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ON REGULAR EXPRESSIONS OVER ONE LETTER AND ON COMMUTATIVE LANGUAGES

BY

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INTRODUCTION

Consider a finite alphabet

$$I_r = \{x_1, \ldots, x_r\} \quad (r \geqq 1)$$

and an infinite alphabet

$$A_w = \{\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, \ldots\}$$

such that I_r and A_w are disjoint. Regular expressions over I_r and A_w are defined in the usual way (cf. [7], p. 3).

If X and Y are regular expressions over the alphabet I_r, we say that the equation $X = Y$ is valid iff X and Y denote the same language, i.e., $|X| = |Y|$. The set of all valid equations between regular expressions over the alphabet I_r is denoted by V_r. Obviously

$$V_1 \subset V_2 \subset V_3 \subset \ldots,$$

where all inclusions are proper. The union of all sets V_r is denoted by V_w.

Let X and Y be regular expressions over the alphabet A_w. By S_r, $r = 1, 2, \ldots,$ we denote the set of equations of the form $X = Y$ such that a valid equation always results when each letter of A_w appearing in X or Y is substituted by some regular expression over I_r. The intersection of all sets S_r is denoted by S_w. It is known (cf. [7], p. 128) that

$$S_2 = S_3 = \ldots = S_w$$

and S_2 is properly included in S_1.

Let c be the operator defined for languages such that $c(L)$ is the language consisting of all such words which are obtained by permuting the letters in some word belonging to L. For regular expressions X and Y over I_r the equation $X = Y$ is said to be c-valid iff the languages $c(|X|)$ and $c(|Y|)$ are equal. By C_r, $r = 1, 2, \ldots,$ we denote the set of equations of the form $X = Y$, where X and Y are regular expressions over A_w such that whenever the letters of A_w appearing in X or Y are substituted by some regular expression over I_r, the resulting equation is c-valid. The intersection of all sets C_r is denoted by C_w. It is proved by Lepistö [2] and Linna [3], that

$$S_1 = C_1 = C_2 = \ldots = C_w .$$

A basis of V_1 is given by Red'ko in [5] and Salomaa in [7, p. 130]. A basis of C_w is given by Red'ko in [6] and Salomaa in [7, p. 139] and also by Pilling in [4] and Conway in [1]. In this paper we consider a basis S of V_1 and a basis T of C_w which are almost identical with those mentioned above. The main purpose of this paper is to prove that the axioms in S and T are independent.

§ 1. The basis S of V_1

1.1. Let S be a subset of S_1 consisting of the following 11 equations, called the axioms:

A$_1$ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
A$_2$ $\alpha(\beta\gamma) = (\alpha\beta)\gamma$,
A$_3$ $\alpha + \beta = \beta + \alpha$,
A$_4$ $\alpha\beta = \beta\alpha$,
A$_5$ $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$,
A$_6$ $\phi^*\alpha = \alpha$,
A$_7$ $\phi\alpha = \phi$,
A$_8$ $(\alpha\beta^*)^* = \phi^* + \alpha\alpha^*\beta^*$,
A$_9$ $(\phi^*)^* = \phi^*$,
A$_{10}$ $(\alpha + \beta)^* = \alpha^*\beta^*$,
A$_{11}$ $\alpha^* = (\alpha^n)^*(\phi^* + \alpha + \ldots + \alpha^{n-1})$, $n = 1, 2, \ldots$.

It is assumed in A$_8$ and A$_{11}$ that A$_1$ and A$_2$ are satisfied and hence the sum and catenations are written without parentheses. A$_{11}$ is in fact an infinite axiom-scheme.

By a substitution instance of an axiom we mean the result of substituting all letters of A_w appearing in the axiom by some regular expression over the alphabet I_1 .

We give also the following inference rule:

R (Replacement). Assume that Y_1 is a well-formed part of a regular expression X_1 and that X_2 is the result of replacing (some occurrence of) Y_1 by a regular expression Y_2 . Then from the equations $X_1 = Z$ and $Y_1 = Y_2$ one may infer the equation $X_2 = Z$.

An equation is *generated* by S , in symbols $\vdash X = Y$, iff there is a finite sequence of equations such that each of them either is a substitution instance of an axiom or may be inferred from some equations occurring earlier in the sequence by R and, furthermore, $X = Y$ is the last equation in the sequence. The set S is a *basis* of V_1 iff every equation in V_1 is generated by S .

1.2. The following lemma is an immediate consequence of A_6 and R, and it is used in subsequent proofs without being explicitly referred to:

Lemma 1.1. *Let* X, Y, Z *and* U *be arbitrary regular expressions over* I_1. *Then* $\vdash X = X$. *If* $\vdash X = Y$ *then* $\vdash Y = X$. *If* $\vdash X = Y$ *and* $\vdash Y = Z$ *then* $\vdash X = Z$. *If* $\vdash X = Z$ *and* $\vdash Y = U$ *then* $\vdash X + Y = Z + U$, $\vdash XY = ZU$ and $\vdash X^* = Z^*$.

Lemma 1.2. *Let* X *be an arbitrary regular expression over* I_1. *Then*

(1.1) $$\vdash X + X = X$$

and

(1.2) $$\vdash X + \phi = X .$$

Proof. If we substitute $\alpha \equiv \beta \equiv \phi$ in A_8, we obtain by A_7

(1.3) $$\vdash \phi^* = \phi^* + \phi .$$

If we substitute $\alpha \equiv \phi^*$, $\beta \equiv \phi$ in A_8, we obtain by A_6 and A_9

(1.4) $$\vdash \phi^* = \phi^* + \phi^* .$$

By A_4—A_7, (1.3) and (1.4) we can verify that the lemma holds true. □.

Note. Only A_4—A_9 are needed for Lemma 1.2.

As an immediate consequence of A_4, A_6, A_9 and A_{10} we obtain

Lemma 1.3. *Let* X *be an arbitrary regular expression over* I_1. *Then*

(1.5) $$\vdash (X + \phi^*)^* = X^* .$$

Using Lemmas 1.2 and 1.3, we may conclude that the following theorem is a consequence of the results of Red'ko [5] and Salomaa [7, p. 130]:

Theorem 1.1. *The set* S, *consisting of the equations* A_1 — A_{11}, *is a basis of* V_1.

§ 2. Independence of the axioms in S

2.1. We shall now show that the axioms in S are independent. This is done by constructing models in which all the axioms are satisfied except one. We prove only that the axiom-scheme A_{11} is independent of the other axioms, i.e., we give a model in which A_1—A_{10} are satisfied but for some values of n A_{11} is not satisfied.

A *model* is an ordered quadruple (E, f_1, f_2, g), where E is a finite non-empty set called the elements, f_1 and f_2 are functions mapping the Cartesian product $E \times E$ into E, called the sum and catenation, and g is a function mapping the set E into E, called the iteration. Moreover it is assumed that E can be divided into two disjoint sets $E_1 = \{\phi, x\}$, where $x \in I_1$, and E_2, which is possibly empty, with the following property. For any $e \in E_2$, there is a regular expression X over E_1 such that $e = |X|$. (Every regular expression X over E can be reduced, interpreting f_1, f_2 and g as mappings, to an element $e \in E$, denoted by $e = |X|$.) The elements of E_2 are denoted by ϕ^*, a and b.

An equation $X = Y$ between two regular expressions X and Y over E is valid in the model iff $|X| = |Y|$. An equation of the form $\xi = \eta$, where ξ and η are regular expressions over the alphabet A_w, is satisfied in the model iff an equation valid in the model always results whenever each letter over A_w appearing in ξ or η is substituted by some regular expression over E.†

If the set E consists of n elements we say that the model is n-valued. If the independence of an axiom can be established using an n-valued model but not using an (n-1)-valued model, we say that the independence model of minimum cardinality in this case consists of n elements. Obviously $n \geqq 2$ for each axiom.

2.2. To establish the independence of A_3 and A_5—A_7 we consider 2-valued models consisting of the elements ϕ and x. In the first model the sum and catenation of the elements are defined in Tables 1 and 2, the iteration is defined by $\alpha^* = x$. In the subsequent tables defining $\alpha + \beta$ and $\alpha\beta$, the values of α determine the row and those of β the column.

Table 1

$\alpha + \beta$	ϕ	x
ϕ	ϕ	ϕ
x	x	x

Table 2

$\alpha\beta$	ϕ	x
ϕ	ϕ	ϕ
x	ϕ	x

It is easy to verify that A_1, A_2 and A_4—A_{11} are satisfied in this model.

† It is assumed that the sum and catenations occurring in A_8 and A_{11} are performed from left to right.

Because $\phi + x = \phi$ but $x + \phi = x$, A_3 is not satisfied in this model and hence A_3 is independent.

2.3. In the second model consisting of the elements ϕ and x the catenation and iteration of the elements are defined as in 2.2, but the sum of the elements is defined in Table 3.

Table 3

$\alpha + \beta$	ϕ	x
ϕ	x	x
x	x	x

Clearly A_1—A_4 and A_6—A_{11} are satisfied in this model, but A_5 is not satisfied, because $\phi(\phi + \phi) = \phi x = \phi$ whereas $\phi\phi + \phi\phi = \phi + \phi = x$.

2.4. In the third 2-valued model the sum and catenation of the elements ϕ and x are defined in Tables 4 and 2, and the iteration is defined by $\alpha^* = \phi$.

Table 4

$\alpha + \beta$	ϕ	x
ϕ	ϕ	x
x	x	x

Obviously A_1—A_5 and A_7—A_{11} are satisfied in this model, but A_6 is not satisfied because $\phi^* x = \phi x = \phi \neq x$.

2.5. In the fourth 2-valued model the sum and catenation of the elements ϕ and x are defined in Tables 4 and 5, and the iteration is defined by $\alpha^* = \alpha$.

Table 5

$\alpha\beta$	ϕ	x
ϕ	ϕ	x
x	x	x

In this model all the axioms except A_7 are obviously satisfied. Because $\phi x = x \neq \phi$, A_7 is independent.

2.6. Suppose that we have an n-valued $(n \geqq 2)$ model in which A_6 and A_7 are satisfied. Let $y \neq \phi$ be an element of the model. By A_6 and A_7 we now obtain $\phi^* y = y$ and $\phi y = \phi$. Thus we must have $\phi^* \neq \phi$ in all models in which A_6 and A_7 are satisfied.

We now consider 2-valued models in which A_3, A_5—A_7 and either A_8 or A_9 and A_{11} are satisfied. Let ϕ and x be the elements of the model. Then $\phi^* = x$ and the only possible way to define the catenation of the elements is given in Table 2.

Because $\phi(\phi + \phi) = \phi$ and $\phi\phi + \phi\phi = \phi + \phi$, we obtain, by A_5, $\phi + \phi = \phi$. Since $\phi^* + \phi\phi^* \phi^* = x + \phi$ and $(\phi\phi)^* (\phi^* + \phi) = x + \phi$, we obtain, by A_3 and either by A_8 or by A_{11}, $\phi + x = x + \phi = x$. Because $\phi^* + xx^* \phi^* = x + x^*$, $x^* = (\phi^*)^*$ and $(xx)^* (\phi^* + x) = x^* (x + x)$, we obtain, either by A_8 or by A_9 and A_{11}, $x^* = x$ and $x + x = x$. Thus the only possible way to define the sum of the elements is given in Table 4.

We have thus obtained the model in which the sum and catenation of the elements are defined in Tables 4 and 2, and the iteration is defined by $\alpha^* = x$. In this model all the axioms are satisfied and hence the independence of A_1, A_2, A_4 and A_8—A_{11} cannot be established using 2-valued models.

2.7. To establish the independence of A_4, A_8 and A_9 we consider 3-valued models consisting of the elements ϕ, ϕ^* and x. In the first model the sum and catenation of the elements are defined in Tables 6 and 7. The iteration is defined by $\alpha^* = \phi^*$.

Table 6

$\alpha + \beta$	ϕ	ϕ^*	x
ϕ	ϕ	ϕ^*	x
ϕ^*	ϕ^*	ϕ^*	ϕ^*
x	x	ϕ^*	x

Table 7

$\alpha\beta$	ϕ	ϕ^*	x
ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x
x	x	x	x

Clearly A_3 and A_6—A_{11} are satisfied in this model.

If we define $\phi < x < \phi^*$, then $\alpha + \beta = \max(\alpha, \beta)$ and we can conclude that A_1 is satisfied.

Obviously A_2 and A_5 are satisfied, if $\alpha = \phi^*$. If $\alpha \neq \phi^*$ then $\alpha(\beta\gamma) = \alpha = (\alpha\beta)\gamma$ and $\alpha(\beta + \gamma) = \alpha = \alpha + \alpha = \alpha\beta + \alpha\gamma$. Thus A_2 and A_5 are satisfied in this model.

A_4 is not satisfied, because $x\phi = x$ whereas $\phi x = \phi$.

2.8. In the second 3-valued model the sum, catenation and iteration of the elements are defined in Tables 8, 9 and 10.

Table 8

$\alpha + \beta$	ϕ	ϕ^*	x
ϕ	ϕ	ϕ^*	x
ϕ^*	ϕ^*	ϕ^*	x
x	x	x	x

Table 9

$\alpha\beta$	ϕ	ϕ^*	x
ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x
x	ϕ	x	x

Table 10

α	α^*
ϕ	ϕ^*
ϕ^*	ϕ^*
x	ϕ

Obviously A_1, A_3, A_4, A_6, A_7 and A_9 are satisfied in this model.

It is clear that A_2 is satisfied if one of the expressions α, β or γ is either ϕ or ϕ^*. Because $x(xx) = x = (xx)x$, A_2 is satisfied otherwise.

If $\beta + \gamma \neq \phi$ then $x(\beta + \gamma) = x = x\beta + x\gamma$. If $\beta + \gamma = \phi$, i.e., $\beta = \gamma = \phi$, then $x(\beta + \gamma) = \phi = x\beta + x\gamma$. Obviously A_5 is satisfied also in the cases where α is either ϕ or ϕ^*.

It is easy to verify that A_{10} and A_{11} are satisfied, because both sides of these equations are simultanously either equal to ϕ^* or equal to ϕ.

In this model we have $\phi^* + xx^*\phi^* = \phi^* + \phi = \phi^*$ and $(x\phi^*)^* = \phi$. Thus A_8 is not satisfied.

2.9. In the third 3-valued model the sum and catenation of the elements are defined as in 2.8, and the iteration is defined in Table 11.

Table 11

α	α^*
ϕ	ϕ^*
ϕ^*	x
x	x

For the same reasons as in 2.8, A_1—A_7 are satisfied in this model.

It is easy to verify that A_8, A_{10} and A_{11} are satisfied, because both sides of these equations are simultanously either equal to ϕ^* or equal to x.

Because $(\phi^*)^* = x \neq \phi^*$, A_9 is not satisfied.

2.10. Consider an n-valued ($n \geqq 3$) model in which A_3—A_9 are satisfied. By the note following Lemma 1.1, we must have $\alpha + \alpha = \alpha$ and $\alpha + \phi = \phi + \alpha = \alpha$. We must also have $\alpha^* \neq \phi$, because if $\alpha^* = \phi$ we obtain, by A_8, $\alpha^* = \phi^* + \alpha\alpha^* = \phi^* + \phi = \phi^*$ but, by 2.6, $\phi^* \neq \phi$.

We consider now 3-valued models in which $A_3 - A_9$ are satisfied. Without loss of generality we may suppose that the elements of these models are ϕ, ϕ^* and x.

Suppose first that $x^* = x$. Then, by A_8, $x = x^* = \phi^* + xx^* = \phi^* + xx$. Thus we must have $xx = x$ and $\phi^* + x = x$. Thus the only possible way to define the sum, catenation and iteration of the elements is given in Tables 8, 9 and 12. But it is easy to verify that all axioms are satisfied in this model.

Table 12

α	α^*
ϕ	ϕ^*
ϕ^*	ϕ^*
x	x

Suppose now that $x^* = \phi^*$. Then, by A_8, $\phi^* + x = \phi^*$, and, by A_4, A_5 and A_6, $x = x\phi^* = x(\phi^* + x) = x + xx$. Thus $xx \neq \phi^*$. Now the only possible way to define the sum is given in Table 6 and the only way to define the iteration is $\alpha^* = \alpha$. The catenation can be defined either by Table 9 or by Table 13. In both cases all the axioms are satisfied.

Table 13

$\alpha\beta$	ϕ	ϕ^*	x
ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x
x	ϕ	x	ϕ

Thus the independence of A_1, A_2, A_{10} and A_{11} cannot be proved using 3-valued models.

2.11. To establish the independence of A_2 we consider a 4-valued model consisting of the elements ϕ, ϕ^*, x and $a = xx$. The sum and catenation of the elements are defined in Tables 14 and 15, and the iteration is defined by $\alpha^* = \phi^*$.

Table 14

$\alpha + \beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ^*	x	a
ϕ^*	ϕ^*	ϕ^*	ϕ^*	ϕ^*
x	x	ϕ^*	x	x
a	a	ϕ^*	x	a

Table 15

$\alpha\beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x	a
x	ϕ	x	a	a
a	ϕ	a	a	ϕ

If we define $\phi < a < x < \phi^*$, then $\alpha + \beta = \max(\alpha, \beta)$. It is now easy to verify that A_1, A_3, A_4 and A_6—A_{11} are satisfied in this model.

Obviously A_5 is satisfied if α is either ϕ or ϕ^*; likewise if $\beta = \gamma$ or either β or γ is ϕ. A_5 is also satisfied in other cases, as seen as follows: $x(\beta + \gamma) = x = x\beta + x\gamma$ if either β or γ is ϕ^* and $x(\beta + \gamma) = a = x\beta + x\gamma$ otherwise. If either β or γ is either ϕ^* or x then $a(\beta + \gamma) = a = a\beta + a\gamma$ and $a(\beta + \gamma) = \phi = a\beta + a\gamma$ otherwise.

Because $x(xa) = xa = a$ whereas $(xx)a = aa = \phi$, A_2 is not satisfied.

2.12. To establish the independence of A_{10} we consider a 4-valued model consisting of the elements ϕ, ϕ^*, x and $a = x^*$. The sum, catenation and iteration of the elements are defined in Tables 16, 17 and 18.

Table 16

$\alpha + \beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ^*	x	a
ϕ^*	ϕ^*	ϕ^*	ϕ^*	a
x	x	ϕ^*	x	a
a	a	a	a	a

Table 17

$\alpha\beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x	a
x	ϕ	x	x	a
a	ϕ	a	a	a

Table 18

α	α^*
ϕ	ϕ^*
ϕ^*	ϕ^*
x	a
a	a

If we define $\phi < x < \phi^* < a$, then $\alpha + \beta = \max(\alpha, \beta)$. It is now easy to verify that A_1—A_4, A_6, A_7 and A_9 are satisfied in this model.

Clearly A_5 is satisfied if α is either ϕ or ϕ^* or $\beta = \gamma$ or either β or γ is ϕ. A_5 is also satisfied in other cases because $x(\beta + \gamma) = x = x\beta + x\gamma$ if neither β nor γ is a and $x(\beta + \gamma) = a = x\beta + x\gamma$ if either β or γ is a, and $a(\beta + \gamma) = a = a\beta + a\gamma$.

A_8 and A_{11} are clearly satisfied if α is either ϕ or ϕ^*; likewise in other cases, because both equations are reduced to the form $a = a$.

We have now $(\phi^* + x)^* = (\phi^*)^* = \phi^*$ and $(\phi^*)^* x^* = \phi^* a = a$. Thus A_{10} is not satisfied.

2.13. We now consider the 4-valued models in which A_2—A_{10} are satisfied. Let ϕ, ϕ^*, x and a be the elements of the model. Because we must have $\phi\alpha = \alpha\phi = \phi$, $\phi^*\alpha = \alpha\phi^* = \alpha$ and $\alpha + \phi = \phi + \alpha = \alpha + \alpha = \alpha$, there are three possibilities to define the fourth element a, namely, $a = xx$, $a = \phi^* + x$ or $a = x^*$.

If $x^* = x$, then, by A_{10}, $x = x^* = (x + x)^* = x^* x^* = xx$ and, by A_8, $x = x^* = \phi^* + xx^* = \phi^* + x$. Hence $xx = \phi^* + x = x^* = x$ and we cannot define a. By 2.10, $x^* \neq \phi$, and hence we must define x^* to be either ϕ^* or a.

If $x^* = \phi^*$, we obtain, by A_8, $\phi^* = x^* = \phi^* + xx^* = \phi^* + x$, and we must define $a = xx$. By A_5 we now obtain $x = x\phi^* = x(\phi^* + x) = x + xx = x + a$.

By A_{10} and A_8, we have then $\phi^* = x^* = (x + a)^* = x^* a^* = a^*$ and $\phi^* = \phi^* + aa^* = \phi^* + a$. Thus we must define the sum of the elements by Table 14, and the iteration by $\alpha^* = \phi^*$. But it is immediately clear that A_1 and A_{11} are satisfied in this model.

If $x^* = a$, we obtain, by A_8 and A_{10}, $a = \phi^* + xx^* = \phi^* + xa$, $a = aa$, $a^* = (\phi^* + xx^*)^* = (xx^*)^* = \phi^* + xx^*x^* = \phi^* + xa = a$ and $a = a^* = \phi^* + aa^* = \phi^* + a$. Because $a = \phi^* + xa$, xa is either a or x. Then, by A_5 and A_4, $xa = x(\phi^* + xa) = x + x(xa)$ and $a = aa = (\phi^* + xa)a = a + xa$ and thus in both cases $x + a = a$. Because $(\phi^* + x)^* = x^* = a$, $\phi^* + x$ is either a or x. Thus we have two possibilities to define the sum, namely, those given in Tables 19 and 20. In both cases it is immediately clear that A_1 is satisfied. Obviously A_{11} is also satisfied if α is either ϕ, ϕ^* or a, or $n = 1$. If the sum is defined in Table 19, $\phi^* + x + \ldots = a$ and A_{11} is satisfied, because $\phi^* a = a$ and $aa = a$. If the sum is defined in Table 20, $x + \phi^* = x$

Table 19

$\alpha + \beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ^*	x	a
ϕ^*	ϕ^*	ϕ^*	a	a
x	x	a	x	a
a	a	a	a	a

Table 20

$\alpha + \beta$	ϕ	ϕ^*	x	a
ϕ	ϕ	ϕ^*	x	a
ϕ^*	ϕ^*	ϕ^*	x	a
x	x	x	x	a
a	a	a	a	a

and, by A_5 and A_4, $xa = (x + \phi^*)a = xa + a = a$ and $xx = (x + \phi^*)x = xx + x$. Thus xx is either x or a, and $(x^n)^* = a$. Thus A_{11} is satisfied also in this case.

In all 4-valued models in which A_2—A_{10} are satisfied also A_1 and A_{11} are satisfied. Thus the independence of A_1 and A_{11} cannot be established using 4-valued models.

2.14. In the following 5-valued models establishing the independence of A_1 and A_{11} we have the elements ϕ, ϕ^*, x, $a = \phi^* + x$ and $b = x^*$. In the first model the sum, catenation and iteration of the elements are defined in Tables 21, 22 and 23.

Table 21

$\alpha + \beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ^*	x	a	b
ϕ^*	ϕ^*	ϕ^*	a	a	b
x	x	a	x	x	b
a	a	a	x	a	b
b	b	b	b	b	b

Table 22

$\alpha\beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x	a	b
x	ϕ	x	x	x	b
a	ϕ	a	x	x	b
b	ϕ	b	b	b	b

Table 23

α	α^*
ϕ	ϕ^*
ϕ^*	ϕ^*
x	b
a	b
b	b

It is easy to verify that A_3, A_4 and A_6—A_{11} are satisfied in this model.

A_2 is obviously satisfied if one of the expressions α, β or γ is either ϕ, ϕ^* or b. In other cases $\alpha(\beta\gamma) = x = (\alpha\beta)\gamma$ and A_2 is satisfied.

A_5 is clearly satisfied if α is either ϕ, ϕ^* or b; likewise in the cases where β or γ is either ϕ or b. But so it is also in other cases, because then $\alpha(\beta + \gamma) = x = \alpha\beta + \alpha\gamma$.

A_1 is not satisfied, because we have now $\phi^* + (x + x) = \phi^* + x = a$ whereas $(\phi^* + x) + x = a + x = x$.

Table 24

$\alpha + \beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ^*	x	a	b
ϕ^*	ϕ^*	ϕ^*	a	a	b
x	x	a	x	a	b
a	a	a	a	a	b
b	b	b	b	b	b

Table 25

$\alpha\beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x	a	b
x	ϕ	x	ϕ^*	a	b
a	ϕ	a	a	a	b
b	ϕ	b	b	b	b

2.15. The sum, catenation and iteration of the elements are now defined in Tables 24, 25 and 23.

Because $a = \phi^* + x$ it is easy to verify that $A_1 - A_{10}$ are satisfied in this model. Because $(x^2)^* \ (\phi^* + x) = \phi^* a = a$ whereas $x^* = b$, A_{11} is not satisfied.

Note. It can be shown that it suffices to suppose that n is a prime number in the axiom-scheme A_{11}. In this case the equations in the scheme are independent of each other, as can be seen from the model below.

Consider a $(k + 3)$-valued model consisting of the elements

$$y_0 , y_1 , \ldots , y_{k+1} \text{ and } y_{k+2} ,$$

where $y_{k+2} = \phi$, $y_0 = \phi^*$, $y_1 = x$, $y_k = \phi^* + x$, $y_{k+1} = x^*$, and $y_i = xy_{i-1}$ for $i = 2 , \ldots , k - 1$. It is assumed that k is a prime number.

The sum, catenation and iteration of the elements are defined as follows:

$y_i + y_i = y_i + y_{k+2} = y_{k+2} + y_i = y_i$, $y_i + y_{k+1} = y_{k+1} + y_i = y_{k+1}$ for $i = 0 , 1 , \ldots , k + 2$, and $y_i + y_j = y_k$ for $i \neq j$ and $0 \leqq i , j \leqq k$. Let $\max(i , j) = m$. Then $y_i y_j = y_m$ if $m \geqq k$, otherwise $y_i y_j = y_l$, where $0 \leqq l < k$ and $l \equiv i + j \pmod{k}$. $(y_0)^* = (y_{k+2})^* = y_0$ and $(y_i)^* = y_{k+1}$ for $i = 1 , \ldots , k + 1$.

In this model $A_1 - A_{10}$ are satisfied and so also is A_{11} if $n \neq k$ (n prime). Because $(y_1^k)^* \ (\phi^* + y_1 + \ldots + y_1^{k-1}) = (y_0)^* y_k = y_k \neq (y_1)^*$, A_{11} is not satisfied if $n = k$.

2.16. We have now obtained the following result:

Theorem 2.1. *Each of the axioms in the basis of V_1 is independent. For A_3 and $A_5 - A_7$ the independence model of minimum cardinality consists of two elements. For A_4, A_8 and A_9 it consists of three elements. For A_2 and A_{10} it consists of four elements. For A_1 and A_{11} it consists of five elements.*

§ 3. The basis T of C_κ

3.1. Consider a subset T of S_1 consisting of the following 11 equations: $A_1 - A_9$ and A_{11}, given in 1.1, and

$$A_{12} \qquad (\alpha + \beta)^* = (\alpha^* + \beta^*) \ (\alpha\beta)^* .$$

Equations $A_1 - A_9$, A_{11} and A_{12} are called the axioms.

By a substitution instance of an axiom we mean the result of substituting all letters of A_w appearing in the axiom by some regular expression over the alphabet A_w. The replacement rule and the notion of generation are defined as in § 1 with S replaced by T. The set T is a basis of C_w iff every equation in C_w is generated by T.

3.2. Lemma 1.1 holds true with the set I_1 replaced by the set A_w, and it is used in subsequent proofs without being explicitly referred to, and exactly as in 1.2, we obtain the following:

Lemma 3.1. *Let X be an arbitrary regular expression over A_w. Then* $\vdash X + X = X$ *and* $\vdash X + \phi = X$.

Lemma 3.2. *Let X and Y be arbitrary regular expressions over A_w. Then*

$$\vdash (X^*)^* = X^* . \tag{3.1}$$

$$\vdash X^*X^* = X^* . \tag{3.2}$$

$$\vdash (X^*Y^*)^* = X^*Y^* . \tag{3.3}$$

$$\vdash (XY^*)^*Y^* = X^*Y^* . \tag{3.4}$$

Proof.[†] By A_8 we obtain the following four equations:

$$\vdash (X\phi^*)^* = \phi^* + XX^*\phi^* . \tag{3.5}$$

$$\vdash (\phi^*X^*)^* = \phi^* + \phi^*(\phi^*)^*X^* . \tag{3.6}$$

$$\vdash (X^*\phi^*)^* = \phi^* + X^*(X^*)^*\phi^* . \tag{3.7}$$

$$\vdash (X^*Y^*)^* = \phi^* + X^*(X^*)^*Y^* . \tag{3.8}$$

By (3.5), A_5 and Lemma 3.1, we can conclude that

$$\vdash X^* = \phi^* + XX^* = \phi^* + X^* = X + X^* \tag{3.9}$$

and

$$\vdash X^*Y^* = X^* + Y^* + X^*Y^* = \phi^* + X^*Y^* = XY + X^*Y^* . \tag{3.10}$$

Using A_9, (3.6) and (3.9), we obtain the equation (3.1). and hence, by (3.7) and (3.9), also the equation (3.2).

The equation (3.3) is an immediate consequence of (3.1), (3.2), (3.8) and (3.10).

Because, by A_5 and (3.2),

$$\vdash (\phi^* + XX^*Y^*)Y^* = Y^* + XX^*Y^*Y^* = Y^* + XX^*Y^* = (\phi^* + XX^*)Y^*$$

[†] References to $A_1 - A_4$ and A_6 are not mentioned in the following proofs.

the equation (3.4) follows, by A_8 and (3.9). □.

Lemma 3.3. *Let* X *and* Y *be arbitrary regular expressions over* A_w. *Then*

$$\vdash (X + Y)^* = X^* Y^* \tag{3.11}$$

and

$$\vdash (X + \phi^*)^* = X^* . \tag{3.12}$$

Proof. By A_5, A_{12}, (3.2) and (3.10) we obtain first

$$\vdash (X + Y)^* = (X + Y)^* (X + Y)^* \tag{3.13}$$

$$= (X^* + Y^*) (XY)^* (X^* + Y^*) (XY)^* = (XY)^* X^* Y^* .$$

Then, by (3.3), (3.10) and (3.13),

$$\vdash X^* Y^* = (X^* Y^*)^* = (XY + X^* Y^*)^* \tag{3.14}$$

$$= (XY)^* (X^* Y^*)^* (XYX^* Y^*)^*$$

$$= (XY)^*(XY(X^*Y^*)^*)^*(X^*Y^*)^* = (XY)^*(XY)^*(X^*Y^*)^* = (XY)^*X^*Y^*.$$

(3.11) now follows by (3.13) and (3.14).

By A_9 the equation (3.12) is an immediate consequence of (3.11). □.

By Lemmas 3.1 and 3.3 Theorem 3.1 follows from the results of Red'ko [6], Salomaa [7] and Pilling [4].

Theorem 3.1. *The set* T, *consisting of the equations* A_1—A_9, A_{11} *and* A_{12}, *is a basis of* C_w.

§ 4. Independence of the axioms in T

4.1. To prove the independence of the axioms in T we use models defined as in 2.1 with the following exception: We demand only that the set E consists of elements one of which is ϕ.

It is easy to verify that A_{12} is satisfied in the models of A_1—A_7, A_9 and A_{11} given in § 2. Thus it suffices to prove only the independence of A_8 and A_{12}. The cardinality of every model mentioned above is not, however, minimum (because now the additional restriction concerning the set E is not made and A_{10} need not to be satisfied), but we do not give new models.

4.2. We now consider the model given in 2.12. In that model A_1—A_9

and A_{11} are satisfied. Because $(\phi^* + x)^* = (\phi^*)^* = \phi^*$ and $((\phi^*)^* + x^*)\,(\phi^* x)^* = (\phi^* + a)a = a$, A_{12} is not satisfied and it is independent.

4.3. To prove the independence of A_8 we consider a 5-valued model consisting of the elements ϕ , ϕ^* , x , a and b . The sum, catenation and iteration of the elements are given in Tables 26, 27 and 28.

Table 26

$\alpha + \beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ^*	x	a	b
ϕ^*	ϕ^*	ϕ^*	x	a	b
x	x	x	x	a	b
a	a	a	a	a	b
b	b	b	b	b	b

Table 27

$\alpha\beta$	ϕ	ϕ^*	x	a	b
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
ϕ^*	ϕ	ϕ^*	x	a	b
x	ϕ	x	x	a	b
a	ϕ	a	a	a	b
b	ϕ	b	b	b	b

Table 28

α	α^*
ϕ	ϕ^*
ϕ^*	ϕ^*
x	a
a	b
b	b

If we define $\phi < \phi^* < x < a < b$, then $\alpha + \beta = \max(\alpha , \beta)$. The catenation can be defined as follows: If $\alpha \neq \phi$ and $\beta \neq \phi$, then $\alpha\beta = \max(\alpha , \beta)$, otherwise $\alpha\beta = \phi$. Now it is easy to verify that $A_1 - A_7$ and A_9 are satisfied in this model.

Obviously A_{11} is satisfied, if $\alpha \neq x$ or $n = 1$. But if $n > 1$, then $(x^n)^*(\phi^* + x + \ldots + x^{n-1}) = ax = a = x^*$. Thus A_{11} is satisfied also in this case.

Because $\phi^* + xx^*x^* = \phi^* + xaa = \phi^* + a = a$ but $(xx^*)^* = (xa)^* = a^* = b$, A_8 is not satisfied.

4.4. We have now obtained the following result:

Theorem 3.2. *Each of the axioms in* T *is independent.*

4.5. We give one further model which establishes the independence of A_{12} in such a way that besides the axioms $A_1 - A_9$ and A_{11}, the additional conditions mentioned in Lemmas 3.1 and 3.3 are satisfied (i.e. all the equations given in the original formulation of Red'ko and Salomaa).

We use a 16-valued model and the numbers 1, 2, . . . , 16 are used to designate the elements of the model $(\phi \equiv 1 , \ \phi^* \equiv 2)$. The sum, catenation and iteration of the elements are given in Tables 29, 30 and 31.

Obviously A_3, A_4, A_6, A_7, A_9 and the equations mentioned in Lemma 3.1 are satisfied in this model.

A_8 is clearly satisfied if α is either 1, 2, 7, 13, 14, 15 or 16, and also if $\beta^* = 16$. That A_8 is satisfied is $\beta^* = 2$ can be seen from the table below:

α	3	4	5	6	8	9	10	11	12
$\alpha\alpha^*$	9	10	11	12	8	9	10	11	12

If $\beta^* = 10$, we obtain after some calculation that, if α is either 3, 4, 8, 9 or 10, A_8 is reduced to the form $10 = 10$, and if α is either 5, 6, 11 or 12, then $(\alpha\beta^*)^* = 16 = \phi^* + \alpha\alpha^*\beta^*$. If $\beta^* = 12$, we can similarly verify that A_8 is satisfied.

It is easy to verify that $(\alpha + \beta)^* = \alpha^*\beta^*$ if $\alpha^* = 2$ or $\beta^* = 2$, i.e., α or β is either 1, 2 or 8. Hence we have in this model $(\alpha + \phi^*)^* = \alpha^*$. We suppose now that $\alpha^* \neq 2$ and $\beta^* \neq 2$. If $\alpha^*\beta^* = 16$, then either $\max(\alpha^*, \beta^*) = 16$ or $\max(\alpha^*, \beta^*) = 12$ and $\min(\alpha^*, \beta^*) = 10$. In the latter case we suppose that $\alpha^* = 10$ and $\beta^* = 12$. Then α is either 3, 4, 9 or 10, and β is either 5, 6, 11 or 12. Hence $\alpha + \beta$ is either 7, 13, 14 or 15 and $(\alpha + \beta)^* = 16$. If $\alpha^*\beta^* = 10$, then $\alpha^* = \beta^* = 10$ and hence α and β must take one of the values 3, 4, 9 or 10, and also $\alpha + \beta$ assumes one of those values. Hence $(\alpha + \beta)^* = 10$. If $\alpha^*\beta^* = 12$ we can in a similar manner establish that $(\alpha + \beta)^* = \alpha^*\beta^*$.

Table 29

$\alpha + \beta$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	2	4	4	6	6	7	2	10	10	12	12	13	14	15	16
3	3	4	3	4	7	7	7	3	9	10	13	13	13	14	15	16
4	4	4	4	4	7	7	7	4	10	10	13	13	13	14	15	16
5	5	6	7	7	5	6	7	5	14	14	11	12	13	14	15	16
6	6	6	7	7	6	6	7	6	14	14	12	12	13	14	15	16
7	7	7	7	7	7	7	7	7	14	14	13	13	13	14	15	16
8	8	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	9	10	9	10	14	14	14	9	9	10	15	15	15	14	15	16
10	10	10	10	10	14	14	14	10	10	10	15	15	15	14	15	16
11	11	12	13	13	11	12	13	11	15	15	11	12	13	15	15	16
12	12	12	13	13	12	12	13	12	15	15	12	12	13	15	15	16
13	13	13	13	13	13	13	13	13	15	15	13	13	13	15	15	16
14	14	14	14	14	14	14	14	14	14	14	15	15	15	14	15	16
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	16
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Table 30

$\alpha\beta$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	3	3	3	8	3	3	8	9	9	11	13	13	9	15	16
4	1	4	3	4	5	7	7	8	9	10	11	13	13	14	15	16
5	1	5	8	5	5	5	5	8	9	14	11	11	11	14	15	16
6	1	6	3	7	5	6	7	8	9	14	11	12	13	14	15	16
7	1	7	3	7	5	7	7	8	9	14	11	13	13	14	15	16
8	1	8	8	8	8	8	8	8	9	9	11	11	11	9	15	16
9	1	9	9	9	9	9	9	9	9	9	16	16	16	9	16	16
10	1	10	9	10	14	14	14	9	9	10	16	16	16	14	16	16
11	1	11	11	11	11	11	11	11	16	16	11	11	11	16	16	16
12	1	12	13	13	11	12	13	11	16	16	11	12	13	16	16	16
13	1	13	13	13	11	13	13	11	16	16	11	13	13	16	16	16
14	1	14	9	14	14	14	14	9	9	14	16	16	16	14	16	16
15	1	15	15	15	15	15	15	15	16	16	16	16	16	16	16	16
16	1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Table 31

α	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
α^*	2	2	10	10	12	12	16	2	10	10	12	12	16	16	16	16

A_{11} is clearly satisfied if $n = 1$. If $n > 1$ and $\alpha \neq 15$ then $\alpha^n = \alpha$ and $(\alpha^n)^*(\phi^* + \alpha + \ldots + \alpha^{n-1}) = \alpha^*(\phi^* + \alpha)$. It is now easy to verify that $\alpha^*(\phi^* + \alpha) = \alpha^*$. If $n > 1$ then $15^n = 16$ and A_{11} is also satisfied in this case.

The fact that A_1, A_3 and A_5 are satisfied in this model is in principle easy to verify, but it needs much work. Hence we have done it by using a computer. The program used and the results obtained are given in the appendix.

If $\alpha = 3$ and $\beta = 5$, we have in this model $(\alpha + \beta)^* = 16$ and $(\alpha^* + \beta^*)\ (\alpha\beta)^* = 15$. Thus A_{12} is not satisfied.

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Appendix

```
C
C        FORTRAN IV PROGRAM MENTIONED IN 4.5
C
         INTEGER SUM (16, 16), CATEN (16, 16)
         LOGICAL A1, A2, A5
C
C        SUM OF THE ELEMENTS
C
         WRITE (2, 10)
   10    FORMAT (4H0I + J)
         DO 1 I = 1, 16
         READ (3, 11) (SUM (I, J), J = 1, 16)
   11    FORMAT (16I2)
         WRITE (2, 12) (SUM (I, J), J = 1, 16)
   12    FORMAT (16I3)
      1  CONTINUE
C
C                            CATENATION OF THE ELEMENTS
C
         WRITE (2, 13)
   13    FORMAT (4H0I*J)
         DO 2 I = 1, 16
         READ (3, 11) (CATEN (I, J), J = 1, 16)
      2  WRITE (2, 12) (CATEN (I, J), J = 1, 16)
C
C                            ARE A1, A2 AND A5 SATISFIED
C
         A1 = .TRUE.
         A2 = .TRUE.
         A5 = .TRUE.
         DO 5 I = 1, 16
         DO 5 J = 1, 16
         DO 5 K = 1, 16
C
C                   IPJ = I + J            JPK = J + K
C               IJ = I*J         IK = I*K          JK = J*K
C
         IPJ = SUM (I, J)
         JPK = SUM (J, K)
         IJ = CATEN (I, J)
         IK = CATEN (I, K)
         JK = CATEN (J, K)
C
C                            IS I + (J + K) = (I + J) + K
C
         IF (SUM (I, JPK). EQ. SUM (IPJ, K)) GOTO 3
         A1 = .FALSE.
         L = 1
```

```
            WRITE (2, 14) L, I, J, K
   14       FORMAT (2H0A, I2, 17H IS NOT SATISFIED, 3I3)
C
C                                       IS I*(J*K) = (I*J)*K
C
       3    IF (CATEN (I, JK). EQ. CATEN (IJ, K)) GOTO 4
            A2 = .FALSE.
            L = 2
            WRITE (2, 14) L, I, J, K
C
C                                       IS I*(J + K) = I*J + I*K
C
       4    IF (CATEN (I, JPK). EQ. SUM (IJ, IK)) GOTO 5
            A5 = .FALSE.
            L = 5
            WRITE (2, 14) L, I, J, K
       5    CONTINUE
C
C                                       OUTPUT IF A1, A2 OR A5 IS SATISFIED
C
            L = 1
            IF (A1) WRITE (2, 15) L
   15       FORMAT (2H0A, I2, 13H IS SATISFIED)
            L = 2
            IF (A2) WRITE (2, 15) L
            L = 5
            IF (A5) WRITE (2, 15) L
            CALL EXIT
            END
```

Input data

01020304050607080910111213141516
02020404060607021010121213141516
03040304070707030910131313141516
04040404070707041010131313141516
05060707050607051414111213141516
06060707060607061414121213141516
07070707070707071414131313141516
08020304050607080910111213141516
09100910141414090910151515141516
10101010141414101010151515141516
11121313111213111515111213151516
12121313121213121515121213151516
13131313131313131515131313151516
14141414141414141414151515141516
15151515151515151515151515151516
16161616161616161616161616161616
01010101010101010101010101010101
01020304050607080910111213141516
01030303080303080909111313091516
01040304050707080910111313141516
01050805050505080914111111141516
01060307050607080914111213141516
01070307050707080914111313141516
01080808080808080909111111091516
01090909090909090909161616091616
01100910141414090910161616141616
01111111111111111616111111161616
01121313111213111616111213161616
01131313111313111616111313161616
01140914141414090914161616141616
01151515151515151616161616161616
01161616161616161616161616161616

Results obtained

I+J

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	4	4	6	6	7	2	10	10	12	12	13	14	15	16
3	4	3	4	7	7	7	3	9	10	13	13	13	14	15	16
4	4	4	4	7	7	7	4	10	10	13	13	13	14	15	16
5	6	7	7	5	6	7	5	14	14	11	12	13	14	15	16
6	6	7	7	6	6	7	6	14	14	12	12	13	14	15	16
7	7	7	7	7	7	7	7	14	14	13	13	13	14	15	16
8	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	10	9	10	14	14	14	9	9	10	15	15	15	14	15	16
10	10	10	10	14	14	14	10	10	10	15	15	15	14	15	16
11	12	13	13	11	12	13	11	15	15	11	12	13	15	15	16
12	12	13	13	12	12	13	12	15	15	12	12	13	15	15	16
13	13	13	13	13	13	13	13	15	15	13	13	13	15	15	16
14	14	14	14	14	14	14	14	14	14	15	15	15	14	15	16
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	16
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

I*J

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	3	3	8	3	3	8	9	9	11	13	13	9	15	16
1	4	3	4	5	7	7	8	9	10	11	13	13	14	15	16
1	5	8	5	5	5	5	8	9	14	11	11	11	14	15	16
1	6	3	7	5	6	7	8	9	14	11	12	13	14	15	16
1	7	3	7	5	7	7	8	9	14	11	13	13	14	15	16
1	8	8	8	8	8	8	8	9	9	11	11	11	9	15	16
1	9	9	9	9	9	9	9	9	9	16	16	16	9	16	16
1	10	9	10	14	14	14	9	9	10	16	16	16	14	16	16
1	11	11	11	11	11	11	11	16	16	11	11	11	16	16	16
1	12	13	13	11	12	13	11	16	16	11	12	13	16	16	16
1	13	13	13	11	13	13	11	16	16	11	13	13	16	16	16
1	14	9	14	14	14	14	9	9	14	16	16	16	14	16	16
1	15	15	15	15	15	15	15	16	16	16	16	16	16	16	16
1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

A 1 IS SATISFIED
A 2 IS SATISFIED
A 5 IS SATISFIED

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