ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

499

BOUNDED ANALYTIC FUNCTIONS WITH LARGE CLUSTER SETS

BY

A. J. LOHWATER and GEORGE PIRANIAN

HELSINKI 1971
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1971.499

Communicated 13 September 1971 by Olli Lehto

KESKUSKIRJAPAINO
HELSINKI 1971

An analytic function $w = f(z)$ in the unit disk D of the z-plane is called an *inner function* if $|f(z)| < 1$ in D and if $f(z)$ possesses radial limits of modulus 1 at almost all points of $|z| = 1$. Such functions were studied extensively by O. Frostman [4] and W. Seidel [6]. In particular, at each point P on $|z| = 1$ the cluster set $C(f, P)$ of an inner function consists either of a single point of modulus 1 or else of the closed disk $|w| \leq 1$ (see [6, Theorem 6 and its corollary]).

Every bounded analytic function in D has a representation

$$f(z) = e^{i\alpha} B(z) \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right), \tag{1}$$

where α is a real constant, $B(z)$ is a Blaschke product extended over the zeros of $f(z)$, and $\mu(t)$ is a nonincreasing function (see [1, p. 40], for example). The function μ has a decomposition $\mu = \mu_1 + \mu_2$, where μ_1 is singular and μ_2 is absolutely continuous. The inner functions are those bounded analytic functions for which μ_2 is constant. Bounded analytic functions that have no zeros and for which μ_1 is constant are called *outer functions.*

The following question arises naturally.: If $f(z)$ is analytic and bounded in D and if at each point P of $|z| = 1$ the cluster set $C(f, P)$ consists either of a single point of modulus 1 or else of the closed disk $|w| \leq 1$, must $f(z)$ be an inner function? In case the answer is negative, do there exist outer functions with this property? Questions such as these have been raised by G. Csordás [2] and L. Rubel (private communication).

In our first theorem, we show that there exists a bounded analytic function whose cluster set $C(f, P)$ is the closed disk $|w| \leq 1$ for every P, but which has no radial limit of modulus 1; this answers the first question of the preceding paragraph. In Theorem 2, we show that an appropriate refinement of the simple construction in Theorem 1 leads to a function with the same property and with the additional feature that the new function has a finite Dirichlet integral. In Theorem 3, we show that there exist outer functions with both properties.

THEOREM 1. *There exists an analytic function $w = f(z)$ in D such that $|f(z)| < 1$ in D, such that for each point P on $|z| = 1$ the cluster set*

G. Piranian acknowledges support from the National Science Foundation.

$C(f, P)$ is the closed disk $|w| \leq 1$, and such that $f(z)$ does not have a radial limit of modulus 1 at any point P.

Proof. We shall construct a special domain G in $|w| < 1$, form the universal covering surface G^∞ over G, and show that each conformal mapping of D onto G generates a function f with the required properties.

We create a domain G_0 by deleting from the disk $|w| < 1$ a curve σ that spirals from the origin toward the circle $|w| = 1$. From G_0, we obtain G by the deletion of a denumerable set E_0 whose derived set consists of the curve σ and the circle $|w| = 1$; we subject E_0 to the additional requirement that if δ is any disk whose center lies on σ, then each component of $\delta \cap G_0$ contains points of E_0. Clearly, G has infinite connectivity, and its boundary consists of the set E_0, the spiral σ, and the circle $|w| = 1$.

Let φ denote a conformal mapping of D onto G^∞, and let f be the composition of φ with the projection of G^∞ onto G. Then f is analytic in D, $|f(z)| < 1$, and $f(z)$ assumes each value in G infinitely often. From the general theory of inverse functions (see [1, Chapter 6], for example) it follows that each point of the set E_0 is the radial limit of $f(z)$ on a set that is dense on $|z| = 1$. From this it follows in turn that for each point P on $|z| = 1$, the set $C(f, P)$ consists of the closed disk $|w| \leq 1$.

To show that no radial limit of f has modulus 1, we merely observe that no path on the Riemann surface G^∞ converges to a single point on $|w| = 1$.

In the statement of the following theorem, we call a point set a *boundary path in D* if it consists of the values $z = g(s)$, where $g(s)$ denotes a continuous function $(0 \leq s < 1)$ such that $|g(s)| < 1$ for all s and $\lim_{s \to 1} |g(s)| = 1$.

THEOREM 2. *There exists an analytic function $w = f(z)$ in D such that*

(i) *$|f(z)| < 1$ in D,*

(ii) *for each point P on $|z| = 1$, the cluster set $C(f, P)$ consists of the closed disk $|w| \leq 1$,*

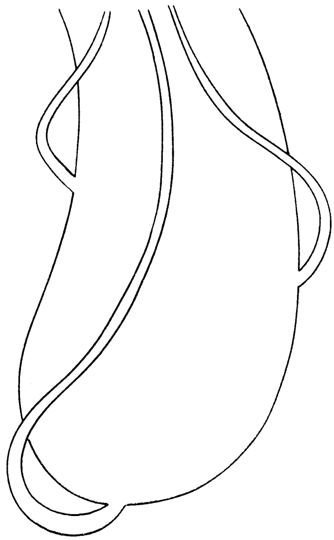
(iii) *for each boundary path λ in D, the cluster set $C_\lambda(f, P)$ either contains no point of the circle $|w| = 1$ or else consists of the entire closed disk $|w| \leq 1$,*

(iv) *the Dirichlet integral of f is finite,*

(v) *the set of Fatou values of f has two-dimensional measure π.*

Proof. Let γ denote a path in $|w| < 1$ that begins at $w_1 = 1/2$ and passes exactly once through each of the points $w_2, w_3, \ldots$, where

$$w_n = \begin{cases} 2^{-n} & (n \text{ odd}), \\ 1 - 2^{-n} & (n \text{ even}) \end{cases}$$



(γ cannot be a boundary path). We may assume that for each n the arc γ_n from w_n to w_{n+1} is simple and rectifiable, and that no point of $|w| < 1$ lies at a distance greater than $1/n$ from γ_n. Let γ have the representation $w = w(s)$ $(0 \leq s < \infty)$, where s denotes arc length on γ. For each s, let D_s denote the disk $|w - w(s)| < 1/8e^s$; without loss of generality, we may assume that each disk D_s lies at a positive distance from the circle $|w| = 1$. Obviously, the disks D_s determine a simply connected Riemann surface G_1 in the form of a ribbon that winds over the disk $|w| < 1$. We divide the portion of the boundary of G_1 over the disk $|w| < 1$ into arcs of diameter at most $1/2$, and along each of these arcs we attach to G_1 a Riemann surface that lies over G_1 (except for a short section near the attachment; see the figure) and follows its contortions indefinitely.

When the Riemann surface G_n has been constructed, we divide the portion of its boundary over the disk $|w| < 1$ into arcs of diameter at most 2^{-n}; along a portion of each of these arcs we attach to G_n a Riemann surface that lies over the corresponding ribbon of G_n (except for a short portion) and follows it indefinitely; and we denote the Riemann surface thus obtained by G_{n+1}. The continuation of the process yields a simply connected Riemann surface G^*, and each conformal mapping of D onto G^* determines a function f with the first two of the properties listed in the theorem. To see that condition (iii) is also satisfied, we merely observe that if the projection of a path in G^* has a limit point on the circle $|w| = 1$, then the projection of path is dense on the unit disk.

To ensure that the functions f associated with G^* satisfy condition (iv), we need only make certain that the total area of the Riemann surfaces comprising G^* is finite.

Finally, to make certain that almost all points in the disk $|w| < 1$ are Fatou values of the functions f, it is simplest to abandon the rectifiability of the arcs γ_n that comprise the path γ (naturally, this also requires modifications in the definition of the disks D_t). We can then choose the path γ and the supporting Riemann surface G_1 so that the projection of the set B_1 of its accessible boundary points has measure at least $3\pi/4$. Whenever we attach a Riemann surface to G_1, we do it in such a way that the set of accessible boundary points of G^* contains a subset of B_1 whose measure is at least $\pi/2$. Similarly, we make certain that the projection of the set of accessible boundary points of the Riemann surface G_m has measure at least $(1 - 2^{-m-1})\pi$, and that a set of measure at least $(1 - 2^{-m})\pi$ survives all later modifications. This completes the proof of Theorem 2.

In connection with condition (iv), we observe that if f is a univalent map of the disk D onto G^*, then at each point $e^{i\Theta}$ the radial cluster set of f consists either of an interior point of the disk $|w| < 1$ or else of the closure of the disk. By a slight modification of the classical proof of a theorem of Fejér (see [3] or [5, Section 13]), we see that at each point $e^{i\Theta}$, the sequence of partial sums of the Taylor series of f either converges or has each point in $|w| \leq 1$ as a limit point.

THEOREM 3. *There exists an outer function with the properties listed in Theorem* 2.

Proof. In the proof of Theorem 2, we can obviously construct the Riemann surface G^* so that the point $w = 0$ is the projection of no interior point of G^* and of no accessible boundary point of G^*. Each of the associated functions f is then an outer function; for if in the decomposition $\mu = \mu_1 + \mu_2$ of the function μ in (1) the component μ_1 were not constant, there would be at least one point t_0 such that $\mu_1'(t_0) = -\infty$, and at the point e^{it_0} the function f would have the radial limit 0.

Case Western Reserve University and
The University of Michigan

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Printed November 1971