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ISOMORPHISMS OF SPACES AND CONVOLUTION ALGEBRAS OF FUNCTIONS

BY

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1. Introduction

Let G_1 and G_2 be locally compact Hausdorff topological groups. In this paper we characterize the isometric and bipositive isomorphisms between the convolution algebras $C_c(G_j)$ of real or complex valued continuous functions with compact support. It turns out that such isomorphisms are precisely those induced by a topological isomorphism between the groups, followed by multiplication with a continuous multiplicative function which is positive in the bipositive and has modulus one in the isometric case (Theorem 4.1). Our method of proof is based on a characterization of the bipositive and isometric linear isomorphisms between the spaces $C_c(S_j)$ for two locally compact Hausdorff spaces S_1 and S_2 (Theorem 3.1). Theorem 4.1 is a sharpening of a result of R. E. Edwards ([2] Theorem 2) who showed that the existence of either type of isomorphism implies that the groups G_j are topologically isomorphic.

In the case of two compact or locally finite discrete groups G_j with normalized Haar measure the general form of a convolution algebra isomorphism $T : C_c(G_1) \to C_c(G_2)$ has been determined by G. V. Wood [7] under the hypothesis that T is merely norm-decreasing. However, in our more general situation this assumption is not even sufficient to guarantee that the underlying groups are algebraically isomorphic. Indeed, there are two non-isomorphic finite groups G_j admitting a (by finite-dimensionality topological) convolution algebra isomorphism $T : C_c(G_1) \to C_c(G_2)$ (see [6] p. 306), and as Wood remarks in [7], p. 775, the Haar measure of G_2 may be so adjusted that T divided by its norm is a norm-decreasing isomorphism.

2. Notation

For a locally compact Hausdorff space S, let $C_c(S, \mathbf{R})$ and $C_c(S, \mathbf{C})$ (resp. $C_0(S, \mathbf{R})$ and $C_0(S, \mathbf{C})$) denote the spaces of the continuous real and complex valued functions on S having compact support (resp. vanishing at infinity). The spaces are regarded as equipped with the uniform norm: $\|f\| = \sup_{x \in S} |f(x)|$. The letter $\mathbf{K}$ is used to refer (consistently) to both the real field $\mathbf{R}$ and the complex field $\mathbf{C}$. We often write for short $C_0(S)$, $C_c(S)$ instead of $C_0(S, \mathbf{K})$, $C_c(S, \mathbf{K})$. For a function f on S we write $f \geq 0$ if f is real valued and $f(x) \geq 0$ for all $x \in S$. An operator $T : C_c(S_1) \to C_c(S_2)$ or $T : C_0(S_1) \to C_0(S_2)$ is *bipositive* if $Tf \geq 0$ is equivalent to $f \geq 0$.

3. Isomorphisms of function spaces

In the isometric case the proof of the next theorem is essentially one of the standard proofs of the well-known Banach-Stone theorem, usually stated for compact spaces (see e.g. [5] pp. 334—335). For classical results in the bipositive case see [3] and [4].

Theorem 3.1. *Let S_1 and S_2 be locally compact Hausdorff spaces and $T : C_0(S_1) \to C_0(S_2)$ a vector space isomorphism.*

(i) *T is isometric if and only if there exist a homeomorphism $\alpha : S_2 \to S_1$ and a continuous $\mathbf{K}$-valued function h on S_2 such that $|h(y)| = 1$ for all $y \in S_2$ and*

$$Tf(y) = h(y)f(\alpha(y)) \, , f \in C_0(S_1) \, , \; y \in S_2 \, . \tag{1}$$

(ii) *T is bipositive if and only if there exist a homeomorphism $\alpha : S_2 \to S_1$ and a continuous everywhere positive function h on S_2 for which the formula* (1) *above holds.*

The theorem remains valid if $C_0(S_j)$ is replaced throughout by $C_c(S_j)$, $j = 1, 2$.

Proof. In each case the sufficiency of the existence of α and h with the stated properties is obvious. To begin the proof of necessity let us observe that, as a consequence of Urysohn's lemma, a set $G \subset S_j$ is open if and only if for any $x \in G$ there is a function f in $C_0(S_j)$ (resp. in $C_c(S_j)$) with $f(x) \neq 0$ and $f(y) = 0$ for all $y \in S_j \setminus G$. It follows that if $T : C_0(S_1) \to C_0(S_2)$ (resp. $T : C_c(S_1) \to C_c(S_2)$) and $\alpha : S_2 \to S_1$ are bijections and $h : S_2 \to \mathbf{K}$ a nowhere vanishing function such that formula (1) holds, then α and α^{-1} map open sets onto open sets, i.e. α is a homeomorphism. Therefore h, too, is continuous, being locally the quotient of two continuous functions. We are left with the task of constructing α and h.

Let us first consider the case of a bipositive isomorphism $T : C_0(S_1, \mathbf{C}) \to C_0(S_2, \mathbf{C})$. The non-zero multiplicative linear forms on the commutative C^*-algebra $C_0(S_j, \mathbf{C})$ are precisely the pure states of $C_0(S_j, \mathbf{C})$ (see [1], 2.5.2). Let P_j denote the convex cone of the positive linear forms on $C_0(S_j, \mathbf{C})$. It follows easily from the definition ([1], 2.5.2) that a non-zero functional $\varphi \in P_j$ belongs to the ray $\{\lambda\gamma \mid \lambda \in \mathbf{R}, \lambda > 0\}$ generated by some pure state γ if and only if $\varphi - \psi \in P_j$ for $\psi \in P_j$ implies that

$\psi = \mu\varphi$ for some real number μ, $0 \leq \mu \leq 1$. This property is obviously preserved by the algebraic transpose T^* of T since it maps P_2 bijectively onto P_1. In particular, for each pure state γ of $C_0(S_2, \mathbf{C})$, $T^*\gamma$ is a positive multiple of a pure state of $C_0(S_1, \mathbf{C})$. Therefore, if we identify S_j via the natural evaluation map with the set of the non-zero multiplicative linear functionals on $C_0(S_j, \mathbf{C})$, we get a bijection $\alpha : S_2 \to S_1$ by dividing T^*y with a positive number $h(y)$ for each $y \in S_2$, i.e.

$$f(\alpha(y)) = \langle f, \frac{1}{h(y)} T^*y \rangle = \frac{1}{h(y)} Tf(y), \ f \in C_0(S_1, \mathbf{C}).$$

The case of a bipositive isomorphism $T : C_0(S_1, \mathbf{R}) \to C_0(S_2, \mathbf{R})$ is reduced to the above by considering $C_0(S_j, \mathbf{R})$ as a real linear subspace of $C_0(S_j, \mathbf{C})$ and extending T to an operator T_c from $C_0(S_1, \mathbf{C})$ to $C_0(S_2, \mathbf{C})$ by setting $T_c(f + ig) = Tf + iTg$.

Suppose next that $T : C_0(S_1, \mathbf{K}) \to C_0(S_2, \mathbf{K})$ is an isometric isomorphism. We make use of the well-known fact that the extreme points of the unit ball U_j of the Banach space adjoint of $C_0(S_j, \mathbf{R})$ (resp. $C_0(S_j, \mathbf{C})$) are precisely the Dirac measures multiplied by a real (resp. complex) number with modulus one. As the transpose of T maps the extreme points of U_2 onto the extreme points of U_1, and the Dirac measures on S_j may be identified with the points of S_j, we obtain as above a bijection $\alpha : S_2 \to S_1$ and a function $h : S_2 \to \mathbf{K}$ with $|h(y)| = 1$ for all $y \in S_2$, such that (1) holds.

Since $C_c(S_j)$ is norm dense in $C_0(S_j)$, the assertion concerning a linear isometry from $C_c(S_1)$ onto $C_c(S_2)$ may be proved by reducing it to the previous case.

Finally, let us assume that $T : C_c(S_1, \mathbf{K}) \to C_c(S_2, \mathbf{K})$ is a bipositive isomorphism. Then a separate argument is needed (see the remark below).

We denote $C_c(S_j)^+ = \{f \in C_c(S_j, \mathbf{R}) \mid f(x) \geq 0 \text{ for all } x \in S_j\}$,

$$P_f = \{x \in S_j \mid f(x) > 0\} \quad \text{for } f \in C_c(S_j),$$

and

$$F_x = \{f \in C_c(S_j)^+ \mid f(x) > 0\} \quad \text{for } x \in S_j, \quad j = 1, 2.$$

The closure of P_f for $f \in C_c(S_j)^+$ is denoted supp(f) as usual. Let us first show that

$$(2) \qquad \bigcap_{f \in F_x} \operatorname{supp}(Tf) \neq \emptyset, \ x \in S_1.$$

As every supp(Tf) is compact, it is enough to prove that $\bigcap_{f \in F} \operatorname{supp}(Tf) \neq \emptyset$ for every finite subset F of F_x. There exist a constant $\delta > 0$, a neighbourhood U of x such that $f(t) > \delta$ on U for each $f \in F$, and a function $g \in C_c(S_1)^+$ with maximum value δ and support contained in U. The positivity of T then implies $\bigcap_{f \in F} \operatorname{supp}(Tf) \supset \operatorname{supp}(Tg) \neq \emptyset$. Next we make use of (2) in proving the stronger statement

$$A_x = \bigcap_{f \in F_x} P_{Tf} \neq \emptyset,\ x \in S_1\,. \tag{3}$$

Indeed, by (2) it is sufficient to find for each $f \in F_x$ a function $g \in F_x$ such that $\operatorname{supp}(Tg) \subset P_{Tf}$. The function g may be constructed as follows. Choose $f_1 \in C_c(S_1)^+$ and $f_2 \in C_c(S_2)^+$ so that $f_1(x) > 0$ and f_2 is positive on $\operatorname{supp}(Tf)$. When the function $f_1 + T^{-1} f_2$ is multiplied with a small enough positive number, we obtain a function $f_3 \in C_c(S_1)^+$ such that $0 < f_3(x) < f(x)$ and Tf_3 is greater than a positive constant on $\operatorname{supp}(Tf)$. Now define $g = f - \inf(f, f_3)$, so that by the linearity and bipositivity of T we have $Tg = Tf - \inf(Tf, Tf_3)$. Then

$$\operatorname{supp}(Tg) \subset \{y \in S_2 | Tf(y) \geq \inf_{t \in \operatorname{supp}(Tf)} Tf_3(t)\} \subset P_{Tf},$$

since the middle set is closed and in its complement $Tg(y) = Tf(y) - Tf(y) = 0$. Clearly, $g \in C_c(S_1)^+$ and $g(x) > 0$. Thus g has all the desired properties, and (3) follows. Suppose now $y \in A_x$. We show that $F_x = T^{-1}(F_y)$. First of all, it is clear that $F_x \subset T^{-1}(F_y)$. Using this after applying (3) to T^{-1} and y, and then observing that $\bigcap_{f \in F_x} P_f = \{x\}$ we have, conversely, for any $g \in T^{-1}(F_y)$

$$\emptyset \neq \bigcap_{f \in T^{-1}(F_y)} P_f \subset P_g \cap (\bigcap_{f \in F_x} P_f) \subset \{x\},$$

i.e. $x \in P_g$ or $g \in F_x$. Thus the sets F_x and F_y are by symmetry in one-to-one correspondence via T, and as F_x determines x uniquely, we get a bijection $\alpha : S_2 \to S_1$ with the property $T^{-1}(F_y) = F_{\alpha(y)}$, $y \in S_2$, which implies

$$P_{Tf} = \alpha^{-1}(P_f)\,,\ f \in C_c(S_1)^+\,. \tag{4}$$

The function h is now defined by choosing for each $y \in S_2$ a function $f \in C_c(S_1)^+$ with $f(\alpha(y)) = 1$ and setting $h(y) = Tf(y)$. It follows from (4) that $h(y) > 0$ and this definition is independent of the choice of f, for if $f_1(\alpha(y)) = f_2(\alpha(y)) = 1$, the non-negative functions $f_1 - f'$ and $f_2 - f'$, where $f' = \inf(f_1, f_2)$, vanish at $\alpha(y)$, so $Tf_1 - Tf'$ and $Tf_2 - Tf'$ vanish at y, i.e. $Tf_1(y) = Tf_2(y)$. Thus, in particular,

$$h(y) f(\alpha(y)) = T\left(\frac{1}{f(\alpha(y))} f\right)(y)\, f(\alpha(y)) = Tf(y)$$

for any $f \in C_c(S_1)^+$ with $f(\alpha(y)) > 0$. If $f \in C_c(S_1)^+$ and $f(\alpha(y)) = 0$, the equation $Tf(y) = h(y)f(\alpha(y))$ is also given by (4). As every $f \in C_c(S_1)$ may be expressed as a linear combination of some elements of $C_c(S_1)^+$, the proof of this last part of the theorem is also complete.

Remark. Unlike the isometric case, the proof of the above result involving a bipositive isomorphism $T : C_c(S_1) \to C_c(S_2)$ cannot be reduced to the part of the theorem concerned with the spaces $C_0(S_j)$, since a bipositive isomorphism $T : C_c(S_1) \to C_c(S_2)$ is not always the restriction to $C_c(S_1)$ of any bipositive isomorphism from $C_0(S_1)$ onto $C_0(S_2)$. For example, it is quickly seen from the above theorem that any bipositive isomorphism $T_0 : C_0(\mathbf{R}, \mathbf{R}) \to C_0(\mathbf{R}, \mathbf{R})$ extending the mapping $T : C_c(\mathbf{R}, \mathbf{R}) \to C_c(\mathbf{R}, \mathbf{R})$, $Tf(x) = (1 + x^2)f(x)$, should also have the form $T_0 f(x) = (1 + x^2)f(x)$, which is impossible, as e.g. the function $x \mapsto (1 + x^2)(1 + x^2)^{-1} = 1$ does not belong to $C_0(\mathbf{R}, \mathbf{R})$.

4. Isomorphisms of convolution algebras

For a locally compact Hausdorff topological group G, let us regard $C_c(G, \mathbf{K})$ as an algebra under the convolution product

$$(f*g)(x) = \int_G f(y)g(y^{-1}x)dy ,$$

integration being with respect to a fixed left Haar measure denoted dy. We prepare the theorem of this section with two lemmas.

Lemma 4.1. *Let G_1 and G_2 be locally compact Hausdorff topological groups, $T : C_c(G_1) \to C_c(G_2)$ a convolution algebra isomorphism and $\alpha : G_2 \to G_1$ a bijection such that, for any $f \in C_c(G_1)$, $Tf(y) = 0$ is equivalent to $f(\alpha(y)) = 0$. Then α is a group isomorphism.*

Proof. Suppose that $\alpha(xy) \neq \alpha(x)\alpha(y)$ for some $x, y \in G_2$. Then there exist neighbourhoods $U \ni \alpha(x)$ and $V \ni \alpha(y)$ such that $\alpha(xy) \notin UV$. Since α is continuous by an argument used at the beginning of the proof of Theorem 3.1, $\alpha^{-1}(U)$ is a neighbourhood of x and $\alpha^{-1}(V)$ one of y. There exist non-negative real functions $f, g \in C_c(G_2)$ with $f(x) > 0$, $g(y) > 0$ such that f vanishes outside $\alpha^{-1}(U)$ and g outside $\alpha^{-1}(V)$. By hypothesis, $T^{-1}f(t) = 0$ for $t \in G_1 \setminus U$ and $T^{-1}g(t) = 0$ for $t \in G_1 \setminus V$. Therefore, $(T^{-1}f) * (T^{-1}g)(t) = \int_{G_1} T^{-1}f(z)T^{-1}g(z^{-1}t)dz = 0$ for all $t \in G_1 \setminus UV$ the integrand being zero both in U and $G_1 \setminus U$. In particular,

$$(1) \qquad (T^{-1}f) * (T^{-1}g)(\alpha(xy)) = 0 .$$

On the other hand, by continuity we can find a symmetric neighbourhood W of the neutral element $e \in G_2$ such that $f(t) > \delta$ for $t \in xW, g(t) > \delta$ for $t \in Wy$, where δ is a positive constant. Thus

$$(f * g)(xy) = \int_{G_2} f(z)g(z^{-1}xy)dz = \int_{G_2} f(xz)g(z^{-1}y)dz \geq \int_{W} \delta^2 dz > 0.$$

The hypothesis then implies

$$(T^{-1}f) * (T^{-1}g)(\alpha(xy)) = T^{-1}(f * g)(\alpha(xy)) \neq 0 ,$$

and this contradiction with (1) proves the lemma.

Lemma 4.2. *Suppose h is a continuous* **K***-valued function on the locally compact Hausdorff topological group G. The mapping $T : C_c(G, \mathbf{K}) \to C_c(G, \mathbf{K}), Tf(x) = h(x)f(x)$ for $x \in G$, is a convolution algebra homomorphism if and only if $h(xy) = h(x)h(y)$ for all $x, y \in G$.*

Proof. If h is multiplicative, a straightforward calculation shows that T preserves convolution. To prove the converse, choose for each compact neighbourhood U of the neutral element e a continuous non-negative real function f_U such that f_U vanishes outside U and $\int_G f_U(x)dx = 1$. We denote $f^u(t) = f(u^{-1}t)$ for a function $f \in C_c(G)$ and define $g_U^{(x,y)} = f_U^x * f_U^y$ for each couple (x , y) of elements of G. Then

$$\int_G g_U^{(x,y)}(t)dt = 1 ,$$

$g_U^{(x,y)}$ is non-negative and vanishes outside $V = xUyU$, and V is eventually contained in any neighbourhood of xy, when the set $\mathcal{U}$ of the compact neighbourhoods of e is given the natural order opposite to inclusion. It follows easily that

$$h(xy) = \lim_U \int_G h(t)g_U^{(x,y)}(t)dt .$$

Similarly,

$$h(u) = \lim_U \int_G h(t)f_U^u(t)dt , \ u \in G .$$

Therefore, if T preserves convolution, Fubini's theorem yields

$$h(xy) = \lim_U \int_G h(t)g_U^{(x,y)}(t)dt = \lim_U \int_G T(f_U^x * f_U^y)(t)dt =$$

$$= \lim_U \int_G (Tf_U^x) * (Tf_U^y)(t)dt = \lim_U \int_G \int_G h(z)f_U^x(z)h(z^{-1}t)f_U^y(z^{-1}t)dzdt =$$

$$= \lim_U \int_G h(z)f_U^x(z)dz \cdot \lim_U \int_G h(t)f_U^y(t)dt = h(x)h(y) .$$

Theorem 4.1. *Let G_1 and G_2 be locally compact Hausdorff spaces and $T : C_c(G_1, \mathbf{K}) \to C_c(G_2, \mathbf{K})$ a bipositive (resp. isometric) vector space isomorphism Then there exist a homeomorphism $\alpha : G_2 \to G_1$ and a continuous everywhere positive function h on G_2 (resp. a continuous function $h : G_2 \to \mathbf{K}$ with $|h(y)| = 1, y \in G_2$) such that*

$$Tf(y) = h(y)f(\alpha(y)) \, , f \in C_c(G_1 \, , \mathbf{K}) \, , y \in G_2 \, .$$

Suppose, in addition, that G_1 and G_2 are topological groups. Then T is a convolution algebra isomorphism if and only if α is a group isomorphism and $h(xy) = h(x)h(y)$ for all $x, y \in G_2$.

Proof. The first assertion is contained in Theorem 3.1, and the second is an immediate consequence of the above lemmas.

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