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ON THE SOLUTIONS OF $\Delta u = Pu$ FOR ACCEPTABLE DENSITIES ON OPEN RIEMANN SURFACES

BY

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Preface

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Helsinki, February 1972

Aatos Lahtinen

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INTRODUCTION

In this paper we study the solutions of the elliptic partial differential equation

$$\Delta u = Pu$$

on open Riemann surfaces for densities P which we call acceptable. This means that there exists a positive P-superelliptic function ω defined on the whole surface. The equation has earlier been studied by M. Ozawa [11], L. Myrberg [6]—[8], H. Royden [12] a.o. with a stronger restriction $P \geqq 0$, which guarantees the validity of the maximum principle contrary to our situation.

After some preliminaries we solve the first boundary value problem in section 2 and construct the Green's function for regular regions in section 3. This will be done using Perron's method. In section 4 it appears that our condition for P is equivalent to the Dirichlet problem to be uniquely solvable in compact regions with regular boundaries. Sufficient conditions for the existence of the Green's function on the whole surface will be given in section 5 after which we introduce solution spaces BP and MP in section 6.

Our main purpose is to compare these Banach spaces with different densities P. This has earlier been done for BP by Royden [12] and Nakai [10], when $P \geqq 0$. We give generalisations of Nakai's result for acceptable densities in section 7. Finally in section 8 we compare the spaces BP and HB when $P \geqq 0$ and state a new isometry condition for them. By using this we can show that Nakai's condition is not a necessary one.

1. PRELIMINARIES

1.1. Notations and definitions

By R we denote an open Riemann surface.

By a *regular region* K we mean an open connected set, whose closure $\bar{K}$ is compact and whose boundary ∂K consists of a finite number of closed analytic curves.

By $\{R_n\}$ we denote an exhaustion of R, which has the following properties: (1) R_n is a regular region, (2) $\bar{R}_n \subset R_{n+1}$, (3) $\bigcup_{n=1}^{\infty} R_n = R$.

The set $C^0(K)$ is the set of all continuous real valued mappings in a region K, $C^0(R) = C^0$. We denote that $u \in C^n(K)$, if the function u has continuous partial derivatives up to the order n, $C^n(R) = C^n$.

By a *density* P we mean a real valued function which belongs to C^1 and is transformed in the change of local parameter in such a way that

$$P(z)\, |dz|^2$$

is invariant.

If P is a density, the elliptic partial differential equation

$$\Delta u = Pu \tag{1.1}$$

is invariantly defined on a Riemann surface.

Definition 1.1.1. *A real valued function u is said to be a P-solution in a region K, if $u \in C^2(K)$ and it is a solution of (1.1) in K.*

Definition 1.1.2. *Let K be a compact region whose boundary is the union of two disjoint sets k_1 and k_2. We say that a P-solution $w_P(K, k_1)$ is the elliptic measure of k_1 with respect to K, if it is identically one on k_1 and zero on k_2. If especially k_2 is empty, we say that $w_P(K, \partial K) = w_P(K)$ is the elliptic measure of K.*

Definition 1.1.3. *We say that a real valued function $G_P(K, z, z_0)$ is the Green's function of (1.1) in a compact region K if*

(1) *$G_P(K, z, z_0)$ is a P-solution in $K-\{z_0\}$ and continuous in $\bar{K}-\{z_0\}$,*

(2) *$G_P(K, z, z_0) + \log |z-z_0|$ is bounded in a neighbourhood of z_0,*

(3) *$G_P(K, z, z_0) = 0$, when $z \in \partial K$, $z_0 \in K$.*

The Green's function of a compact region K is usually denoted by $G_P(K)$.

If we handle an exhaustion $\{R_n\}$, we denote for short $w_P(R_n) = w_P^n$, $G_P(R_n) = G_P^n$ etc..

1.2. Some auxiliary results

We state some important results we shall need later on. The first one deals with the situation in the small.

Lemma 1.2.1. *To every parametric disc (V, z) there exists a reduced disc (V_0, z), $V_0 \subset V$, such that in V_0 the following are valid:*

(1) *There exists one and only one P-solution which is equal to a given function $f \in C^0(\partial V_0)$ on ∂V_0.*

(2) *Equation (1.1) has the Green's function* $G_P(V_0)$.

(3) $G_P(V_0)$ *has the following properties:*

(a) $\frac{\partial}{\partial x}(G_P(V_0, z, z_0) + \log|z-z_0|)$ *and* $\frac{\partial}{\partial y}(G_P(V_0, z, z_0) + \log|z-z_0|)$ *are bounded.*

(b) $G_P(V_0, z, z_0) \geqq 0$.

(c) $\inf_{z\in\partial V_0} \frac{\partial}{\partial n} G_P(V_0, z, z_0) > 0$ *and* $\sup_{z\in\partial V_0} \frac{\partial}{\partial n} G_P(V_0, z, z_0) < \infty$, *where* n *is the direction of the inward normal.*

For the proofs see e.g. [4] pp. 20, 66, 77 and 80 regarding 1, 2 and 3 a and [9] regarding 3 c as well as the existence and continuity of the normal derivative. 3 b follows from 1 and 3 a.

By the expression *reduced disc* we shall always mean a parametric disc V_0 for which lemma 1.2.1 is valid.

Next we present Harnack's inequalities for P-solutions. They are proved by using lemma 1.2.1 and the compactness of K quite as for non-negative densities (Cf. [9]).

Lemma 1.2.2. *Let* K *be a compact and* L *a regular region with* $\bar{L} \subset K$. *Then there exists a positive constant* k *such that for each pair of points* (z_1, z_2) *in* $\bar{L}$ *and for each non-negative P-solution* u *in* K *we have*

$$k^{-1}u(z_1) \leqq u(z_2) \leqq ku(z_1).$$

Remark 1.2.3. *If* u *is a non-negative P-solution which vanishes at one point, then by lemma 1.2.2 it vanishes identically.*

A corollary of these inequalities is the Harnack's principle, which by obvious modifications can be shown in the same way as for harmonic functions (Cf. [1] p. 236 and [2] p. 134).

Lemma 1.2.4. *Let* U *be a non-empty family of P-solutions on a Riemann surface with the following property: If* $u_1, u_2 \in U$, *there exists a function* $u \in U$ *such that* $u \geqq \max(u_1, u_2)$. *Then the function* u_0,

$$u_0 = \sup\{u \mid u \in U\}$$

is either a P-solution or identically $+\infty$.

1.3. Subelliptic functions

When solving the first boundary value problem we shall use auxiliary functions which we call *subelliptic*. They can be defined in the same way as for non-negative densities and also have similar properties (Cf. [8]).

Their role will be the same as the subharmonic functions have in the theory of harmonic functions.

Definition 1.3.1. *A real valued function* v *is said to be P-subelliptic in a region* K, *if* $v \in C^0(K)$ *and to any point* $z_0 \in K$ *there exists a reduced disc* $(V_0, z_0) = \{z \mid |z-z_0| < r_0\}$, $\bar{V}_0 \subset K$ *such that in the disc* $(V_r, z_0) = \{z \mid |z-z_0| < r\}$

$$v(z_0) \leqq I_v^P(V_r, z_0) \quad , \quad 0 < r \leqq r_0 ,$$

where

$$I_v^P(V_r, z_0) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\phi}) \frac{\partial}{\partial n} G_P(V_r, re^{i\phi}, z_0)\, r d\phi .$$

v *is said to be P-superelliptic, if* $-v$ *is P-subelliptic.*

If especially $v \in C^2(K)$, we can deduce from Green's formulas

$$(1.3.1) \quad v(z_0) = I_v^P(V_r, z_0) - \frac{1}{2\pi} \iint_{V_r} G_P(V_r, z, z_0)\, (\Delta v(z) - P(z)\, v(z))\, dxdy .$$

This implies the following result.

Remark 1.3.2. *A function* $v \in C^2(K)$ *is P-subelliptic in* K *if and only if*

$$(1.3.2) \qquad \Delta v - Pv \geqq 0 .$$

A function $v \in C^2(K)$ *is a P-solution in* K *if and only if for each* (V_0, z_0)

$$v(z_0) = I_v^P(V_r, z_0) \quad , \quad 0 < r \leqq r_0 .$$

Remark 1.3.3. *Let* P *and* Q *be densities with* $P \leqq Q$. *Then every non-negative Q-subelliptic function is P-subelliptic.*

The next two lemmas are direct consequences of the definition.

Lemma 1.3.4. *If* v_1 *and* v_2 *are P-subelliptic and* α *a non-negative constant, then* $v_1 + v_2$, αv_1 *and* $\max(v_1, v_2)$ *are P-subelliptic.*

Lemma 1.3.5. *If* v *is P-subelliptic in a region* K *and* V_0 *is a reduced disc,* $\bar{V}_0 \subset K$, *then the function* v_0,

$$v_0 = \begin{cases} v & in \quad K - V_0 \\ I_v^P & in \quad V_0 \end{cases}$$

is P-subelliptic in K.

The function v_0 is called the *P-modification* of v (in V_0).

In the continuation we have great use of the next forms of the maximum principle.

Lemma 1.3.6. *Let* ω *be a positive P-superelliptic function on a Riemann surface* R. *If for a P-subelliptic function* v

$$0 \leqq \sup_R \frac{v}{\omega} = M < \infty \,,$$

then either $v < M\omega$ *or* $v \equiv M\omega$.

Proof: If there exists a point z_0 with $v(z_0) = M\omega(z_0)$, then in a reduced disc (V_0 , z_0)

$$v(z_0) = M\omega(z_0) \geqq I^P_{M\omega} (V_r , z_0) \geqq I^P_v(V_r , z_0) \geqq v(z_0) \,, \quad 0 < r \leqq r_0 \,.$$

Therefore

$$I^P_{M\omega - v}(V_r , z_0) = 0 \,, \qquad 0 < r \leqq r_0 \,.$$

By lemma 1.2.1 and definition 1.3.1 this is only possible, when $M\omega - v \equiv 0$. This proves the lemma.

Lemma 1.3.7. *Let* K *be a compact region,* ω *in* K *a P-superelliptic function, positive and continuous in* $\bar{K}$. *If for a P-subelliptic function* v *defined in* K $\sup_K v \geqq 0$ *and*

$$\overline{\lim_{z \to \zeta \in \partial K}} \frac{v(z)}{\omega(z)} \leqq M < \infty \,,$$

then either $v < M\omega$ *or* $v \equiv M\omega$.

The proof will be quite similar to the special case $P \geqq 0, \omega = w_P(K)$, (Cf. [8]).

We often use the latter in the following form.

Corollary 1.3.8. *Let* K *be a compact region,* ω *in* K *a P-superelliptic function, positive and continuous in* $\bar{K}$. *If for a P-subelliptic function* v *defined in* K

$$\overline{\lim_{z \to \zeta \in \partial K}} v(z) \leqq 0$$

then $v \leqq 0$ *in* K .

Finally we give the definition of a Perron family and state its characteristic property which follows from Harnack's principle, lemma 1.2.4, quite as in the harmonic case.

Definition 1.3.9. *A non-empty family* F_P *of P-subelliptic functions* v *on a Riemann surface is called a Perron family, if the following two conditions are fulfilled.*

(1) *If* $v_1 , v_2 \in F_P$, *then* $\max(v_1 , v_2) \in F_P$.

(2) *If* $v \in F_P$, *then every P-modification* $v_0 \in F_P$.

Lemma 1.3.10. *If* F_P *is a Perron family, the function*

$$u_0 = \sup \{v \mid v \in F_P\}$$

is either a P-solution or identically $+ \infty$.

2. THE FIRST BOUNDARY VALUE PROBLEM

2.1. Acceptable densities and the uniqueness of a solution

It is well known that the solution of Dirichlet's problem is not always unique for an arbitrary density. Therefore we present a restrictive condition, which guarantees the uniqueness by making use of corollary 1.3.8.

Definition 2.1.1. *A density P is acceptable by ω on a Riemann surface R, if there exists a real valued positive P-superelliptic function ω. ω is called the accepting function of P.*

If a density P is acceptable by $\omega \in C^2$, then we have by remark 1.3.2 a lower bound for P

$$P \geqq \frac{\Delta \omega}{\omega} . \tag{2.1.1}$$

This shows that acceptable densities can also have negative values and they form a wider class than non-negative densities.

From here on any density we use will always be acceptable by its accepting function ω. This function will play a somewhat similar role as the positive constants have in the theory of non-negative densities.

Remark 2.1.2. *Every non-negative density is acceptable by 1.*

Remark 2.1.3. *If Q is acceptable by ω and P is a density with $P \geqq Q$, then P is also acceptable by ω.*

Now we show that our condition guarantees the uniqueness of Dirichlet's problem.

Theorem 2.1.4. *Let P be acceptable and K a compact region. Then the first boundary value problem has at most one solution.*

Proof: If u_1 and u_2 are P-solutions with $u_1 = u_2$ on ∂K, then by corollary 1.3.8 both $u_1 - u_2$ and $u_2 - u_1$ are non-positive in K. Therefore $u_1 = u_2$.

The theory of non-negative densities is strongly based on the maximum principle which states that for a P-solution u in a compact region K, $u \in C^0(\bar{K})$,

$$\sup_K |u| = \sup_{\partial K} |u| . \tag{2.1.2}$$

Now the situation is more complicated. We have by restricting ourselves to the acceptable densities achieved the uniqueness of the first boundary value problem. This does, however, not imply the existence of an extremum principle for the boundary values as the following simple example shows.

Example 2.1.5. We choose $K = \{z \mid |z| < 1\}$ and

$$P(z) = 4(8\,|z|^2 - 3)\,(2\,|z|^4 - 3\,|z|^2 + 2)^{-1} .$$

P is acceptable by $u(z) = 2\,|z|^4 - 3\,|z|^2 + 2$ which is a P-solution. On ∂K $u \equiv 1$ but in K u has both greater and smaller values, because $u(0) = 2$ and $u(\frac{1}{2}\sqrt{3}) = \frac{7}{8}$.

This happens because acceptable densities can have both positive and negative values. In fact, if $P \leqq 0$ is acceptable, then for a P-solution u in a compact region K, $u \in C^0(\bar{K})$,

$$\inf_K |u| = \inf_{\partial K} |u| . \tag{2.1.3}$$

2.2. Existence of a solution

We start by defining what we mean by a *regular boundary.*

Definition 2.2.1. *Let K be a compact region. We say that ∂K is P-regular, if for any continuous boundary values the Dirichlet problem has a unique solution which is a P-solution. If especially the solution is harmonic, we say that ∂K is regular.*

For non-negative densities the solvability of the first boundary value problem has been thoroughly investigated. Therefore we do not enter deeper into it, but cite the following result (Cf. [13] p. 741 and 759).

Lemma 2.2.2. *Let K be a compact region. If ∂K is regular, it is also P-regular for each non-negative P.*

There is, however, not much literature about the case where the maximum principle is not valid, which is just the situation we have. That is why we have to examine this possibility more closely. We intend to keep lemma 2.2.2 as known and advance from it by Perron's method. First we define in the usual way the family $V(f)$ and its least upper bound.

Definition 2.2.3. *Let P be acceptable, K a compact region and f a real valued continuous function defined on ∂K. Then*

$$V(f) = \{v \mid v \ \textit{P-subelliptic in } K \textit{ and } \overline{\lim_{z \to \zeta \in \partial K}}\, v(z) \leqq f(\zeta)\}$$

and

$$u_f = \sup \ \{v \mid v \in V(f)\} .$$

It is easy to see that $V(f)$ is a Perron family, or empty. Now we are able to extend lemma 2.2.2 to acceptable densities.

Theorem 2.2.4. *Let K be a compact region. If ∂K is regular, it is also P-regular for each acceptable density P.*

Proof: If $P \geqq 0$ then this is true by lemma 2.2.2. Therefore we suppose that P has also negative values.

Let first $f \in C^0(\partial K)$, $f \geqq 0$. If P is acceptable by ω, then there exists a positive constant M such that $M\omega \geqq f$ on ∂K.

Let Q be a non-negative density such that $Q \geqq P$ in K. By lemma 2.2.2 there exist Q-solutions v_1 and v_2 such that on ∂K

$$v_1 = M\omega - f \geqq 0 \, , \quad v_2 = f \geqq 0 \, .$$

The function $\omega_1 = M\omega - v_1$ is in K P-superelliptic (Remark 1.3.3) and is equal to f on ∂K. If now $v \in V(f)$ then $\overline{\lim}_{z \to \zeta \in \partial K} v(z) - \omega_1(\zeta) \leqq 0$. By corollary 1.3.8 $v \leqq \omega_1$ in K. Therefore $V(f)$ is bounded from above by ω_1.

On the other hand v_2 is P-subelliptic and belongs to $V(f)$ which is thus non-empty. By lemma 1.3.10 u_f is a P-solution and by construction

$$f(\zeta) = \underline{\lim}_{z \to \zeta} \; v_2(z) \leqq \underline{\lim}_{z \to \zeta} \; u_f(z) \leqq \overline{\lim}_{z \to \zeta} \; u_f(z) \leqq \omega_1(\zeta) = f(\zeta) \, ,$$

that is,

$$\lim_{z \to \zeta \in \partial K} u_f(z) = f(\zeta) \, , \quad f \geqq 0 \, .$$

For non-negative boundary values u_f is thus the solution of Dirichlet's problem.

If $f \in C^0(\partial K)$ and has also negative values, then we can use the decomposition

$$f = f^+ - f^- \, , \quad f^\pm \in C^0(\partial K) \, , \quad f^\pm \geqq 0 \, .$$

By the preceding part there exist P-solutions u_{f+} and u_{f-} such that on ∂K $u_{f\pm} = f^\pm$. The P-solution $u_f = u_{f+} - u_{f-}$ has then the right boundary values.

So u_f is the solution of the first boundary value problem and by theorem 2.1.4 the only one.

3. THE GREEN'S FUNCTION IN REGULAR REGIONS

3.1. The existence of the Green's function

By lemma 1.2.1 every reduced disc has the Green's function. We shall now show by using Perron's method that this implies the existence of the Green's function in a regular region if the density is acceptable.

Definition 3.1.1. *Let P be acceptable, K a regular region and $z_0 \in K$. We say that a P-subelliptic function v defined in $K - \{z_0\}$ belongs to W_P if*

(1) $\overline{\lim}_{z \to \zeta \in \partial K} \; v(z) \leqq 0$,

(2) *in a reduced disc* (V_0, z_0), $\bar{V}_0 \subset K$, *the expression*

$$v(z) + G_P(V_0, z, z_0)$$

is non-positive and bounded from below.

Remark 3.1.2. *If* $v \in W_P$, *then* $v < 0$ *in* $K - \{z_0\}$.

Lemma 3.1.3. W_P *is a Perron family.*

Proof: We start by showing that W_P is not empty. Let (V_0, z_0), (V_1, z_0) be reduced discs with $\bar{V}_1 \subset V_0$, $\bar{V}_0 \subset K$ and $K_1 = K - \bar{V}_1$. We denote by

$$a = \sup_{z \in \partial V_1} G_P(V_0, z, z_0), \quad 0 < a < \infty .$$

Let moreover u be a P-solution in V_0 with $u = b\, w_P(K_1, \partial K)$ on ∂V_0, where b is a constant chosen so that $u \geqq a$ on ∂V_1 and let g be the following function

$$g(z, z_0) = \begin{cases} b(w_P(K_1, \partial K, z) - w_P(K, z)) & \text{in } K - V_0 \\ u(z) - G_P(V_0, z, z_0) - b w_P(K, z) & \text{in } V_0 - \{z_0\} . \end{cases}$$

Then g is P-subelliptic. This is clear if $z \in K - \bar{V}_0$ or $z \in V_0 - \{z_0\}$. If $z \in \partial V_0$, let (V, z) be a reduced disc, $\bar{V} \subset K_1$. Then

$$\begin{aligned} g(z, z_0) &= b(w_P(K_1, \partial K, z) - w_P(K, z)) \\ &= I^P_{b(w_P(K_1, \partial K) - w_P(K))} \leqq I^P_g(z) . \end{aligned}$$

From the construction follows that

(1) $\lim_{z \to \zeta \in \partial K} g(z, z_0) = 0$

(2) $g(z, z_0) + G_P(V_0, z, z_0) = u(z) - b w_P(K, z)$ in V_0.

Because $u - b w_P(K) < 0$ and bounded from below in V_0, $g \in W_P$.

Next we notice that if $v_1, v_2 \in W_P$, then clearly $\max(v_1, v_2) \in W_P$, too.

Let then $v \in W_P$ and v_0 be its P-modification in the disc V_2. Now in V_0

$$v_0(z) + G_P(V_0, z, z_0) \leqq 0 .$$

In fact, let $z \in V_0$. If $z \notin V_2$, then

$$v_0(z) + G_P(V_0, z, z_0) = v(z) + G_P(V_0, z, z_0) \leqq 0 .$$

If $z \in V_2$, then

$$v_0(z) + G_P(V_0, z, z_0) = \begin{cases} v(z) + G_P(V_0, z, z_0) \leqq 0 & \text{on } (\partial V_2) \cap \bar{V}_0 , \\ v_0(z) < 0 & \text{on } V_2 \cap (\partial V_0) . \end{cases}$$

Because $v_0 + G_P(V_0)$ is a P-solution in $V_0 \cap V_2$ it must be there non-positive.

After this it is obvious that $v_0 \in W_P$.

This shows W_P to be a Perron family.

Now we can prove the existence of $G_P(K)$.

Theorem 3.1.4. *Let P be acceptable and K a regular region. Then the Green's function $G_P(K)$ exists and*

$$G_P(K, z, z_0) = -\sup \{v \mid v \in W_P\}.$$

Proof: We have shown that W_P is a Perron family which is bounded above by zero. Therefore

$$u(z, z_0) = \sup \{v \mid v \in W_P\}$$

is a P-solution in $K - \{z_0\}$. Moreover by definition and lemma 3.1.3 $u(z, z_0) - \log |z - z_0|$ is bounded in a neighbourhood of z_0 and

$$\lim_{z \to \zeta \in \partial K} u(z, z_0) = 0 .$$

Thus $-u(z, z_0) = G_P(K, z, z_0)$.

Remark 3.1.5. *The Green's function is symmetric, that is*

$$G_P(K, z, z_0) = G_P(K, z_0, z) .$$

3.2. The linear mapping T_{PQ}^K

We define here the linear mapping T_{PQ}^K. It will be used in proving the properties of the linear transformation T_{PQ} with which we examine the isometry of solution spaces in sections 7 and 8.

The proof of the following lemma is a direct consequence of Green's formulas and theorems 2.2.4 and 3.1.4.

Lemma 3.2.1. *Let K be a regular region, P and Q acceptable, and $u \in C^0(\bar{K})$ a P-solution. We define the linear transformation $T_{PQ}^K u$ as follows:*

$$(3.2.1) \qquad T_{PQ}^K u(z_0) = u(z_0) + \frac{1}{2\pi} \iint_K (P(z) - Q(z))\, G_Q(K, z, z_0) u(z)\, dx dy .$$

Then

(1) $T_{PQ}^K u \in C^0(\bar{K})$ *and is a Q-solution in K.*

(2) $T_{PQ}^K u = u$ *on* ∂K.

Remark 3.2.2. *Let u, $u_1 \in C^0(\bar{K})$ be P-solutions and c a positive constant.*

(1) *If* $|u| \leqq cu_1$, *then* $|T^K_{PQ}u| \leqq cT^K_{PQ}u_1$.

(2) $T^K_{QP}\,T^K_{PQ}u = u$.

(3) *Let* P *and* Q *be acceptable by* ω . *If* $|u| \leqq c\omega$, *then* $|T^K_{PQ}u| \leqq c\omega$.

We usually denote $T^{R_n}_{PQ} = T^n_{PQ}$.

3.3. The uniqueness of the Green's function

We want to prove that the Green's function $G_P(K)$ is uniquely determined. For that we need some auxiliary results.

Lemma 3.3.1. *Let* P *be acceptable,* L *a closed set and* K *a regular region containing* L. *If the harmonic measure of* L *with respect to* K *is zero, then also its elliptic measure* $w_P(K, L)$ *is zero.*

Proof: Let $\{K_n\}$ be an exhaustion of $K - L$ by regular regions K_n , $\partial K_n = \partial K \cup k_n$ and $h(k_n)$ the harmonic measure of k_n with respect to K_n . We denote $w_P(K_n, k_n) = w^n_P$, $T^{K_n}_{OP} = T^n_{OP}$ and $G_P(K_n) = G^n_P$. Then $w^n_P = T^n_{OP}h(k_n)$. Because K is compact, $|P|\, G_P(K)$ is integrable and

$$|PG^n_P\, h(k_n)| \leqq |P|\, G_P(K) \, .$$

If we continue the domain of $G^n_P\, h(k_n)$ over the whole K by setting $G^n_P\, h(k_n) \equiv 0$ in $K - K_n$, we get by Lebesgue's theorem of dominated convergence (Cf. e.g. [5] p. 234)

$$\lim_{n\to\infty} \iint_{K_n} P\, G^n_P\, h(k_n)\, dxdy = \iint_K P \lim_{n\to\infty} G^n_P\, h(k_n)\, dxdy = 0 \, ,$$

because $\lim\limits_{n\to\infty} h(k_n) = 0$. Therefore

$$w_P(K, L) = \lim_{n\to\infty} w^n_P = \lim_{n\to\infty} T^n_{OP}h(k_n) = 0 \, ,$$

which proves the theorem.

By using this result we can prove the following lemma quite in the same way as for the non-negative densities (Cf. [7]).

Lemma 3.3.2. *Let* K *be a regular region and* L *a closed set contained in* K *with the harmonic measure in respect to* K *zero. If* P *is acceptable and* u *a P-solution which vanishes on* ∂K *and is bounded in* $K - L$, *then* $u \equiv 0$. *Moreover every bounded P-solution defined in* $K - L$ *is a P-solution in the whole* K .

Now we are able to show the uniqueness.

Theorem 3.3.3. *If* P *is acceptable and* K *a regular region,* $G_P(K)$ *is uniquely determined.*

Proof: If $g_P(K)$ is also a Green's function in K, then $g_P(K, z, z_0) - G_P(K, z, z_0)$ vanishes on ∂K and is bounded in $K - \{z_0\}$. Therefore $g_P(K, z, z_0) - G_P(K, z, z_0) \equiv 0$.

4. THE EXISTENCE OF SOLUTIONS ON THE WHOLE SURFACE

4.1. Solutions for acceptable densities

We shall show that every acceptable density P, $P \not\equiv 0$, has a non-constant solution defined on the whole surface. The proof of the auxiliary lemma and the main result are again quite similar to the ones for non-negative densities (Cf. [6] and [9]).

Lemma 4.1.1. *Let P be acceptable, $\{R_n\}$ an exhaustion of R and $\{u_n\}$ a sequence of P-solutions bounded on every compact set, each u_n defined in R_n. Then there exists a subsequence, which converges uniformly on every compact subset of R towards a P-solution.*

If we now choose a point $z_0 \in R_1$ and if w_P^n is the elliptic measure of R_n, then the sequence

(4.1.1) $$\{u_n \mid u_n(z) = (w_P^n(z_0))^{-1}\, w_P^n(z)\}$$

is by lemma 1.2.2 bounded in every compact set because $u_n(z_0) = 1$ for each n. By lemma 4.1.1 we have

Theorem 4.1.2. *Every acceptable density P, $P \not\equiv 0$, has a positive non-constant P-solution defined on the whole surface.*

4.2. Acceptable densities and Dirichlet problem

We restricted the inspection to the acceptable densities in order to guarantee the uniqueness of Dirichlet's problem. With help of the preceding result we are now able to notice that the restriction is not too stringent.

Theorem 4.2.1. *Let K be a compact region with a regular boundary and P a density. ∂K is P-regular if and only if P is acceptable.*

Proof: If P is acceptable, then ∂K is P-regular by theorem 2.2.4. If ∂K is P-regular, then by analysing the proofs which led to theorem 4.1.2 we see that it guarantees the existence of a positive P-solution u defined on R. P is then acceptable by u.

5. THE GREEN'S FUNCTION ON THE WHOLE SURFACE

5.1. The definition of the Green's function

Let P be acceptable, $\{R_n\}$ an exhaustion of R and $z_0 \in R_1$. The sequence $\{G_P^n(z, z_0)\}$ is strictly increasing wherefore by lemma 1.2.4 it either increases uniformly to $+\infty$ on every compact set or there exists a limit function G_P

$$G_P(z, z_0) = \lim_{n\to\infty} G_P^n(z, z_0) , \tag{5.1.1}$$

which we call the Green's function of P on R.

It is known that every acceptable density has not G_P (Cf. [9]).

Definition 5.1.1. *A density P is said to be completely acceptable, if it is acceptable and has the Green's function on R.*

The function G_P does not depend on exhaustion but is uniquely determined by the following property:

Lemma 5.1.2. *Let P be completely acceptable. Then $G_P(z, z_0)$ is the smallest of functions $u(z, z_0)$ such that*

(1) *$u(z, z_0)$ is a non-negative P-solution on $R - \{z_0\}$,*

(2) *$u(z, z_0) + \log |z - z_0|$ is bounded in a neighbourhood of z_0.*

Proof: By lemma 3.3.2 $v_n(z, z_0) = G_P^n(z, z_0) - u(z, z_0)$ is a P-solution in R_n. Because $v_n \leqq 0$ on ∂R_n, $v_n \leqq 0$ in R_n. Therefore

$$G_P(z, z_0) - u(z, z_0) = \lim_{n\to\infty} v_n(z, z_0) \leqq 0 .$$

Remark 3.1.5 can be generalized to G_P.

Remark 5.1.3. *The Green's function G_P is symmetric, that is,*

$$G_P(z, z_0) = G_P(z_0, z) .$$

This implies that the existence of G_P does not depend on the pole z_0 but only on the density and surface.

Before going more closely to the existence problem we give a couple of inequalities for G_P for later use.

Lemma 5.1.4. *Let P be completely acceptable by ω and K, L regular regions with $\bar{L} \subset K$. Then there exist positive constants k and m such that for every $z_0, z_1 \in L$, and $z \in R - K$ we have*

(1) $k^{-1} G_P(z, z_0) \leqq G_P(z, z_1) \leqq k\, G_P(z, z_0)$,

(2) $G_P(z, z_0) \leqq m\, \omega(z)$.

Proof: The first inequalities follow directly from lemma 1.2.2 and remark 5.1.3. For the second formula let

$$s = \sup_{\substack{z \in \partial K \\ z_0 \in L}} G_P(z, z_0) \ , \qquad a = \inf_{z \in \partial K} \omega(z) \ .$$

Then in $R_n - K$ we have

$$G_P^n(z, z_0) \leqq \frac{s}{a} \, \omega(z) \ ,$$

which gives

$$G_P(z, z_0) = \lim_{n \to \infty} G_P^n(z, z_0) \leqq \frac{s}{a} \, \omega(z) \ .$$

Remark 5.1.5. *Let P be a density, Q completely acceptable and $u \in C^0$. Then by lemma 5.1.4 (1) the convergence of the integral*

$$\iint_R |P(z)| \ G_Q(z, z_0) \ |u(z)| \ dxdy$$

does not depend on the pole z_0.

5.2. The existence of G_P

When examining isometric relations of solution spaces we need the Green's function G_P. Therefore we investigate on which conditions densities are completely acceptable. The first result is that every density P having an acceptable minorant $Q, Q \not\equiv P$, is completely acceptable. The second says that a density P acceptable by ω is completely acceptable if a kind of elliptic measure of the ideal boundary with respect to ω is positive.

Theorem 5.2.1. *A density P is completely acceptable, if there exists an acceptable density Q with $P \geqq Q$, $P \not\equiv Q$.*

Proof: P is acceptable by remark 2.1.3. According to theorem 4.1.2 there exists a positive Q-solution u defined on the whole surface. If $\{R_n\}$ is an exhaustion of R, then by lemma 3.2.1 $T_{QP}^n u$ is a P-solution in R_n with $T_{QP}^n u = u$ on ∂R_n. Therefore $0 < T_{QP}^n u < u$ in R_n and from (3.2.1) we get the inequality

$$(5.2.1) \qquad \frac{1}{2\pi} \iint_{R_n} (P(z) - Q(z)) \, G_P^n(z, z_0) \, u(z) \, dxdy < u(z_0) \ .$$

The integrand is non-negative and does not vanish identically. The sequence $\{G_P^n(z, z_0)\}$ cannot then increase towards infinity uniformly in every compact set. This guarantees the existence of G_P.

Next we give two corollaries in order to show how this theorem can be used to prove the existence of the Green's function in different cases.

Corollary 5.2.2. *Every non-negative density* P , $P \not\equiv 0$, *is completely acceptable.*

Proof: Choose $Q = \frac{1}{2} P$ (Cf. [7]) .

Corollary 5.2.3. *Let* P *be acceptable by* ω . *If there exists a parametric disc* V *such that* ω *is not a P-solution at any point of* V , *then* P *is completely acceptable.*

Proof: For any disc (V_0, z_0) , $\bar{V}_0 \subset V$, we have

$$I^P_\omega(V_0 , z_0) < \omega(z_0) .$$

Solutions of (1.1) in V with boundary values ω are continuously depending on the density P . Therefore we can choose a density Q with $Q = P$ in $R - V$, $Q \leqq P$ in V such that $Q \not\equiv P$ and

$$I^Q_\omega(V_0 , z_0) \leqq \omega(z_0)$$

for any disc (V_0, z_0) , $\bar{V}_0 \subset V$. This density Q is now acceptable by ω and a minorant of P . P is then completely acceptable.

This situation happens e.g. when $\omega \in C^2(V)$ and it fails to be a P-solution at one point of V .

If we cannot find any acceptable minorant to the given density, we have to solve the existence of the Green's function otherwise. In order to find a relatively simple condition we introduce a suitable auxiliary function.

Let P be acceptable by ω , K a regular region and $\{R_n\}$ an exhaustion of R with $\bar{K} \subset R_1$. We define on $R - \bar{K}$ a P-solution ω_K as follows:

Let ω_K^n be a P-solution in $R_n - \bar{K}$ with

$$\omega_K^n = \begin{cases} 0 & \text{on } \partial K \\ \omega & \text{on } \partial R_n . \end{cases}$$

Then $0 \leqq \omega_K^n \leqq \omega$ for each n . By lemma 4.1.1 there exists a subsequence $\{\omega_K^{n_i}\}$ converging to a P-solution ω_K on $R - \bar{K}$ uniformly in compact sets. It is easily seen that ω_K does not depend on exhaustion or subsequence. By remark 1.2.3 either $\omega_K \equiv 0$ or $\omega_K > 0$.

Theorem 5.2.4. *Let* P *be acceptable by* ω . *If there exists a regular region* K *such that* $\omega_K > 0$, *then* P *is completely acceptable.*

Proof: We construct an upper bound to the sequence $\{G_P^n(z, z_0)\}$ by using the same method we had in the proof of lemma 3.1.3.

Let L be a regular region with $\bar{K} \subset L$ and $z_0 \in K$. Let moreover $\{R_n\}$ be an exhaustion of R such that $\lim_{n\to\infty} \omega_K^n = \omega_K$ and $L \subset R_1$. We define in L a P-solution u_L^n by its boundary values: $u_L^n = b\,\omega_K^n$ on ∂L, where b is a constant chosen so that $u_L^n \geqq \sup_{z\in\partial K} G_P(L, z, z_0)$ on ∂K for every n. This is possible because $\lim_{n\to\infty} \omega_K^n = \omega_K > 0$.

Next we form in $R_n - \{z_0\}$ a P-subelliptic function $g_n(z, z_0)$:

$$g_n(z, z_0) = \begin{cases} b(\omega_K^n(z) - \omega(z)) & \text{in } R_n - L \\ u_L^n(z) - G_P(L, z, z_0) - b\,\omega(z) & \text{in } L - \{z_0\}\,. \end{cases}$$

As in the proof of lemma **3.1.3** we see that

$$G_P^n(z, z_0) \leqq - g_n(z, z_0)\,.$$

Because $\lim_{n\to\infty} \omega_K^n > 0$ there exists $\lim_{n\to\infty} u_L^n = u_L$ and

$$\lim_{n\to\infty} G_P^n(z, z_0) \leqq - \lim_{n\to\infty} g_n(z, z_0)$$

$$= \begin{cases} b(\omega(z) - \omega_K(z)) & \text{in } R - L \\ b\,\omega(z) + G_P(L, z, z_0) - u_L(z) & \text{in } L - \{z_0\}\,. \end{cases}$$

Therefore the sequence $\{G_P^n(z, z_0)\}$ has a limit function G_P .

6. THE CLASSIFICATION OF DENSITIES AND SOLUTIONS

6.1. The elliptic measure

By definition 1.1.2 the elliptic measure w_P^n of a regular region R_n is a P-solution defined in R_n with $w_P^n = 1$ on ∂R_n. By using this we define the elliptic measure of R for P .

Definition 6.1.1. *Let P be acceptable. $P \not\equiv 0$. If there exists a non-negative P-solution w_P on R such that*

$$\lim_{n\to\infty} w_P^n = w_P$$

for every exhaustion $\{R_n\}$, where the sequence $\{w_P^n\}$ of elliptic measures of R_n is converging uniformly on every compact set, we say that P is normal and w_P is the elliptic measure of R .

Remark 6.1.2. *If P is normal, then by remark 1.2.3 either $w_P > 0$ or $w_P \equiv 0$. In the former case we say that P is hyperbolic and in the latter parabolic.*

If $P \geqq 0$, it is always parabolic on parabolic surfaces and either parabolic or hyperbolic on hyperbolic surfaces (Cf. [12]). If P has also negative values the existence and uniqueness of $\lim_{n\to\infty} w_P^n$ are not always sure. This is partly illustrated by the following duality: Let P be normal. Then

(6.1.1) $\quad w_P = \max\{u \mid u$ is a P-solution and $u \leqq 1\}$, if $P \geqq 0$

and

(6.1.2) $\quad w_P = \min\{u \mid u$ is a P-solution and $u \geqq 1\}$, if $P \leqq 0$.

Another deviation from non-negative densities is that w_P is not always bounded. This is seen from the following.

Example 6.1.3. We choose

$$R = \{z \mid 0 < |z| < 1\}\,,\ R_n = \left\{z \;\Big|\; \frac{1}{n} < |z| < 1 - \frac{1}{n}\right\}$$

and

$$P(z) = -2\,(3\,|z|\,\log\tfrac{1}{2}|z|)^{-2}\,.$$

P is acceptable by $-(\log\frac{1}{2}|z|)^{1/3}$. $\{R_n\}$ is an exhaustion of R and

$$w_P^n(z) = (a_n + b_n)\,(\log\tfrac{1}{2}|z|)^{1/3} - a_n\, b_n\,(\log\tfrac{1}{2}|z|)^{2/3}\,,$$

where

$$a_n = \left(\log\frac{1}{2n}\right)^{-1/3}\,,\quad b_n = \left(\log\tfrac{1}{2}\left(1 - \frac{1}{n}\right)\right)^{-1/3}.$$

Now w_P exists and

$$w_P(z) = \lim_{n\to\infty} w_P^n(z) = (\log\tfrac{1}{2})^{-1/3}\,(\log\tfrac{1}{2}|z|)^{1/3}\,.$$

This is not bounded for $\lim_{|z|\to 0} w_P(z) = \infty$.

These properties cause that the elliptic measure is not in general as useful as for non-negative densities.

Let $u \in C^0$. We say that

$$\overline{\lim_{\partial R}}\,|u| = s\,,$$

if for every $\varepsilon > 0$ there exists a compact region K_ε such that $|u| \leqq s + \varepsilon$ on $R - \bar{K}_\varepsilon$ and for any compact region K we have $|u(z)| > s - \varepsilon$ at least at one point $z \in R - \bar{K}$.

Theorem 6.1.4. *Let P be normal and u a bounded P-solution. Then*

$$|u| \leqq (\overline{\lim_{\partial R}}\,|u|)\, w_P\,.$$

Proof: Let $\{R_n\}$ be an exhaustion with $\lim_{n\to\infty} w_P^n = w_P$. Then in R_n

$$|u| \leqq (\sup_{\partial R_n} |u|)\, w_P^n ,$$

and on R

$$|u| \leqq \overline{\lim_{n\to\infty}} (\sup_{\partial R_n} |u|\, w_P^n) \leqq \overline{\lim_{n\to\infty}} (\sup_{\partial R_n} |u|)\, w_P \leqq (\overline{\lim_{\partial R}} |u|)\, w_P .$$

Corollary 6.1.5. *Let P be normal, u a P-solution and c a positive constant. If $|u| \leqq c$, then $|u| \leqq c\, w_P$.*

From the elliptic measure we always get an upper bound for bounded solutions. This implies a.o. that if P is parabolic, the only bounded P-solution is the constant zero. Another corollary leads to the non-existence of w_P.

Corollary 6.1.6. *Let P be acceptable. If there exists a bounded P-solution u with $\lim_{z\to\partial R} u(z) = 0$, $u \not\equiv 0$, then P is not normal.*

This implies that even completely acceptable densities are not always normal.

Example 6.1.7. We choose

$$R = \{z \mid |z| < 1\} , \quad R_n = \left\{z \mid |z| < 1 - \frac{1}{n}\right\}$$

and

$$P(z) = \tfrac{1}{4} (|z|^2 - 4)\, (1 - |z|^2)^{-2} .$$

P is acceptable by $\omega = (1 - |z|^2)^{1/4}$, $\{R_n\}$ is an exhaustion of R and

$$G_P^n(z, 0) = \tfrac{1}{2} (1 - |z|^2)^{1/4} \log \left[\frac{n - \sqrt{2n - 1}}{n + \sqrt{2n - 1}}\, \frac{1 + \sqrt{1 - |z|^2}}{1 - \sqrt{1 - |z|^2}}\right] .$$

Then G_P exists and

$$G_P(z, 0) = \tfrac{1}{2} (1 - |z|^2)^{1/4} \log \left[\frac{1 + \sqrt{1 - |z|^2}}{1 - \sqrt{1 - |z|^2}}\right] .$$

Therefore P is completely acceptable. However, by corollary 6.1.6 P is not normal for ω is a P-solution with $|\omega| \leqq 1$ and $\lim_{|z|\to 1} \omega(z) = 0$.

6.2. Banach spaces of solutions

Let SP be the set of all P-solutions defined on R. We define Banach space BP with norm $\|u\|$.

$$BP = \{u \in SP \mid \|u\| = \sup_R |u| < \infty\} .$$

If P is normal we can define another Banach space MP with norm $\|u\|_P$.

$$MP = \begin{cases} \left\{u \in SP \mid \|u\|_P = \sup\limits_R \dfrac{|u|}{w_P} < \infty \right\}, & \text{if } P \text{ hyperbolic} \\ \{0\} \text{ with norm } 0 , & \text{if } P \text{ parabolic} . \end{cases}$$

In section 7 we are going to examine how these spaces change when P varies. For that we need still one Banach space AP_ω with the norm $\|u\|_\omega$.

$$AP_\omega = \left\{ u \in SP \mid P \text{ acceptable by } \omega \text{ and } \|u\|_\omega = \sup_R \frac{|u|}{\omega} < \infty \right\} .$$

For later use we present some simple relations between these spaces and their norms.

Remark 6.2.1. *The norms depend on the behaviour near the ideal boundary as follows:*

$$\|u\| \geqq \overline{\lim_{\partial R}} |u| , \text{ where the equality holds if } P \geqq 0 .$$

$$\|u\|_P = \overline{\lim_{\partial R}} \frac{|u|}{w_P} , \text{ if } P \text{ is hyperbolic} .$$

$$\|u\|_\omega = \overline{\lim_{\partial R}} \frac{|u|}{\omega} .$$

Remark 6.2.2. *From the definitions follow:*

(1) *If* P *is normal, then* $BP \subset MP$ *and* $\|u\|_P \leqq \|u\|$.

(2) *If* $P \geqq 0$, *then* $BP = MP$ *and* $\|u\|_P = \|u\|$.

(3) *If* P *is normal and* $\inf\limits_R \omega > 0$, *then* $MP \subset AP_\omega$.

(4) *If* P *is acceptable and* $\inf\limits_R \omega > 0$, $\sup\limits_R \omega < \infty$, *then* $BP = AP_\omega$.

7. ISOMETRIC SOLUTIONS SPACES

7.1. The linear mapping T_{PQ}

Our main tool in the examination of isometric relations will be the transformation T_{PQ} which is a natural generalisation of the mapping T_{PQ}^K introduced in 3.2.

Let P be acceptable, Q completely acceptable and u a continuous function. If

$$\iint_R |P(z) - Q(z)| \; G_Q(z, z_0) \; |u(z)| \, dxdy < \infty \tag{7.1.1}$$

at some point $z_0 \in R$, then it holds at all points of R by remark 5.1.5.

Definition 7.1.1. *Let P be acceptable, Q completely acceptable and $u \in C^0$ a function for which (7.1.1) is true at some point $z_0 \in R$. Then the linear transformation $T_{PQ}u$ of u is well defined by*

$$T_{PQ}u(z_0) = u(z_0) + \frac{1}{2\pi} \iint_R (P(z) - Q(z)) \, G_Q(z, z_0) u(z) \, dxdy \,. \tag{7.1.2}$$

The following result will often be used in the continuation.

Lemma 7.1.2. *Let P be acceptable, Q completely acceptable, u a P-solution and $\{u_n\}$ a sequence of P-solutions each defined in R_n so that $\{R_n\}$ is an exhaustion of R and $\lim_{n\to\infty} u_n = u$. If there exists a function $v \in C^0$ such that $|u_n| \leqq v$ for each n and v fulfils (7.1.1) at some point $z_0 \in R$, then $T_{PQ}u$ is well defined and*

(1) $\lim_{n\to\infty} T^n_{PQ}u_n = T_{PQ}u$,

(2) $T_{PQ}u$ *is a Q-solution.*

Proof: By definition 7.1.1 $T_{PQ}u$ is well defined.

(1) For each n and $z_0 \in R$

$$|(P(z) - Q(z)) \; G^n_Q(z, z_0) \, u_n(z)| \leqq |P(z) - Q(z)| \; G_Q(z, z_0) \, v(z)$$

and the majorant is integrable. We define $G^n_Q u_n$ on the whole surface by setting its value to be identically zero in $R - R_n$. Now we can use Lebesgue's theorem of dominated convergence (Cf. e.g. [5] p. 234) and get

$$\lim_{n\to\infty} \iint_{R_n} (P(z) - Q(z)) \, G^n_Q(z, z_0) \, u_n(z) \, dxdy =$$

$$\iint_R (P(z) - Q(z)) \, G_Q(z, z_0) \, u(z) \; dxdy \,.$$

This shows that

$$\lim_{n\to\infty} T^n_{PQ}u_n = T_{PQ}u \,.$$

(2) Because (7.1.1) holds for v at every point z_0, we get an estimate

$$|T^n_{PQ}u_n(z_0)| \leqq v(z_0) + \frac{1}{2\pi} \iint_R |P(z) - Q(z)| \; G_Q(z, z_0) \, v(z) \; dxdy < \infty$$

which shows $\{T^n_{PQ}u_n\}$ to be bounded in every compact set. By lemma 4.1.1 there exists a subsequence $\{T^{n_i}_{PQ}u_{n_i}\}$ which converges uniformly on every compact set towards a Q-solution v_Q. Then we have

$$T_{PQ}u = \lim_{n\to\infty} T^n_{PQ}u_n = \lim_{n_i\to\infty} T^{n_i}_{PQ}u_{n_i} = v_Q ,$$

which shows $T_{PQ}u$ to be a Q-solution.

The lemma is now proved.

By this lemma and remark 3.2.2 the following properties of T_{PQ} are obvious.

Remark 7.1.3. *Let P be acceptable, Q completely acceptable, u and u_1 P-solutions for which (7.1.1) is true and c a positive constant.*

(1) *If $|u| \leqq cu_1$, then $|T_{PQ}u| \leqq c\, T_{PQ}u_1$.*

(2) *If P and Q are acceptable by ω, then $|u| \leqq c\,\omega$ implies $|T_{PQ}u| \leqq c\,\omega$.*

7.2. The spaces AP_ω and AQ_ω

We examine by using the transformation T_{PQ} when the Banach spaces AP_ω and AQ_ω are isometric.

Theorem 7.2.1. *Let P and Q be completely acceptable by ω. If*

$$\iint_R |P(z) - Q(z)|\, \omega^2(z)\, dxdy < \infty , \tag{7.2.1}$$

then AP_ω and AQ_ω are isometric.

Proof: We begin by showing that (7.1.1) holds for ω. Let $z_0 \in R$ and K be a regular region with $z_0 \in K$. By lemma 5.1.4 there exists a constant m such that $G_Q(z, z_0) \leqq m\, \omega(z)$, $z \in R - K$. Thus

$$\iint_R |P(z) - Q(z)|\, G_Q(z, z_0)\, \omega(z)\, dxdy \leqq$$

$$\iint_K |P(z) - Q(z)|\, G_Q(z, z_0)\, \omega(z)\, dxdy + m \iint_{R-K} |P(z) - Q(z)|\, \omega^2(z)\, dxdy < \infty .$$

If $u \in AP_\omega$, then $|u| \leqq \|u\|_\omega\, \omega$. $T_{PQ}u$ is now well defined and by remark 7.1.3

$$|T_{PQ}u| \leqq \|u\|_\omega\, \omega .$$

Therefore $T_{PQ}u \in AQ_\omega$ and

$$\|T_{PQ}u\|_\omega \leqq \|u\|_\omega .$$

This makes T_{PQ} a linear mapping from AP_ω to AQ_ω which does not increase norms.

By changing the roles of P and Q we see that T_{QP} is a linear mapping from AQ_ω to AP_ω which does not increase norms.

Next we show that if $u \in AP_\omega$, then

$$T_{QP}\, T_{PQ} u = u \, .$$

In fact, let $u \in AP_\omega$. Then by lemma 7.1.2

$$\lim_{n \to \infty} T_{PQ}^n u = T_{PQ} u$$

and $T_{PQ}^n u$ is a Q-solution in R_n with

$$|T_{PQ}^n u| \leqq \|u\|_\omega \, \omega \, .$$

By using lemma 7.1.2 once more we now get

$$\lim_{n \to \infty} T_{QP}^n\, T_{PQ}^n u = T_{QP}\, T_{PQ} u \, .$$

On the other hand by remark 3.2.2

$$\lim_{n \to \infty} T_{QP}^n\, T_{PQ}^n u = u \, .$$

By changing the roles of P and Q again we get that if $v \in AQ_\omega$, then

$$T_{PQ}\, T_{QP} v = v \, .$$

These facts make T_{PQ} an isomorphism from AP_ω onto AQ_ω and T_{QP} its inverse mapping. Because they do not increase norms, they must be isometries. This proves the theorem.

7.3. The spaces MP and MQ

We now use theorem 7.2.1 in a special case in order to get a condition for the isometry of MP and MQ.

Theorem 7.3.1. *Let* P *and* Q *be completely acceptable by* ω *so that* $\inf_R \omega > 0$ *and*

$$\iint_R |P(z) - Q(z)| \, \omega^2(z) \, dxdy < \infty \, . \tag{7.2.1}$$

If P *is normal, then* Q , *too, is and* MP *is isometric with* MQ.

Proof: Let P be normal. We first show that P and Q are both hyperbolic or both parabolic.

If $\{R_n\}$ is an exhaustion with $\lim_{n \to \infty} w_P^n = w_P$, then

$$w_P^n \leqq (\inf_R \omega)^{-1} \omega$$

and the upper bound fulfils (7.1.1). Because $w_Q^n = T_{PQ}^n w_P^n$, there exists by lemma 7.1.2

$$\lim_{n\to\infty} w_Q^n = \lim_{n\to\infty} T_{PQ}^n w_P^n = T_{PQ} w_P .$$

Therefore Q is normal and

$$w_Q = T_{PQ} w_P .$$

By changing the roles of P and Q we get that also

$$w_P = T_{QP} w_Q .$$

This causes that P can be hyperbolic if and only if Q is, and that P is parabolic exactly when Q is.

Then we consider the isometry. If P and Q are parabolic the case is trivial because

$$MP = \{0\} = MQ .$$

Let us then suppose that P and Q are both hyperbolic. We have now $MP \subset AP_\omega$, $MQ \subset AQ_\omega$, and AP_ω is isometric with AQ_ω. If $u \in MP$, then

$$|u| \leqq \|u\|_P \, w_P$$

and

$$|T_{PQ} u| \leqq \|u\|_P \, T_{PQ} w_P = \|u\|_P \, w_Q .$$

Therefore $T_{PQ} u \in MQ$ and

$$\|T_{PQ} u\|_Q \leqq \|u\|_P .$$

In the same way: If $v \in MQ$, then $T_{QP} v \in MP$ and

$$\|T_{QP} v\|_P \leqq \|v\|_Q .$$

This makes T_{PQ} an isomorphism from MP onto MQ and T_{QP} its inverse mapping. Because they do not increase norms, they have to be isometries.

Remark 7.3.2. *The integral condition (7.2.1) in theorems 7.2.1 and 7.3.1 can be substituted by a weaker one:*

$$(7.3.1) \qquad \iint_R |P(z) - Q(z)| \, (G_P(z, z_0) + G_Q(z, z_1)) \, \omega(z) \, dxdy < \infty$$

at some points $z_0 , z_1 \in R$.

7.4. The spaces BP and BQ

If we use another special case of theorem 7.2.1, we have a condition of the isometry of the spaces BP and BQ. However, it has to be noticed that the isometry does not in general hold for the norm $\|u\|$ but for $\|u\|_\omega$.

Theorem 7.4.1. *Let P and Q be completely acceptable by ω. If*

$$\inf_R \omega > 0 , \quad \sup_R \omega < \infty$$

and

$$\iint_R |P(z) - Q(z)| \, dxdy < \infty , \tag{7.4.1}$$

then BP and BQ are isometric with regard to the ω-norm.

Proof: In this case $BP = AP_\omega$ and $BQ = AQ_\omega$ and ω fulfils (7.2.1). The statement then follows from theorem 7.2.1.

If we especially choose $\omega \equiv 1$, then P and Q are non-negative, $\|u\| = \|u\|_\omega$ and we get Nakai's result (Cf. [10]).

Corollary 7.4.2. *Let P and Q be non-negative densities and (7.4.1) valid. Then BP and BQ are isometric.*

Remark 7.4.3. *The condition (7.4.1) can be replaced by a weaker one:*

$$\iint_R |P(z) - Q(z)| \; (G_P(z, z_0) + G_Q(z, z_1)) \, dxdy < \infty \tag{7.4.2}$$

at some points $z_0 , z_1 \in R$.

7.5. Densities equaling outside a compact region

Finally we consider the possibility that P and Q are equal outside a compact region. Because we have not the maximum principle, this is not as restrictive a condition as for non-negative densities.

Theorem 7.5.1. *Let P and Q be completely acceptable and $P \equiv Q$ outside a compact region K. Then the following are true:*

(1) *SP and SQ are isomorphic.*

(2) *If P is normal, then Q is normal and MP is isometric with MQ.*

(3) *If BP contains a positive solution, then BQ, too, contains a positive solution and BP is isomorphic with BQ.*

Proof: In this case mappings $T_{PQ} : SP \to SQ$ and $T_{QP} : SQ \to SP$ are well defined.

(1) If $u \in SP$, then

$$\lim_{n\to\infty} T^n_{PQ}u = T_{PQ}u .$$

Because

$$|T^n_{PQ}u(z_0)| \leqq |u(z_0)| + \frac{1}{2\pi} \iint\limits_K |P(z) - Q(z)| \, G_Q(z, z_0) \, |u(z)| \, dxdy$$

and the upper bound fulfils (7.1.1), we get by lemma 7.1.2

$$u = \lim_{n\to\infty} T^n_{QP} T^n_{PQ}u = T_{QP} T_{PQ}u .$$

If $v \in SQ$, we get in the same way that

$$v = \lim_{n\to\infty} T^n_{PQ} T^n_{QP}v = T_{PQ} T_{QP}v .$$

Therefore SP and SQ are isomorphic by T_{PQ}.

(2) If P is normal, we get from lemma 7.1.2 by the compactness of K that Q is also normal and

$$w_Q = T_{PQ}w_P , \quad w_P = T_{QP}w_Q .$$

Now we get the statement as in theorem 7.3.1.

(3) Because BP contains a positive solution u_P, the integral

$$\iint\limits_K |P(z) - Q(z)| \, G_P(z, z_0) \, dxdy \tag{7.5.1}$$

is by the symmetry of G_P and lemma 5.1.4 (2) uniformly bounded. If now $u \in BP$, then by part (1) $T_{PQ}u \in SQ$ and $T_{QP}(T_{PQ}u) = u$. Therefore

$$|T_{PQ}u(z_0)| \leqq |u(z_0)| + \sup_K |T_{PQ}u| \frac{1}{2\pi} \iint\limits_K |P(z) - Q(z)| \, G_P(z, z_0) \, dxdy ,$$

which implies that $T_{PQ}u \in BQ$.

Especially $T_{PQ}u_P$ is a positive Q-solution.

By changing the roles of P and Q we then get that if $v \in BQ$, then $T_{QP}v \in BP$.

By part (1) in the proof BP and BQ are isomorphic.

The proof is now complete.

8. THE ISOMETRY OF BP AND HB WHEN $P \geqq 0$

8.1. Introduction

Let HB be the space of bounded harmonic functions with the norm $\|h\| = \sup_R |h|$. Nakai has shown (Cf. [10] p. 271 and also corollary 7.4.2) that if P is a non-negative hyperbolic density on a hyperbolic surface and if

$$\iint_R P(z)\, G_O(z, z_0)\, dxdy < \infty \tag{8.1.1}$$

at some point $z_0 \in R$, then BP and HB are isometric.

According to him it is an open question whether (8.1.1) is also a necessary condition.

Our aim is to present a different condition for the isometry of BP and HB and by using it to show that (8.1.1) is not a necessary condition. Through the whole section P will be non-negative and R hyperbolic.

8.2. The least harmonic majorant

If $P \geqq 0$, then w_P is the greatest P-solution to be smaller than one. Now we on the contrary consider the existence of the smallest harmonic function to be greater than w_P.

Lemma 8.2.1. *If $P \geqq 0$ and R is hyperbolic, then there exists the least harmonic majorant h_P of the elliptic measure w_P with $h_P \leqq 1$. Moreover*

$$h_P = T_{PO} w_P \quad \text{and} \quad w_P = T_{OP} h_P\,.$$

Proof: If P is parabolic, then $w_P \equiv 0$ and $h_P \equiv 0$.

If P is hyperbolic, the sequence $\{P(z)\, G_O^n(z, z_0)\, w_P(z)\}$ is non-negative and non-decreasing. Because

$$\frac{1}{2\pi} \iint_{R_n} P(z)\, G_O^n(z, z_0)\, w_P(z)\, dxdy = T_{PO}^n w_P(z_0) - w_P(z_0) \leqq 1$$

we have (Cf. e.g. [5] p. 186)

$$\frac{1}{2\pi} \iint_R P(z)\, G_O(z, z_0)\, w_P(z)\, dxdy \leqq 1 \tag{8.2.1}$$

and by lemma 7.1.2 $T_{PO} w_P$ is harmonic. In R_n $w_P \leqq T_{PO}^n w_P \leqq 1$ which implies

$$w_P \leqq T_{PO} w_P \leqq 1\,.$$

If h' is another harmonic majorant of w_P, then on ∂R_n

$$T_{PO}^n w_P = w_P \leqq h'$$

which means that in R_n $T_{PO}^n w_P \leqq h'$. Therefore

$$T_{PO}w_P = \lim_{n\to\infty} T^n_{PO}w_P \leqq h'$$

and $T_{PO}w_P$ is the least harmonic majorant of w_P. $T_{PO}w_P = h_P$.

On the other hand harmonic functions $h_n = T^n_{PO}w_P$ form a non-decreasing sequence wherefore also the sequence $\{P(z)\, G^n_P(z, z_0)\, h_n(z)\}$ is non-decreasing. Moreover

$$\frac{1}{2\pi}\iint\limits_{R_n} P(z)\, G^n_P(z, z_0)\, h_n(z)\, dxdy = h_n(z_0) - w_P(z_0) \leqq 1$$

and therefore

$$\frac{1}{2\pi}\iint\limits_{R} P(z)\, G_P(z, z_0)\, h_P(z)\, dxdy \leqq 1 \tag{8.2.2}$$

and

$$w_P(z_0) = \lim_{n\to\infty} T^n_{OP}h_n(z_0) = T_{OP}h_P(z_0)\,.$$

8.3. The spaces BP and HB_P

Let $P \geqq 0$ be hyperbolic. We define an auxiliary Banach space HB_P with the norm $\|h\|_{OP}$

$$HB_P = \left\{ h \mid h \text{ harmonic and } \|h\|_{OP} = \sup_R \frac{|h|}{h_P} < \infty \right\}.$$

Because $h_P \leqq 1$, $HB_P \subset HB$ and we can also use in HB_P the norm

$$\|h\| = \sup_R |h|\,.$$

Clearly $\|h\| \leqq \|h\|_{OP}$. Later on it appears that they are in fact equal.

The meaning of HB_P appears from the following result.

Theorem 8.3.1. *Let $P \geqq 0$ be a hyperbolic density on a hyperbolic surface R. Then T_{PO} is an isometry from BP onto HB_P and T_{OP} its inverse mapping with regard to both norms of HB_P.*

Proof: We start by showing that T_{PO} is a mapping from BP to HB_P and T_{OP} a mapping from HB_P to BP.

If $u \in BP$, then $|u| \leqq \|u\|\, w_P$ and by formula (8.2.1) and lemma 7.1.2 $T_{PO}u$ is harmonic. Moreover

$$|T_{PO}u| \leqq \|u\|\, T_{PO}w_P = \|u\|\, h_P\,,$$

that is, $T_{PO}u \in HB_P$ and

(8.3.1) $$\|T_{PO}u\| \leqq \|T_{PO}u\|_{OP} \leqq \|u\| .$$

If $h \in HB_P$, then $|h| \leqq \|h\|_{OP} h_P$ and by formula (8.2.2) and lemma 7.1.2 $T_{OP}h \in SP$. Furthermore

$$|T_{OP}h| \leqq \|h\|_{OP} T_{OP} h_P = \|h\|_{OP} w_P \leqq \|h\|_{OP} .$$

Therefore $T_{OP}h \in BP$ and

(8.3.2) $$\|T_{OP}h\| \leqq \|h\|_{OP} .$$

Next we prove that T_{PO} is an isomorphism with the inverse mapping T_{OP}.

Let $u \in BP$. Then $T^n_{PO}u$ is harmonic in R_n and

$$\lim_{n\to\infty} T^n_{PO}u = T_{PO}u .$$

Because $\{T^n_{PO}w_P\}$ is non-decreasing

$$|T^n_{PO}u| \leqq \|u\| \, h_P .$$

By this and (8.2.2) we can use lemma 7.1.2 again to get

$$u = \lim_{n\to\infty} T^n_{OP}\, T^n_{PO}u = T_{OP}\, T_{PO}u .$$

Let then $h \in HB_P$. Now we cannot use the preceding method, because $T^n_{OP}h$ does not always have a majorant fulfilling (7.1.1), but we have to prove it otherwise.

We define $G^n_P \equiv 0$ in $R - R_n$. Firstly we estimate the difference $T_{OP}h - T^n_{OP}h$.

$$\begin{aligned} |T_{OP}h - T^n_{OP}h| &= \frac{1}{2\pi}\left| \iint_R P\,(G_P - G^n_P)\, h\, dxdy \right| \\ &\leqq \|h\|_{OP} \frac{1}{2\pi} \iint_R P\,(G_P - G^n_P)\, h_P\, dxdy \\ &= \|h\|_{OP}(T^n_{OP}h_P - T_{OP}h_P) . \end{aligned}$$

By using this we secondly estimate the difference $T^n_{PO}\, T_{OP}h - h$.

$$\begin{aligned} |T^n_{PO}\, T_{OP}h - h| &= |T^n_{PO}(T_{OP}h - T^n_{OP}h)| \\ &\leqq \|h\|_{OP} T^n_{PO}(T^n_{OP}h_P - T_{OP}h_P) \\ &= \|h\|_{OP}(h_P - T^n_{PO}\, T_{OP}h_P) . \end{aligned}$$

Because $T_{OP}h \in BP$, we have

$$\lim_{n\to\infty} T^n_{PO}\, T_{OP}h = T_{PO}\, T_{OP}h \in HB_P$$

and when n goes to infinity we have by lemma 8.2.1

$$|T_{PO}\,T_{OP}h - h| \leqq \|h\|_{OP}\,(h_P - T_{PO}\,T_{OP}h_P) = 0$$

that is

$$T_{PO}\,T_{OP}h = h\,.$$

In order to show the isometry we finally prove that T_{PO} and T_{OP} do not increase norms.

If $h \in HB_P$, then

$$|T^n_{OP}h| \leqq \sup_{R_n} |h| \leqq \|h\|$$

and

$$|T_{OP}h| = \lim_{n\to\infty} |T^n_{OP}h| \leqq \|h\|\,.$$

Therefore

$$\|T_{OP}h\| \leqq \|h\|\,.$$

This together with (8.3.1) and (8.3.2) shows that

$$\|u\| = \|T_{PO}u\| = \|T_{PO}u\|_{OP}\,, \qquad \|h\| = \|T_{OP}h\| = \|h\|_{OP}\,. \tag{8.3.3}$$

The proof is thus complete.

Remark 8.3.2. *The formulas (8.3.3) imply that if* $h \in HB_P$, *then*

$$\|h\| = \|h\|_{OP}\,.$$

This equality implies an auxiliary result.

Lemma 8.3.3. *If* $h \in HB_P$, *then* $|h| \leqq \|h\|\,h_P$.

As a by-product of theorem 8.3.1 we finally can give a new condition for the isometry of BP and BQ when densities are non-negative.

Corollary 8.3.4. *Let* P *and* Q *be non-negative hyperbolic densities on a Riemann surface. If* $h_P = h_Q$, *that is, if their elliptic measures have a common least harmonic majorant, then* BP *and* BQ *are isometric.*

8.4. The isometry of BP and HB

We can now exactly say when the mapping T_{PO} makes BP and HB isometric.

Theorem 8.4.1. *Let* P *be a non-negative hyperbolic density on a hyperbolic surface* R. T_{PO} *is an isometry from* BP *onto* HB *if and only if* h_P, *the least harmonic majorant of* w_P, *is identically one.*

Proof: We show that $h_P \equiv 1$ if and only if $HB = HB_P$.

If $h_P \equiv 1$, then trivially $HB = HB_P$.

If $HB = HB_P$, then $1 \in HB_P$ and by lemmas 8.2.1 and 8.3.3

$$1 \leqq h_P \leqq 1 .$$

The statement now follows from theorem 8.3.1.

In order to make the strength of this theorem more clear we give two sufficient conditions for h_P to be identically one.

Lemma 8.4.2. *Let P be a non-negative hyperbolic density on a Riemann surface R. If $\inf_R h_P > 0$, then $h_P \equiv 1$.*

Proof: Because $\inf_R h_P \in HB_P$ also $1 \in HB_P$ and $1 \leqq h_P \leqq 1$.

Lemma 8.4.3. *Let P be a non-negative hyperbolic density on a hyperbolic surface R. If*

$$\iint_R P(z)\, G_O(z, z_0)\, dxdy < \infty \tag{8.1.1}$$

at some point $z_0 \in R$, then $h_P \equiv 1$.

Proof: By lemmas 7.1.2 and 8.2.1

$$1 = \lim_{n\to\infty} T^n_{PO} w^n_P = T_{PO} w_P = h_P .$$

Notice that lemma 8.4.3 is just Nakai's result and we have shown it to be a special case of theorem 8.4.1. We now demonstrate with an example that theorem 8.4.1 is really stronger than lemma 8.4.3.

Example 8.4.4. We choose

$$R = \{z \mid 0 < |z| < 1\} , \quad R_n = \left\{ z \mid \frac{1}{n} < |z| < 1 - \frac{1}{n} \right\}$$

and

$$P(z) = |z|^{-2}.$$

$\{R_n\}$ is an exhaustion of R and the elliptic measure of R_n is

$$w^n_P(z) = |z| + \frac{n-1}{n^2} \frac{1}{|z|} .$$

Thus

$$w_P(z) = |z| .$$

In this case $h_P \equiv 1$ and BP is isometric with HB by theorem 8.4.1. However, Nakai's condition (8.1.1) is not valid. In fact, let us consider a point $z_0 \in R$. Because

$$\lim_{z\to 0} G_O(z, z_0) > 0$$

there exists a positive constant δ such that

$$G_O(z, z_0) \geqq \delta\,, \quad \text{when} \quad |z| < \tfrac{1}{2}\,|z_0|\,.$$

If now ε is a positive constant with $\varepsilon < \frac{1}{2}\,|z_0|$ and if we denote

$$R_\varepsilon = \{z \mid \varepsilon < |z| < \tfrac{1}{2}\,|z_0|\}\,,$$

then

$$\iint_R P(z)\, G_O(z, z_0)\, dxdy \geqq \lim_{\varepsilon \to 0} \delta \iint_{R_\varepsilon} P(z)\, dxdy$$

$$= \lim_{\varepsilon \to 0} 2\pi\delta \log \frac{|z_0|}{2\varepsilon} = \infty\,.$$

Thus we have also shown that Nakai's condition (8.1.1) is not necessary for the isometry of BP and HB.

Finally we show with another example that T_{PO} does not always make BP and HB isometric because h_P is not always identically one.

Example 8.4.5. We choose

$$R = \{z \mid 1 < |z| < 2\}\,, \quad R_n = \left\{z \mid 1 + \frac{1}{n} < |z| < 2 - \frac{1}{n}\right\}$$

and

$$P(z) = 2\,(|z| \log |z|)^{-2}\,.$$

Then $\{R_n\}$ is an exhaustion and

$$w_P^n(z) = (a_n^2 + a_n b_n + b_n^2)^{-1} \left(a_n b_n\, (a_n + b_n)\, \frac{1}{\log |z|} + \log^2 |z|\right),$$

where

$$a_n = \log\left(1 + \frac{1}{n}\right), \quad b_n = \log\left(2 - \frac{1}{n}\right).$$

Therefore

$$w_P(z) = \left[\frac{\log |z|}{\log 2}\right]^2$$

and

$$h_P(z) = \frac{\log |z|}{\log 2}\,.$$

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LITERATURE

[1] AHLFORS, L.: Complex Analysis, 2. ed. McGraw-Hill, New York — Toronto — London, 1966.

[2] —»— and SARIO, L.: Riemann Surfaces. Princeton University Press, Princeton, 1960.

[3] COURANT, R. and HILBERT, D.: Methods of Mathematical Physics, Vol. II. Interscience, New York, 1962.

[4] MIRANDA, C.: Partial Differential Equations of Elliptic Type, 2. rev.ed. Springer — Verlag, Berlin — Heidelberg — New York, 1970.

[5] MUNROE, M.: Introduction to Measure and Integration. Addison — Wesley, Reading Mass., 1959.

[6] MYRBERG, L.: Über die Integration der Differentialgleichung $\Delta u = c(P)u$ auf offenen Riemannschen Flächen. Math. Scand. 2 (1954) pp. 142—152.

[7] —»— Über die Existenz der Greenschen Funktion der Gleichung $\Delta u = c(P)u$ auf Riemannschen Flächen. Ann. Acad. Sci. Fennicae A I, 170 (1954).

[8] —»— Über subelliptische Funktionen. Ann. Acad. Sci. Fennicae A I, 290 (1960).

[9] —»— Über die Integration der Gleichung $\Delta u = c(z)u$ auf einer Riemannschen Fläche im indefiniten Fall. Ann. Acad. Sci. Fennicae A I, 514 (1972).

[10] NAKAI, M.: The Space of Bounded Solutions of the Equation $\Delta u = pu$ on a Riemann Surface. Proc. of the Japan Academy 36 nr 5 (1960) pp. 267—272.

[11] OZAWA, M.: Classification of Riemann surfaces. Kodai Math. Sem. Rep. 4 (1952) pp. 63—76.

[12] ROYDEN, H.: The equation $\Delta u = Pu$ and the classification of open Riemann surfaces. Ann. Acad. Sci. Fennicae A I, 271 (1959).

[13] TAUTZ, G.: Zur Theorie der elliptischen Differentialgleichungen II. Math. Ann. 118 (1943) pp. 733—770.

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