $T \geqq 2$, $D \geqq 1$, or there exists a triple (α', T', D') with

$$\tfrac{1}{2} \leqq \alpha' \leqq 1 - 5\varepsilon , \ T' \geqq 2 , \ D' \geqq 1 , \ D'T' > B_1 \tag{2.3}$$

such that (2.2) holds for all $\alpha \geqq \alpha' + \varepsilon$, $T \geqq 2$, $D \geqq 1$, but

$$N(\alpha', T', D') > B_2 (D'T')^{\xi(1-\alpha')} l'^{B_3} , \ l' = \log D'T' . \tag{2.4}$$

Suppose that the second alternative occurs. Then, for the proof of the theorem, we need an estimate for $\alpha' = \alpha'(\xi)$. Since the theorem is interesting for $\alpha > \frac{3}{4}$ only, we may suppose that $\alpha' > \frac{3}{4}$. The triple (α', T', D') will be fixed during the proof, and will henceforth be written simply (α, T, D).

Consider the zeros $\varrho = \beta + i\tau$ of all L-functions (mod D) in the rectangle $R(\alpha, T)$, and pick out a subset A (the »good» zeros) of these zeros from the following conditions:

(i) $$\alpha \leqq \beta \leqq \alpha + \varepsilon\, , 0 \leqq \tau \leqq T\, ,$$

(ii) for the zeros $\varrho_j = \beta_j + i\tau_j \in A$ of $L(s, \chi_j)$, respectively, we have

(2.5) $$|\tau_i - \tau_j| \geqq 2l^2$$

if $\chi_i = \chi_j, i \neq j$, and, furthermore,

(2.6) $$\tau \geqq 2l^2$$

for all zeros of $L(s, \chi_0)$ (i.e. of $\zeta(s)$) in A.
(iii) the region

(2.7) $$\sigma \geqq \alpha + \varepsilon\, , |t - \tau| \leqq 2l^2$$

does not contain any zero of $L(s, \chi)$ if ϱ is counted into A as a zero of $L(s, \chi)$.

Lemma 1. *The class* A *can be selected in such way that its cardinality* $|A|$ *satisfies*

(2.8) $$|A| \gg l^{-3} N(\alpha, T, D)\, .$$

Proof. By the definition of α we have

$$N(\alpha + \varepsilon, T, D) \leqq B_2 (DT)^{\xi(1-\alpha-\varepsilon)}\, l^{B_3}\, .$$

If B_1 (and so DT) is sufficiently large, the expression on the right is

$$\leqq \tfrac{1}{2}\, B_2 (DT)^{\xi(1-\alpha)}\, l^{B_3}$$

(note that a lower bound for B_1 can be given as a function of ε only). Hence, in view of (2.4), we see that at least a fourth of the zeros in $R(\alpha, T)$ satisfy the condition (i).

Next drop away from the zeros, satisfying (i), the zeros of $\zeta(s)$ for which $\tau \leqq 2l^2$, at most $\ll l^3$ in number. We may suppose that after this at least a half of the zeros are left. From the remaining zeros pick out a set satisfying the condition (2.5). So it is easily seen that at least

(2.9) $$\gg l^{-3} N(\alpha, T, D)$$

zeros satisfy both (i) and (ii).

Finally, using again the definition of α, we have

$$N(\alpha + \varepsilon , T + 2l^2 , D) \leqq B_2(D(T + 2l^2))^{\xi(1-\alpha-\varepsilon)} l^{B_3}$$
$$\ll l^{-4} N(\alpha , T , D) .$$

Hence, for DT sufficiently large, at least a half of the zeros, counted in (2.9), satisfy (iii), too. This completes the proof.

2.2 Construction of the polynomials B(s, χ). In this section we shall construct the Dirichlet polynomials $B(s , \chi)$, mentioned in the introduction.

Let

$$X = (DT)^{\varepsilon} , \; Y = (DT)^{1/2+7\varepsilon} , \tag{2.10}$$

$$M(s , \chi) = \sum_{n \leqq X} \mu(n)\chi(n)n^{-s} , \tag{2.11}$$

and for $n \geqq 1$ let

$$d_n = \sum_{\substack{d \mid n \\ d \leqq X}} \mu(d) .$$

Then $d_1 = 1$, $d_n = 0$ for $2 \leqq n \leqq X$, $|d_n| \leqq \tau(n)$ for $n > X$.

Let ϱ be counted into A as a zero of $L(s , \chi)$. Then we have the identity (see [7], p. 104)

$$\begin{aligned} & e^{-1/Y} + \sum_{n > X} d_n \chi(n) n^{-\varrho} e^{-n/Y} \\ & = \varepsilon(\chi)\varphi(D)D^{-1}M(1 , \chi)Y^{1-\varrho}\Gamma(1-\varrho) + \\ & + (2\pi i)^{-1} \int_{-1/2-i\infty}^{-1/2+i\infty} L(w + \varrho , \chi)M(w + \varrho , \chi)Y^{w}\Gamma(w)dw , \end{aligned} \tag{2.12}$$

where $\varepsilon(\chi) = 1$ for $\chi = \chi_0$, and $\varepsilon(\chi) = 0$ otherwise.

If DT is sufficiently large, then the first term on the right of (2.12) is $\leqq \frac{1}{8}$ in absolute value, in view of the condition (2.6) in the definition of A. Also, the contribution of the integers $n > Y_1 = Y(DT)^{\varepsilon} = (DT)^{1/2+8\varepsilon}$ may be supposed to be $\leqq \frac{1}{8}$ in absolute value. We may also suppose that $e^{-1/Y} \geqq \frac{3}{4}$. Then, writing $I(\varrho)$ for the integral in (2.12), we have

$$\Big| \sum_{X < n \leqq Y_1} b_n \chi(n) n^{-\varrho} \Big| \geqq \tfrac{1}{2} - |I(\varrho)| , \; b_n = d_n e^{-n/Y} . \tag{2.13}$$

Lemma 2. *For the »good» zeros we have*

$$\Big| \sum_{X < n \leqq Y_1} b_n \chi(n) n^{-\varrho} \Big| \geqq \tfrac{1}{4} .$$

Proof. Remove in $I(\varrho)$ the integration to the line $\sigma = \eta$ with $\eta = 1 - 2\alpha - 3\varepsilon$. Now $\frac{3}{4} \leqq \alpha \leqq 1 - 5\varepsilon$, so that

$$- 1 + 7\varepsilon \leqq \eta \leqq - \tfrac{1}{2} - 3\varepsilon .$$

Hence no singularity of the integrand lies between the lines $\sigma = - \frac{1}{2}$ and $\sigma = \eta$.

The integral over $|\mathrm{Im}\, w| \geqq l^2$ is $\leqq \frac{1}{8}$ in absolute value, provided DT is sufficiently large. It remains to estimate the integral over $|\mathrm{Im}\, w| \leqq l^2$.

If $|\mathrm{Im}\, w| \leqq l^2$, Re $w = \eta$, then $w + \varrho$ lies in the rectangle

$$1 - \alpha - 3\varepsilon \leqq \sigma \leqq 1 - \alpha - 2\varepsilon \,, \; |t - \tau| \leqq l^2 . \tag{2.14}$$

As noted in (1.7)—(1.9), we have

$$|L(\sigma + it \,, \chi)| \ll_\varepsilon (DT)^\varepsilon$$

in the rectangle

$$\alpha + 2\varepsilon \leqq \sigma \leqq 1 \,, \; |t - \tau| \leqq l^2$$

(owing to the zero-freeness condition (iii) in the definition of A). Consequently, by the functional equation for L-functions (see [8], p. 207), we have

$$|L(\sigma + it \,, \chi)| \ll_\varepsilon (DT)^{1/2 - \sigma + \varepsilon} \tag{2.15}$$

in the rectangle (2.14).

To estimate $I(\varrho)$, note further that $|\Gamma(w)| \leqq B_4$, that by (2.10)—(2.11) trivially

$$|M(\varrho + w)| \leqq (DT)^\varepsilon ,$$

and that $|Y^w| = (DT)^{\eta'}$ with

$$\eta' = (\tfrac{1}{2} + 7\varepsilon)\eta = \tfrac{1}{2} - \alpha + \tfrac{11}{2}\,\varepsilon - 14\alpha\varepsilon - 21\varepsilon^2 \leqq \tfrac{1}{2} - \alpha - 5\varepsilon - 21\varepsilon^2 .$$

Then we have

$$|I(\varrho)| \leqq \tfrac{1}{8} + B_5 (DT)^{-21\varepsilon^2} l^2 \leqq \tfrac{1}{4}$$

if DT is sufficiently large. This estimate combined with (2.13) gives the desired result.

Now construct the polynomials

$$B_j(s \,, \chi) = \sum_{2^{j-1} X < n \leqq \min(2^j X, \, Y_1)} b_n \chi(n) n^{-s} \,, j = 1 \,, 2 \,, \ldots , b$$

with $2^{b-1}X < Y_1$, $2^b X \geqq Y_1$. Given a zero ϱ of $L(s, \chi)$ in A, we have by lemma 2

$$|B_j(\varrho, \chi)| \geqq (4b)^{-1} \tag{2.16}$$

for at least one index j. There exists an index j' such that the condition (2.16) holds with $j = j'$ for at least $b^{-1}|A|$ zeros from A. Let these zeros be enumerated as $\varrho_1, \varrho_2, \ldots, \varrho_J$, let $A_1 = \{\varrho_1, \ldots, \varrho_J\}$, and write for simplicity $B(s, \chi) = B_{j'}(s, \chi)$.

By lemma 1 we obviously have

$$N(\alpha, T, D) \ll l^4 J. \tag{2.17}$$

The rest of the paper is devoted to obtaining an estimate for J.

3. The Halász-Montgomery method

3.1. A lemma on Dirichlet polynomials. The corollary of lemma 1 of [6] can be stated in the following sharpened form (here J is a general symbol, not related to the set A_1).

Lemma 3. *For* $1 \leqq j \leqq J$ *let* χ_j *be any character,* $s_j = \sigma_j + it_j$ *any complex number, and let* $\sigma = \min \sigma_j$. *Let* a_n, $n = 1, \ldots, N$ *be any complex numbers, and write*

$$f(s, \chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s}.$$

Let

$$K = J^{-2} e (2\pi)^{-1} \sum_{j \neq k} \left| \int_{2-i\infty}^{2+i\infty} L(s_j + \bar{s}_k - 2\sigma + w, \chi_j \bar{\chi}_k) N^w \Gamma(w) dw \right|. \tag{3.1}$$

If

$$V^2 \geqq 4K \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}, \tag{3.2}$$

and if for $1 \leqq j \leqq J$

$$|f(s_j, \chi_j)| \geqq V > 0 \tag{3.3}$$

then

$$J \ll N V^{-2} \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}. \tag{3.4}$$

Proof. This is, essentially, a combination of lemma 1, its corollary, and an identity in the proof of lemma 3 of [6]; note only that our quantity K is a constant multiple of the *mean-value* instead of the *maximum* of the integral in (3.1). This sharpening is justified by the inequality (28) of [6].

3.2. An estimate for K. We shall apply lemma 3 in the case $s_j = \varrho_j$, $j = 1, \ldots, J$, ϱ_j running over the zeros in the set A_1, and χ_j being the respective character for which $L(\varrho_j, \chi_j) = 0$.

Lemma 4. *In the case* $s_j = \varrho_j, j = 1, \ldots, J$ *we have*

$$K \ll_\varepsilon (DT)^{\alpha-1/2+6\varepsilon} N^{1-\alpha} . \tag{3.5}$$

Proof. Let j_1 be an index such that the pairs $(j_1, k), k = 1, \ldots, J, k \neq j_1$, give to K the largest contribution. Then by (3.1) we have

$$K \leqq J^{-1} e (2\pi)^{-1} \sum_{k \neq j_1} \left| \int_{2-i\infty}^{2+i\infty} L(\varrho_{j_1} + \overline{\varrho}_k - 2\alpha + w, \chi_{j_1} \overline{\chi}_k) N^w \Gamma(w) dw \right| . \tag{3.6}$$

Now we classify the indices k in (3.6) into classes $C_0, C_1, \ldots$ from the following condition: $k \in C_\nu$ if ν is the least non-negative integer such that the region

$$\sigma \geqq \alpha + (\nu + 1)\varepsilon , |t - (\tau_{j_1} - \tau_k)| \leqq 2l^2 \tag{3.7}$$

is free of zeros of $L(s, \overline{\chi}_{j_1} \chi_k)$.

For the cardinality $|C_\nu|$ of C_ν we have by the definition of α and by the condition (ii) in the definition of A the estimate

$$|C_\nu| \leqq 3B_2(D(T + 2l^2))^{\xi(1-\alpha-\nu\varepsilon)} l^{B_3} \leqq 6B_2(DT)^{\xi(1-\alpha-\nu\varepsilon)} l^{B_3+2} \tag{3.8}$$

for $\nu = 1, 2, \ldots$ Further, trivially, $|C_0| \leqq J$.

Next, for $k \in C_\nu$, we need an estimate for the integral in (3.6). To this end, we remove the integration to the line $\sigma = \eta_1$ with $\eta_1 = \max (1 - \alpha - (\nu + 4)\varepsilon, \varepsilon)$. Then

$$\max (1 - \alpha - (\nu + 4)\varepsilon , \varepsilon) \leqq \operatorname{Re} (\varrho_{j_1} + \overline{\varrho}_k - 2\alpha + w) \leqq \max (1 - \alpha - (\nu + 2)\varepsilon , 3\varepsilon) . \tag{3.9}$$

The residue, arising from the pole of $L(s, \chi_0)$ at $s = 1$ is easily seen to be negligible, as well as the integral over $|\operatorname{Im} w| \geqq l^2$.

Now let $\operatorname{Re} w = \eta_1$, $|\operatorname{Im} w| \leqq l^2$. Then we have by the zero-freeness condition (3.7) and by (3.9) (as in (2.15)) the estimate

$$L(\varrho_{j_1} + \overline{\varrho}_k - 2\alpha + w, \chi_{j_1} \overline{\chi}_k) \ll_\varepsilon (DT)^{1/2-\eta_1+\varepsilon} .$$

Hence the integral in consideration is

$$\ll_{\varepsilon} (DT)^{1/2-\eta_1+\varepsilon} N^{\eta_1} l^2 . \tag{3.10}$$

Combining (3.8) and (3.10) we conclude that the indices k in C_ν contribute to (3.6) at most

$$B_6 B_2 J^{-1}(DT)^{1/2-\eta_1+\varepsilon+\xi(1-\alpha-\nu\varepsilon)} N^{\eta_1} l^{B_3+4} . \tag{3.11}$$

for $\nu \geqq 1$, and for $\nu = 0$

$$\ll_{\varepsilon} (DT)^{\alpha-1/2+5\varepsilon} N^{1-\alpha-4\varepsilon} l^2 \tag{3.12}$$

since $\max(1-\alpha-4\varepsilon , \varepsilon) = 1-\alpha-4\varepsilon$ (recall that $\alpha \leqq 1-5\varepsilon$).

In any case, we have $\alpha + (\nu+1)\varepsilon \leqq 1+\varepsilon$, so that

$$1-\alpha-\nu\varepsilon-4\varepsilon \leqq \eta_1 \leqq 1-\alpha-\nu\varepsilon+\varepsilon .$$

Using this as well as (2.4) and (2.17), we see that the expression in (3.11) is

$$\ll_{\varepsilon} (DT)^{\alpha-1/2-(\xi-1)\nu\varepsilon+5\varepsilon} N^{1-\alpha-\nu\varepsilon+\varepsilon} l^8 .$$

Summing here over $\nu = 1, 2, \ldots,$ and taking into account the estimate (3.12), we obtain (3.5).

3.3. The final inequality. We combine lemmas 3 and 4 in the case

$$f(s, \chi) = (B(s, \chi))^u , \; 1 \leqq u \leqq 2\varepsilon^{-1} . \tag{3.13}$$

$$V = (4b)^{-u} . \tag{3.14}$$

Then we have $N = 2^{j'u} X^u$,

$$a_n = 0 \; for \; n < 2^{-u} N , \tag{3.15}$$

$$\sum_{n=1}^{N} |a_n|^2 \ll_{\varepsilon} N(\log N)^{B_7} . \tag{3.16}$$

The condition (3.3) of lemma 3 is satisfied by (2.16), (3.13), and (3.14). Also, the condition (3.2) will be satisfied if u is chosen (if possible) in such way that

$$(4b)^{-2u} \geqq 4K \sum_{n=1}^{N} |a_n|^2 n^{-2\alpha} . \tag{3.17}$$

From lemma 4 and (3.15), (3.16) we see that (3.17) holds if

$$(4b)^{-4\varepsilon^{-1}} \geqq B_8 (DT)^{\alpha-1/2+6\varepsilon} N^{2-3\alpha} l^{B_9} .$$

This gives for N a condition of the type

$$N \geqq N_0 = B_{10} (DT)^{f(\alpha)} l^{B_{11}} , \tag{3.18}$$

(3.19) $$f(\alpha) = (\alpha - \tfrac{1}{2} + 6\varepsilon) \,/\, (3\alpha - 2)\,.$$

Then, by (3.4) and (3.14)—(3.16), we have

(3.20) $$J \ll_\varepsilon N^{2-2\alpha}\, l^{B_{12}}\,.$$

4. Proof of the theorem

Let us choose for u the least integer such that the condition (3.18) is satisfied. By (3.19) we have for $\frac{3}{4} \leqq \alpha \leqq 1$

(4.1) $$f(\alpha) \leqq (\alpha - \tfrac{1}{2}) \,/\, (3\alpha - 2) + 24\varepsilon\,.$$

Hence $f(\alpha) \leqq 1 + 24\varepsilon < \frac{3}{2}$, so that the condition $u \leqq 2\varepsilon^{-1}$ in (3.13) holds.

If $u \leqq 2$, then $N \leqq 4Y_1^2 = 4(DT)^{1+16\varepsilon}$; if $u \geqq 3$, then $N \leqq N_0^{3/2}$. So, in any case, by (3.20) we have

(4.2) $$J \ll_\varepsilon \max\,(N_0^{3(1-\alpha)}\,,\,(DT)^{(2+32\varepsilon)(1-\alpha)})\; l^{B_{12}}\,.$$

From (3.18), (4.1), (4.2), and (2.17) we obtain for $N(\alpha\,,\,T\,,\,D)$ an estimate of the type (1.2) with $\omega(\alpha)$ being given by (1.4) and (1.5), and the constants $C\,,B$ being independent of $B_1\,,B_2\,,B_3\,,$ and ξ.

We conclude that if $\xi \geqq 2 + 73\varepsilon$ is given, and $\alpha = \alpha(\xi)$, obtained from the basic assumption in the beginning of the proof, is such that $\omega(\alpha) + 73\varepsilon \leqq \xi$, then the estimates (2.4) and (4.2) give a contradiction, provided $B_1\,,B_2\,,$ and B_3 are supposed to be sufficiently large. This completes the proof of the theorem.

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