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CONTINUOUS BOUNDARY EXTENSION OF QUASICONFORMAL MAPPINGS

BY

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1. Introduction

In the study of the boundary correspondence under n-dimensional quasiconformal mappings, the object of investigation has usually been either homeomorphic boundary extension or prime end correspondence. (See, for example, Väisälä [3], [4, §17], and Zorič [5], [6].) With the exception of some scattered results, no serious effort has been made to consider *continuous* boundary extension. It is the purpose of the present paper to try to eliminate this deficiency. We consider the possibility of extending n-dimensional quasiconformal mappings continuously to the boundaries in the case where the mappings are defined between two domains, one of which, D_0, satisfies the following smoothness condition: each boundary point of D_0 has arbitrarily small neighborhoods U such that $U \cap D_0$ can be mapped quasiconformally onto a ball.

We show in Section 3 that a quasiconformal mapping of a domain D onto D_0 can be extended to a continuous mapping between the closures if and only if D satisfies the following simple *metric* condition: any two connected sets in D whose distance measured in the whole space is zero have relative distance zero in D, i.e. they can be joined in D by an arc whose diameter is arbitrarily small. In the other direction we show that a quasiconformal mapping of D_0 onto a domain D can be extended to a continuous mapping between the closures if and only if D satisfies the following simple *topological* condition: each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ contains only a finite number of components. We also consider the extension problem from the point of view of the *extremal length.*

The present paper is a continuation of [1] where some problems on cluster sets and boundary extension of quasiconformal mappings of more general domains were considered.

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2. Smoothness properties of boundaries

2.1. Notation and terminology. We will use the same notation and terminology as in [1]. Unless otherwise stated, all point sets are assumed to lie in the compactified n-space $\bar{R}^n = R^n \cup \{\infty\}$, $n \geq 2$. Besides the Euclidean metric, defined in R^n, we use the spherical metric q and the relative spherical metric q_D in $\bar{R}^n$. The latter is needed only for domains D and it is defined by setting for $x_1, x_2 \in D$,

$$q_D(x_1, x_2) = \inf q(|\gamma|),$$

where the infimum is taken over the spherical diameters of the loci of all paths γ in D joining x_1 to x_2. The modulus of a path family Δ is designated by $M(\Delta)$, and $\Delta(E, F : D)$ will denote the family of all paths which join the sets E and F in D.

2.2. Quasiconformal collaredness. A domain D is said to be *quasiconformally collared* on the boundary if each point of ∂D has arbitrarily small neighborhoods U such that $U \cap D$ can be mapped quasiconformally onto a ball.

We give some general remarks on domains that possess this smoothness property. Our first lemma shows that the above definition for quasiconformal collaredness on the boundary is equivalent to those definitions employed in [1] and in Väisälä [4, §17].

2.3. **Lemma.** *Given a domain* D, *the following statements are equivalent*:

(1) D *is quasiconformally collared on the boundary.*

(2) *For each point* $b \in \partial D$ *there is a neighborhood* U *of* b *and a quasiconformal mapping* $f : U \cap D \to B^n_+$ *such that* $\lim_{x \to b} f(x) = 0$ and $\lim_{y \to 0} f^{-1}(y) = b$.

(3) *For each point* $b \in \partial D$ *there is a neighborhood* U *of* b *and a homeomorphism* $f : U \cap \bar{D} \to B^n_+ \cup B^{n-1}$ *such that* f *is quasiconformal in* $U \cap D$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial, while an n-dimensional analogue of the proof of Theorem 4.7 in [1] shows that (1) implies (3).

As a corollary we obtain

2.4. **Lemma.** *If* D *is a domain which is quasiconformally collared on the boundary, then* $\partial D = \partial \bar{D}$ *and* ∂D *contains only a finite number of components.*

2.5. **Lemma.** *A plane domain is quasiconformally collared on the boundary if and only if its boundary consists of a finite number of disjoint Jordan curves.*

Proof. The sufficiency part is obvious. For the necessity part, suppose that a domain D is quasiconformally collared on the boundary. Then D is uniformly locally connected on the boundary. Since all components of ∂D must be non-degenerate, they are necessarily Jordan curves. Lemma 2.4 then completes the proof.

2.6. REMARK. A bounded domain D whose boundary consists of finitely many $(n-1)$-manifolds satisfying the following geometric condition is quasiconformally collared on the boundary ([1, 1.19]): For each point $b \in \partial D$ there exists a neighborhood U of b and a vector e such that, given any two points $b_1, b_2 \in U \cap \partial D$, the acute angle which the segment $b_1 b_2$ makes with e is never less than some $\alpha > 0$. Thus, for example, bounded convex domains, polyhedrons, domains bounded by finitely many, disjoint, differentiable $(n-1)$-manifolds, etc. are quasiconformally collared on the boundary. In particular, every ball is quasiconformally collared on the boundary.

2.7. OTHER BOUNDARY PROPERTIES. We next define a number of concepts weaker than the quasiconformal collaredness allowing us to measure the regularity of the boundary of a more general domain. These concepts will be employed in the next section when studying the continuous boundary extension of quasiconformal mappings.

2.8. DEFINITION. Let D be a domain.

(1) D is *flat* on the boundary if $q_D(F, F^*) = 0$ whenever F and F^* are non-degenerate connected subsets of D with $q(F, F^*) = 0$.

(2) D *is quasiconformally flat* on the boundary if $M(\Delta(F, F^* : D)) = \infty$ whenever F and F^* are non-degenerate connected subsets of D with $q(F, F^*) = 0$.

(3) D is *quasiconformally connected* on the boundary if for each point $b \in \partial D$ there is a decreasing sequence of neighborhoods U_k of b such that $U_k \cap D$ is connected and $\lim M(\Delta(A, U_k \cap D : D)) = 0$ for some (and hence for each (cf. [2])) continuum A in D.

(4) D is *finitely connected* on the boundary if each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ consists of a finite number of components.

(5) Each boundary point of D is *accessible from all sides* from D if, given a sequence (x_k) of points in D accumulating at ∂D, there exists

a closed Jordan arc lying in D except for one end point and containing a subsequence of (x_k).

(6) D is a *uniform domain* if for each $r > 0$ there is a $\delta > 0$ such that $M(\Delta(F, F^* : D)) \geq \delta$ whenever F and F^* are connected subsets of D with $q(F) \geq r$ and $q(F^*) \geq r$.

2.9. Remark. The concept of quasiconformal flatness, due to Väisälä, and the finite connectedness property have been previously employed in [1] and in Väisälä [4, §17] for the study of the boundary behavior of quasiconformal mappings. Uniform domains are considered in [2]. As far as the writer is aware, the remaining three properties of 2.8 have not been previously published.

2.10. **Lemma.** *A domain which is quasiconformally collared on the boundary has properties* (1)—(6) *as defined in* 2.8.

Proof. In view of Lemma 2.3, the assertions concerning properties (1), (3), (4), and (5) are obvious. Property (2) has been proved by Väisälä [4, §17], while property (6) has been established in [2].

3. Extension theorems

3.1. **Theorem.** *Let D_0 be a domain which is quasiconformally collared on the boundary and let $f: D \to D_0$ be a quasiconformal mapping. Then the following statements are equivalent*:

(1) *f can be extended to a continuous mapping $\bar{f}: \bar{D} \to \bar{D}_0$.*

(2) *D is flat on the boundary.*

(3) *D is quasiconformally flat on the boundary.*

(4) *D is quasiconformally connected on the boundary.*

Proof. To prove that (1) implies (4), fix a point $b \in \partial D$ and choose sequences (V_k) and (W_k) of neighborhoods of b and $\bar{f}(b)$, respectively, so that $W_k \cap D_0$ is connected, $\bigcap W_k = \{\bar{f}(b)\}$, and $f(V_k \cap D) \subset W_k$. Setting $U_k = V_k \cup f^{-1}(W_k \cap D_0)$ we obtain the desired sequence of neighborhoods of b.

Conversely, to show that (4) implies (1), fix a point $b \in \partial D$ and choose a continuum $A \subset D$ and a decreasing sequence of neighborhoods U_k of b so that $U_k \cap D$ is connected and

$$\lim M(\Delta(A, U_k \cap D : D)) = 0 .$$

By the quasiconformality of f,

$$\lim M(\Delta(fA\,, f(U_k \cap D) : fD)\,) = 0\,.$$

Since $fD = D_0$ is a uniform domain,

$$\lim\ q(f(U_k \cap D)\,) = 0\,.$$

Thus the cluster set of f at b reduces to a single point, i.e. f has a limit at b.

To complete the proof we show that (1) ⇒ (3) ⇒ (2) ⇒ (1). The implication (1) ⇒ (3) follows from the fact that D_0 is quasiconformally flat on the boundary, while (3) implies (2) because $M(\Delta(F\,, F^* : D)) < \infty$ whenever F and F^* are non-degenerate connected subsets of D with $q_D(F\,, F^*) > 0$.

In order to prove the implication (2) ⇒ (1), suppose, contrary to the assertion, that there exist points $b \in \partial D\,, b_1 \in \partial D_0\,, b_2 \in \partial D_0\,, b_1 \neq b_2$, and sequences $(b_{1k})\,, (b_{2k})$ of points in D with $b_{ik} \to b\,, f(b_{ik}) \to b_i\,, i = 1\,, 2$. Choose a continuum A in D and for $i = 1\,, 2$ choose a neighborhood U_i of b_i so that $U_i \cap D_0$ is connected and $\bar U_i \cap fA = \emptyset = \bar U_1 \cap \bar U_2$. Let $F_i = f^{-1}(U_i \cap D_0)$. Since F_i is connected and since $b \in \bar F_1 \cap \bar F_2$, there exists, by hypothesis, a sequence of paths γ_k in D joining F_1 to F_2 so that $\lim q(|\gamma_k|) = 0$. Obviously

$$M(\Delta\,(A\,, |\gamma_k| : D)\,) \to 0$$

because $q(A\,, |\gamma_k|) \geq q_0$ for some $q_0 > 0$ and for all k sufficiently large. But $fD = D_0$ is a uniform domain and therefore

$$M\,(\Delta\,(fA\,, f|\gamma_k| : fD)\,) \not\to 0\,,$$

which contradicts the quasiconformality of f. The proof is complete.

3.2. **Theorem.** *Let D_0 be a domain which is quasiconformally collared on the boundary and let $f : D_0 \to D$ be a quasiconformal mapping. Then the following statements are equivalent*:

(1) *f can be extended to a continuous mapping $\bar f : \bar D_0 \to \bar D$.*
(2) *D is finitely connected on the boundary.*
(3) *Each boundary point of D is accessible from all sides from D.*
(4) *D is a uniform domain.*

Proof. To prove that (1) implies (4), fix $r > 0$. Since f is uniformly continuous, there is an $r_0 > 0$ such that $q(f^{-1}F) \geq r_0$ whenever F is a connected set in D with $q(F) \geq r$. Since $D_0 = f^{-1}D$ is a uniform domain,

$$M(\Delta(F\,, F^* : D)\,) \geq M(\Delta(f^{-1}F\,, f^{-1}F^* : f^{-1}D)\,)\,/\,K(f) \geq \delta_0/K(f)$$

whenever F and F^* are connected sets in D with $q(F) \geq r$ and $q(F^*) \geq r$, where $K(f) < \infty$ is the maximal dilatation of f in D_0 and $\delta_0 > 0$ is a constant corresponding to the domain D_0 and the number r_0 in the definition of a uniform domain. This shows that D is a uniform domain.

Conversely, (4) implies (1) by virtue of the argument given for the implication (4) $\Rightarrow$ (1) in Theorem 3.1.

Since D_0 is quasiconformally flat on the boundary, (2) implies (1) by Väisälä [4, §17], while (1) implies (3) because D_0 has the topological property described in (3), and (3) implies (2) because if U is a neighborhood of a boundary point b of D such that $V \cap D$ contains infinitely many components for each neighborhood $V \subset U$ of b, there exists a sequence (b_k) converging to b so that the points b_k belong to different components of $U \cap D$, which contradicts (3). The proof is complete.

As a corollary of Theorems 3.1 and 3.2 we obtain

3.3. **Theorem.** *Let D_0 be a domain which is quasiconformally collared on the boundary and let D be a second domain. Then each or no quasiconformal mapping of D onto D_0 (respectively of D_0 onto D) can be extended to a continuous mapping between the closures.*

For plane domains we have:

3.4. **Theorem.** *Let D_0 be a plane domain bounded by a finite number of disjoint Jordan curves and let $f: D \to D_0$ be a quasiconformal mapping. Then f can be extended to a continuous mapping $\bar{f}: \bar{D} \to \bar{D}_0$ if and only if D has one (each) of the properties* (1) — (3) *as defined in* 2.8. *The inverse mapping f^{-1} can be extended to a continuous mapping $\bar{f}^{-1}: \bar{D}_0 \to \bar{D}$ if and only if D has one (each) of the properties* (4) — (6) *as defined in* 2.8.

Proof. The assertions follow from Lemma 2.5 and Theorems 3.1 and 3.2.

3.5. Remarks. (1) Theorem 3.1 yields the following extremal length result: Let D_0 be a domain which is quasiconformally collared on the boundary, let D be quasiconformally equivalent to D_0, and let F and F^* be non-degenerate connected sets in D. Then $M(\Delta(F, F^* : D)) = \infty$ if and only if $q_D(F, F^*) = 0$.

(2) Let D be a plane domain with finitely many boundary components. Then the properties (1) — (3) in 2.8 are equivalent. Similarly, the properties (4) — (6) are equivalent.

(3) One obtains several different characterizations of Jordan domains in the plane from Theorems 3.1 and 3.2. As an example we give the following metric-topological characterization: A simply connected plane domain with a non-degenerate boundary is a Jordan domain if and only if it is flat and finitely connected on the boundary.

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